The FriCAS System for Computer Mathematics

2026-10-18

General note

This is the original Axiom book from 1992 with title "*AXIOM—The Scientific Computation System*" by Jenks and Sutor in its adaptation for the FriCAS fork of the Axiom source code.

It counts as the official version for the FriCAS project.

The original sources (including the text and scripts) for this book were released in 2002, under the modified BSD license that is found in the file LICENSE.txt in the source code repository of FriCAS.

Website: https://fricas.github.io

Repository: https://github.com/fricas/fricas

FriCAS version: 1.4.0

Warning

Since this book is a moving target, each part of its generation is automated. In particular, the output of FriCAS commands (including their typesetting in LaTeX) is automated. However, that sometimes means that you see boxes with output that does not fit the width of the page. We use the breqn LaTeX package to break equations appropriately. Unfortunately, that does not always work nicely.

If you find other inaccuracies, please report them to the mailing list fricas-devel@googlegroups.com.

Authors

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Chapter 0

Introduction to FriCAS

Welcome to the world of FriCAS. We call FriCAS a scientific computation system: a self-contained toolbox designed to meet your scientific programming needs, from symbolics, to numerics, to graphics.

This introduction is a quick overview of what FriCAS offers.

0.1 Symbolic computation

FriCAS provides a wide range of simple commands for symbolic mathematical problem solving. Do you need to solve an equation, to expand a series, or to obtain an integral? If so, just ask FriCAS to do it.

Integrate $\frac{1}{(x^3\ (a+bx)^{1/3})}$ with respect to x.

```
integrate(1/(x^3 * (a+b*x)^(1/3)),x)
```

$$\frac{-2\,b^2\,x^2\,\sqrt{3}\,\log\left(\sqrt[3]{a}\,\sqrt[3]{b\,x+a}^{2}+\sqrt[3]{a}^{2}\,\sqrt[3]{b\,x+a}+a\right)+4\,b^2\,x^2\,\sqrt{3}\,\log\left(\sqrt[3]{a}^{2}\,\sqrt[3]{b\,x+a}-a\right)+12\,b^2\,x^2\,\arctan\left(\frac{2\sqrt{3}\,\sqrt[3]{a}^{2}\,\sqrt[3]{b\,x+a}+a\sqrt{3}}{3\,a}\right.}{18\,a^2\,x^2\,\sqrt{3}\,\sqrt[3]{a}} \tag{1}$$

Union(Expression(Integer), ...)

FriCAS provides state-of-the-art algebraic machinery to handle your most advanced symbolic problems. For example, FriCAS's integrator gives you the answer when an answer exists. If one does not, it provides a proof that there is no answer. Integration is just one of a multitude of symbolic operations that FriCAS provides.

0.2 Numeric computation

FriCAS has a numerical library that includes operations for linear algebra, solution of equations, and special functions. For many of these operations, you can select any number of floating point digits to be carried out in the computation.

Solve $x^{49} - 49x^4 + 9$ to 49 digits of accuracy.

```
digits(49); solve(x^49-49*x^4+9 = 0,1.e-49)
```

$$\begin{aligned}[x &= -0.65465367069042711367181221050959847618512243315 56,\\ x &= 1.08692139565385950849393903595489328900921338876 3,\\ x &= 0.65465367072552717396946860661367648353614876076 61]\end{aligned} \tag{1}$$

List (Equation(Polynomial(Float)))

The output of a computation can be converted to FORTRAN to be used in a later numerical computation. Besides floating point numbers, FriCAS provides literally dozens of kinds of numbers to compute with. These range from various kinds of integers, to fractions, complex numbers, quaternions, continued fractions, and to numbers represented with an arbitrary base.

What is 10 to the 100^{th} power in base 32?

```
radix(10^100,32)
```

$$4I9LKIP9GRSTC5IF164PO5V72ME827226JSLAP462585Q7H00000000000000000000 \tag{2}$$

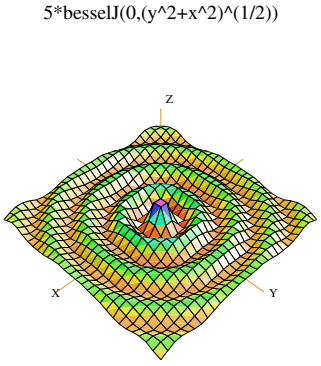
RadixExpansion(32)

0.3 Graphics

You may often want to visualize a symbolic formula or draw a graph from a set of numerical values. To do this, you can call upon the FriCAS graphics capability.

Draw $J_0(\sqrt{x^2+y^2})$ for $-20 \le x, y \le 20$.

```
draw(5*besselJ(0,sqrt(x^2+y^2)), x=-20..20, y=-20..20)
```



Graphs in FriCAS are interactive objects you can manipulate with your mouse. Just click on the graph, and a control panel pops up. Using the mouse and the control panel, you can translate, rotate, zoom, change the coloring, lighting, shading, and perspective on the picture. You can also generate a PostScript copy of your graph to produce hard-copy output.

0.4 HyperDoc

HyperDoc presents you with windows on the world of FriCAS, offering on-line help, examples, tutorials, a browser, and reference material. HyperDoc gives you on-line access to this book in a "hypertext" format. Words that appear in a different font (for example, **Matrix**, factor, and *category*) are generally mouse-active; if you click on one with your mouse, HyperDoc shows you a new window for that word.

As another example of a HyperDoc facility, suppose that you want to compute the roots of $x^{49} - 49x^4 + 9$ to 49 digits (as in our previous example) and you don't know how to tell FriCAS to do this. The "basic command" facility of HyperDoc leads the way. Through the series of HyperDoc windows shown in Figure 1 and the specified mouse clicks, you and HyperDoc generate the correct command to issue to compute the answer.

Figure 1: Computing the roots of $x^{49} - 49x^4 + 9$.

0.5 Interactive Programming

FriCAS's interactive programming language lets you define your own functions. A simple example of a user-defined function is one that computes the successive Legendre polynomials. FriCAS lets you define these polynomials in a piece-wise way.

The first Legendre polynomial.

```
p(0) == 1
```

The second Legendre polynomial.

```
p(1) == x
```

The n^{th} Legendre polynomial for $(n > 1)$.

```
p(n) == ((2*n-1)*x*p(n-1) - (n-1) * p(n-2))/n
```

In addition to letting you define simple functions like this, the interactive language can be used to create entire application packages. All the graphs in the FriCAS Images section in the center of the book, for example, were created by programs written in the interactive language.

The above definitions for `p` do no computation—they simply tell FriCAS how to compute `p(k)` for some positive integer `k`. To actually get a value of a Legendre polynomial, you ask for it.

What is the tenth Legendre polynomial?

```
p(10)
```

```
   Compiling function p with type Integer -> Polynomial(Fraction(Integer))
   Compiling function p as a recurrence relation.
```

$$\frac{46189}{256}x^{10} - \frac{109395}{256}x^{8} + \frac{45045}{128}x^{6} - \frac{15015}{128}x^{4} + \frac{3465}{256}x^{2} - \frac{63}{256} \tag{4}$$

Polynomial(Fraction (Integer))

FriCAS applies the above pieces for `p` to obtain the value of `p(10)`. But it does more: it creates an optimized, compiled function for `p`. The function is formed by putting the pieces together into a single piece of code. By *compiled*, we mean that the function is translated into basic machine-code. By *optimized*, we mean that certain transformations are performed on that code to make it run faster. For `p`, FriCAS actually translates the original definition that is recursive (one that calls itself) to one that is iterative (one that consists of a simple loop).

What is the coefficient of x^{90} in `p(90)`?

```
coefficient(p(90),x,90)
```

$$\frac{5688265542052017822223458237426581853561497449095175}{773712524553362671811952 64} \tag{5}$$

Polynomial(Fraction (Integer))

In general, a user function is type-analyzed and compiled on first use. Later, if you use it with a different kind of object, the function is recompiled if necessary.

0.6 Data Structures

A variety of data structures are available for interactive use. These include strings, lists, vectors, sets, multisets, and hash tables. A particularly useful structure for interactive use is the infinite stream:

Create the infinite stream of derivatives of Legendre polynomials

```
[D(p(i),x) for i in 1..]
```

$$\left[1,\ 3\,x,\ \frac{15}{2}\,x^2-\frac{3}{2},\ \frac{35}{2}\,x^3-\frac{15}{2}\,x,\ \frac{315}{8}\,x^4-\frac{105}{4}\,x^2+\frac{15}{8},\right.$$
$$\left.\frac{693}{8}\,x^5-\frac{315}{4}\,x^3+\frac{105}{8}\,x,\ \frac{3003}{16}\,x^6-\frac{3465}{16}\,x^4+\frac{945}{16}\,x^2-\frac{35}{16},\ \ldots\right] \tag{5}$$

Stream(Polynomial(Fraction(Integer)))

Streams display only a few of their initial elements. Otherwise, they are "lazy": they only compute elements when you ask for them.

Data structures are an important component for building application software. Advanced users can represent data for applications in optimal fashion. In all, FriCAS offers over forty kinds of aggregate data structures, ranging from mutable structures (such as cyclic lists and flexible arrays) to storage efficient structures (such as bit vectors). As an example, streams are used as the internal data structure for power series.

What is the series expansion of $\log(\cot(x))$ about $x = \pi/2$?

```
series(log(cot(x)),x = %pi/2)
```

$$\log\left(\frac{-2\,x+\pi}{2}\right)+\frac{1}{3}\left(x-\frac{\pi}{2}\right)^2+\frac{7}{90}\left(x-\frac{\pi}{2}\right)^4+\frac{62}{2835}\left(x-\frac{\pi}{2}\right)^6+O\left(\left(x-\frac{\pi}{2}\right)^8\right) \tag{6}$$

GeneralUnivariatePowerSeries (Expression (Integer), x, %pi/2)

Series and streams make no attempt to compute *all* their elements! Rather, they stand ready to deliver elements on demand.

What is the coefficient of the 50^{th} term of this series?

```
coefficient(%,50)
```

$$\frac{44590788901016030052447242300856550965644}{71314692864386691115840908813093603545813591308593 75} \tag{7}$$

Expression (Integer)

0.7 Mathematical Structures

FriCAS also has many kinds of mathematical structures. These range from simple ones (like polynomials and matrices) to more esoteric ones (like ideals and Clifford algebras). Most structures allow the construction of arbitrarily complicated "types."

Even a simple input expression can result in a type with several levels.

```
matrix [[x + %i,0], [1,-2]]
```

$$\begin{bmatrix} x+i & 0 \\ 1 & -2 \end{bmatrix} \tag{1}$$

Matrix(Polynomial(Complex(Integer)))

The FriCAS interpreter builds types in response to user input. Often, the type of the result is changed in order to be applicable to an operation.

The inverse operation requires that elements of the above matrices are fractions.

```
inverse(%)
```

$$\begin{bmatrix} \frac{1}{x+i} & 0 \\ \frac{1}{2\,x+2\,i} & -\frac{1}{2} \end{bmatrix} \tag{2}$$

Union(Matrix(Fraction(Polynomial(Complex(Integer)))), ...)

0.8 Pattern Matching

A convenient facility for symbolic computation is "pattern matching." Suppose you have a trigonometric expression and you want to transform it to some equivalent form. Use a `rule` command to describe the transformation rules you need. Then give the rules a name and apply that name as a function to your trigonometric expression.

Introduce two rewrite rules.

```
sinCosExpandRules := rule
  sin(x+y) == sin(x)*cos(y) + sin(y)*cos(x)
  cos(x+y) == cos(x)*cos(y) - sin(x)*sin(y)
  sin(2*x) == 2*sin(x)*cos(x)
  cos(2*x) == cos(x)^2 - sin(x)^2
```

$$\begin{aligned} &\{\sin(y+x) == \cos(x)\,\sin(y) + \cos(y)\,\sin(x),\ \cos(y+x) == -\sin(x)\,\sin(y) + \cos(x)\,\cos(y), \\ &\ \sin(2\,x) == 2\,\cos(x)\,\sin(x),\ \cos(2\,x) == -\sin(x)^2 + \cos(x)^2\} \end{aligned} \tag{1}$$

Ruleset(Integer , Integer , Expression(Integer))

Apply the rules to a simple trigonometric expression.

```
sinCosExpandRules(sin(a+2*b+c))
```

$$\begin{aligned} &\left(-\cos(a)\,\sin(b)^2 - 2\,\cos(b)\,\sin(a)\,\sin(b) + \cos(a)\,\cos(b)^2\right)\,\sin(c) \\ &\ -\cos(c)\,\sin(a)\,\sin(b)^2 + 2\,\cos(a)\,\cos(b)\,\cos(c)\,\sin(b) + \cos(b)^2\,\cos(c)\,\sin(a) \end{aligned} \tag{2}$$

Expression(Integer)

Using input files, you can create your own library of transformation rules relevant to your applications, then selectively apply the rules you need.

0.9 Polymorphic Algorithms

All components of the FriCAS algebra library are written in the FriCAS library language. This language is similar to the interactive language except for protocols that authors are obliged to follow. The library language permits you to write "polymorphic algorithms," algorithms defined to work in their most natural settings and over a variety of types.

Define a system of polynomial equations S.

```
S := [3*x^3 + y + 1 = 0,y^2 = 4]
```

$$\left[y + 3\,x^3 + 1 = 0,\ y^2 = 4\right] \tag{1}$$

List (Equation(Polynomial(Integer)))

Solve the system S using rational number arithmetic and 30 digits of accuracy.

```
digits(30); solve(S,1/10^30)
```

$$\left[\left[y = -2,\ x = \frac{576022010662480853494356513425685 09}{830767497365572420564879412675215 36}\right],\ [y = 2,\ x = -1]\right] \tag{2}$$

List (List (Equation(Polynomial(Fraction(Integer)))))

Solve S with the solutions expressed in radicals.

```
radicalSolve(S)
```

$$\left[[y = 2,\ x = -1],\ \left[y = 2,\ x = \frac{\sqrt{-3}+1}{2}\right],\ \left[y = 2,\ x = \frac{-\sqrt{-3}+1}{2}\right],\ \left[y = -2,\ x = \frac{-\sqrt{-1}\,\sqrt{3}-1}{2\,\sqrt[3]{3}}\right],\ \left[y = -2,\ x = \frac{\sqrt{-1}\,\sqrt{3}-1}{2\,\sqrt[3]{3}}\right],\ \left[y = -2,\ x = \frac{1}{\sqrt[3]{3}}\right]\right] \tag{3}$$

List (List (Equation(Expression(Integer))))

While these solutions look very different, the results were produced by the same internal algorithm! The internal algorithm actually works with equations over any "field." Examples of fields are the rational numbers, floating point numbers, rational functions, power series, and general expressions involving radicals.

0.10 Extensibility

Users and system developers alike can augment the FriCAS library, all using one common language. Library code, like interpreter code, is compiled into machine binary code for run-time efficiency.

Using this language, you can create new computational types and new algorithmic packages. All library code is polymorphic, described in terms of a database of algebraic properties. By following the language protocols, there is an automatic, guaranteed interaction between your code and that of colleagues and system implementers.

A Technical Introduction to FriCAS

FriCAS has both an *interactive language* for user interactions and a *programming language* for building library modules. Like Modula 2, PASCAL, FORTRAN, and Ada, the programming language emphasizes strict type-checking. Unlike these languages, types in FriCAS are dynamic objects: they are created at run-time in response to user commands.

Here is the idea of the FriCAS programming language in a nutshell. FriCAS types range from algebraic ones (like polynomials, matrices, and power series) to data structures (like lists, dictionaries, and input files). Types combine in any meaningful way. You can build polynomials of matrices, matrices of polynomials of power series, hash tables with symbolic keys and rational function entries, and so on.

Categories define algebraic properties to ensure mathematical correctness. They ensure, for example, that matrices of polynomials are OK, but matrices of input files are not. Through categories, programs can discover that polynomials of continued fractions have a commutative multiplication whereas polynomials of matrices do not.

Categories allow algorithms to be defined in their most natural setting. For example, an algorithm can be defined to solve polynomial equations over *any* field. Likewise a greatest common divisor can compute the "gcd" of two elements from *any* Euclidean domain. Categories foil attempts to compute meaningless "gcds", for example, of two hashtables. Categories also enable algorithms to be compiled into machine code that can be run with arbitrary types.

The FriCAS interactive language is oriented towards ease-of-use. The FriCAS interpreter uses type-inferencing to deduce the type of an object from user input. Type declarations can generally be omitted for common types in the interactive language.

So much for the nutshell. Here are these basic ideas described by ten design principles:

Types are Defined by Abstract Datatype Programs

Basic types are called *domains of computation*, or, simply, *domains.* Domains are defined by FriCAS programs of the form:

```
Name(...): Exports == Implementation
```

Each domain has a capitalized `Name` that is used to refer to the class of its members. For example, **Integer** denotes "the class of integers," **Float**, "the class of floating point numbers," and **String**, "the class of strings."

The "`...`" part following `Name` lists zero or more parameters to the constructor. Some basic ones like

Integer take no parameters. Others, like **Matrix**, **Polynomial** and **List**, take a single parameter that again must be a domain. For example, **Matrix(Integer)** denotes "matrices over the integers," **Polynomial (Float)** denotes "polynomial with floating point coefficients," and **List (Matrix (Polynomial (Integer)))** denotes "lists of matrices of polynomials over the integers." There is no restriction on the number or type of parameters of a domain constructor.

The `Exports` part specifies operations for creating and manipulating objects of the domain. For example, type **Integer** exports constants `0` and `1`, and operations `+`, `-`, and `*`. While these operations are common, others such as `odd?` and `bit?` are not.

The `Implementation` part defines functions that implement the exported operations of the domain. These functions are frequently described in terms of another lower-level domain used to represent the objects of the domain.

The Type of Basic Objects is a Domain or Subdomain

Every FriCAS object belongs to a *unique* domain. The domain of an object is also called its *type*. Thus the integer `7` has type `Integer` and the string `"daniel"` has type `String`.

The type of an object, however, is not unique. The type of integer `7` is not only **Integer** but `NonNegativeInteger`, `PositiveInteger`, and possibly, in general, any other "subdomain" of the domain `Integer`. A *subdomain* is a domain with a "membership predicate". `PositiveInteger` is a subdomain of `Integer` with the predicate "is the integer `> 0`?".

Subdomains with names are defined by abstract datatype programs similar to those for domains. The *Export* part of a subdomain, however, must list a subset of the exports of the domain. The `Implementation` part optionally gives special definitions for subdomain objects.

Domains Have Types Called Categories

Domain and subdomains in FriCAS are themselves objects that have types. The type of a domain or subdomain is called a *category*. Categories are described by programs of the form:

```
Name(...): Category == Exports
```

The type of every category is the distinguished symbol `Category`. The category `Name` is used to designate the class of domains of that type. For example, category **Ring** designates the class of all rings. Like domains, categories can take zero or more parameters as indicated by the "`...`" part following `Name`. Two examples are **Module(R)** and **MatrixCategory(R,Row,Col)**.

The `Exports` part defines a set of operations. For example, **Ring** exports the operations `0`, `1`, `+`, `-`, and `*`. Many algebraic domains such as **Integer** and **Polynomial (Float)** are rings. **String** and **List (R)** (for any domain `R`) are not.

Categories serve to ensure the type-correctness. The definition of matrices states `Matrix(R: Ring)` requiring its single parameter `R` to be a ring. Thus a "matrix of polynomials" is allowed, but "matrix of lists" is not.

Operations Can Refer To Abstract Types

All operations have prescribed source and target types. Types can be denoted by symbols that stand for domains, called "symbolic domains." The following lines of FriCAS code use a symbolic domain `R`:

```
R: Ring
power: (R, NonNegativeInteger): R -> R
power(x, n) == x ** n
```

Line 1 declares the symbol `R` to be a ring. Line 2 declares the type of `power` in terms of `R`. From the definition on line 3, `power(3,2)` produces 9 for `x = 3` and `R =` **Integer**. Also, `power(3.0,2)` produces `9.0` for `x = 3.0` and `R =` **Float**. `power("oxford",2)` however fails since `"oxford"` has type **String** which is not a ring.

Using symbolic domains, algorithms can be defined in their most natural or general setting.

Categories Form Hierarchies

Categories form hierarchies (technically, directed-acyclic graphs). A simplified hierarchical world of algebraic categories is shown below in Figure 2. At the top of this world is **SetCategory**, the class of algebraic sets. The notions of parents, ancestors, and descendants is clear. Thus ordered sets (domains of category **OrderedSet**) and rings are also algebraic sets. Likewise, fields and integral domains are rings and algebraic sets. However fields and integral domains are not ordered sets.

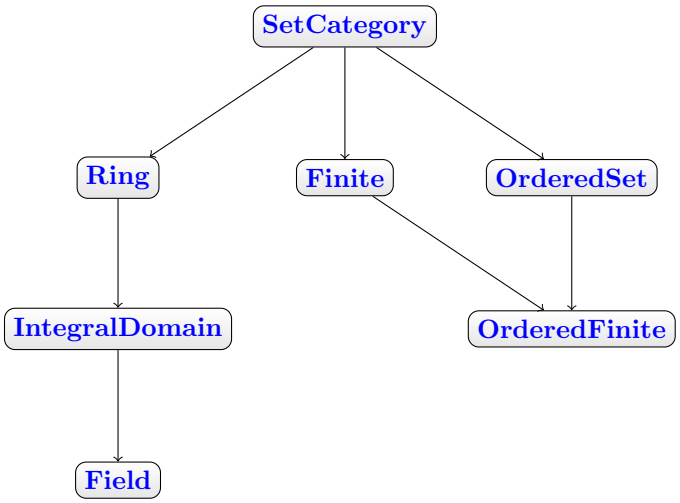


Figure 2: A simplified category hierarchy.

Domains Belong to Categories by Assertion

A category designates a class of domains. Which domains? You might think that **Ring** designates the class of all domains that export `0`, `1`, `+`, `-`, and `*`. But this is not so. Each domain must *assert* which categories it belongs to.

The `Export` part of the definition for **Integer** reads, for example:

```
Join(OrderedSet, IntegralDomain,  ...) with ...
```

This definition asserts that **Integer** is both an ordered set and an integral domain. In fact, **Integer** does not explicitly export constants `0` and `1` and operations `+`, `-` and `*` at all: it inherits them all from `Ring`! Since **IntegralDomain** is a descendant of `Ring`, **Integer** is therefore also a ring.

Assertions can be conditional. For example, **Complex(R)** defines its exports by:

```
Ring with ... if R has Field then Field ...
```

Thus **Complex(Float)** is a field but **Complex(Integer)** is not since **Integer** is not a field.

You may wonder: "Why not simply let the set of operations determine whether a domain belongs to a given category?". FriCAS allows operation names (for example, norm) to have very different meanings in different contexts. The meaning of an operation in FriCAS is determined by context. By associating operations with categories, operation names can be reused whenever appropriate or convenient to do so. As a simple example, the operation `<` might be used to denote lexicographic-comparison in an algorithm. However, it is wrong to use the same `<` with this definition of absolute-value: `abs(x) == if x < 0 then -x else x.` Such a definition for `abs` in FriCAS is protected by context: argument `x` is required to be a member of a domain of category **OrderedSet**.

Packages Are Clusters of Polymorphic Operations

In FriCAS, facilities for symbolic integration, solution of equations, and the like are placed in "packages". A *package* is a special kind of domain: one whose exported operations depend solely on the parameters of the constructor and/or explicit domains.

If you want to use FriCAS, for example, to define some algorithms for solving equations of polynomials over an arbitrary field `F`, you can do so with a package of the form:

```
MySolve(F: Field): Exports == Implementation
```

where `Exports` specifies the solve operations you wish to export and `Implementation` defines functions for implementing your algorithms. Once FriCAS has compiled your package, your algorithms can then be used for any `F`: floating-point numbers, rational numbers, complex rational functions, and power series, to name a few.

The Interpreter Builds Domains Dynamically

The FriCAS interpreter reads user input then builds whatever types it needs to perform the indicated computations. For example, to create the matrix

$$M = \begin{pmatrix} x^2+1 & 0 \\ 0 & x/2 \end{pmatrix}$$

the interpreter first loads the modules **Matrix**, **Polynomial**, **Fraction**, and **Integer** from the library, then builds the *domain tower* "matrices of polynomials of rational numbers (fractions of integers)".

Once a domain tower is built, computation proceeds by calling operations down the tower. For example, suppose that the user asks to square the above matrix. To do this, the function `*` from **Matrix** is passed `M` to compute `M * M`. The function is also passed an environment containing `R` that, in this case, is **Polynomial (Fraction (Integer))**. This results in the successive calling of the `*` operations from **Polynomial**, then from **Fraction**, and then finally from **Integer** before a result is passed back up the tower.

Categories play a policing role in the building of domains. Because the argument of **Matrix** is required to be a ring, FriCAS will not build nonsensical types such as "matrices of input files".

FriCAS Code is Compiled

FriCAS programs are statically compiled to machine code, then placed into library modules. Categories provide an important role in obtaining efficient object code by enabling:

- static type-checking at compile time;
- fast linkage to operations in domain-valued parameters;
- optimization techniques to be used for partially specified types (operations for "vectors of `R`", for instance, can be open-coded even though `R` is unknown).

FriCAS is Extensible

Users and system implementers alike use the FriCAS language to add facilities to the FriCAS library. The entire FriCAS library is in fact written in the FriCAS source code and available for user modification and/or extension.

FriCAS's use of abstract datatypes clearly separates the exports of a domain (what operations are defined) from its implementation (how the objects are represented and operations are defined). Users of a domain can thus only create and manipulate objects through these exported operations. This allows implementers to "remove and replace" parts of the library safely by newly upgraded (and, we hope, correct) implementations without consequence to its users.

Categories protect names by context, making the same names available for use in other contexts. Categories also provide for code-economy. Algorithms can be parameterized categorically to characterize their correct and most general context. Once compiled, the same machine code is applicable in all such contexts.

Finally, FriCAS provides an automatic, guaranteed interaction between new and old code. For example:

- if you write a new algorithm that requires a parameter to be a field, then your algorithm will work automatically with every field defined in the system; past, present, or future.
- if you introduce a new domain constructor that produces a field, then the objects of that domain can be used as parameters to any algorithm using field objects defined in the system; past, present, or future.

These are the key ideas. For further information, we particularly recommend your reading chapters 11, 12, and 13, where these ideas are explained in greater detail.

Part I

Basic Features of FriCAS

Chapter 1

An Overview of FriCAS

Welcome to the FriCAS environment for interactive computation and problem solving. Consider this chapter a brief, whirlwind tour of the FriCAS world. We introduce you to FriCAS's graphics and the FriCAS language. Then we give a sampling of the large variety of facilities in the FriCAS system, ranging from the various kinds of numbers, to data types (like lists, arrays, and sets) and mathematical objects (like matrices, integrals, and differential equations). We conclude with the discussion of system commands and an interactive "undo."

Before embarking on the tour, we need to brief those readers working interactively with FriCAS on some details. Others can skip immediately to Section 1.2 on page 22.

1.1 Starting Up and Winding Down

You need to know how to start the FriCAS system and how to stop it. We assume that FriCAS has been correctly installed on your machine (as described in another FriCAS document).

To begin using FriCAS, issue the command **fricas** to the operating system shell. There is a brief pause, some start-up messages, and then one or more windows appear.

If you are not running FriCAS under the X Window System, there is only one window (the console). At the lower left of the screen there is a prompt that looks like

```
(1) ->
```

When you want to enter input to FriCAS, you do so on the same line after the prompt. The "1" in "(1)" is the computation step number and is incremented after you enter FriCAS statements. Note, however, that a system command such as `)clear all` may change the step number in other ways. We talk about step numbers more when we discuss system commands and the workspace history facility.

If you are running FriCAS under the X Window System, there may be two windows: the console window (as just described) and the HyperDoc main menu. HyperDoc is a multiple-window hypertext system that lets you view FriCAS documentation and examples on-line, execute FriCAS expressions, and generate graphics. If you are in a graphical windowing environment, it is usually started automatically when FriCAS begins. If it is not running, issue `)hd` to start it. We discuss the basics of HyperDoc in Chapter 3.

To interrupt an FriCAS computation, hold down the **Ctrl** (control) key and press **c**. This brings you back to the FriCAS prompt.

To exit from FriCAS, move to the console window, type `)quit` at the input prompt and press the **Enter** key. You will probably be prompted with the following message:

Please enter **y** or **yes** if you really want to leave the
interactive environment and return to the operating system

You should respond **yes**, for example, to exit FriCAS.

We are purposely vague in describing exactly what your screen looks like or what messages FriCAS displays. FriCAS runs on a number of different machines, operating systems and window environments, and these differences all affect the physical look of the system. You can also change the way that FriCAS behaves via *system commands* described later in this chapter and in Appendix A. System commands are special commands, like `)set`, that begin with a closing parenthesis and are used to change your environment. For example, you can set a system variable so that you are not prompted for confirmation when you want to leave FriCAS.

1.1.1 Clef

If you are using FriCAS under the X Window System, the Clef command line editor is probably available and installed. With this editor you can recall previous lines with the up and down arrow keys.

To move forward and backward on a line, use the right and left arrows. You can use the **Insert** key to toggle insert mode on or off. When you are in insert mode, the cursor appears as a large block and if you type anything, the characters are inserted into the line without deleting the previous ones.

If you press the **Home** key, the cursor moves to the beginning of the line and if you press the **End** key, the cursor moves to the end of the line. Pressing **Ctrl**–**End** deletes all the text from the cursor to the end of the line.

Clef also provides FriCAS operation name completion for a limited set of operations. If you enter a few letters and then press the **Tab** key, Clef tries to use those letters as the prefix of an FriCAS operation name. If a name appears and it is not what you want, press **Tab** again to see another name.

You are ready to begin your journey into the world of FriCAS. Proceed to the first stop.

1.2 Typographic Conventions

In this book we have followed these typographical conventions:

- Categories, domains and packages are displayed in a sans-serif typeface: **Ring**, **Integer**, **DiophantineSolutionPackage**.
- Prefix operators, infix operators, and punctuation symbols in the FriCAS language are displayed in the text like this: `+`, “`$`”, “`+->`”.

- FriCAS expressions or expression fragments are displayed in a monospace typeface: `inc(x) == x + 1`.
- For clarity of presentation, TEX is often used to format expressions: $g(x) = x^2 + 1$.
- Function names and HyperDoc button names are displayed in the text in a bold typeface: factor, integrate, **Lighting**.
- Italics are used for emphasis and for words defined in the glossary: *category*.

This book contains over 2500 examples of FriCAS input and output. All examples were run through FriCAS and their output was created in LATEX form for this book by the FriCAS **FormatLaTeX** package.

1.3 The FriCAS Language

The FriCAS language is a rich language for performing interactive computations and for building components of the FriCAS library. Here we present only some basic aspects of the language that you need to know for the rest of this chapter. Our discussion here is intentionally informal, with details unveiled on an "as needed" basis. For more information on a particular construct, we suggest you consult the index at the back of the book.

1.3.1 Arithmetic Expressions

For arithmetic expressions, use the + and - *operators* as in mathematics. Use * for multiplication, and ^ for exponentiation. To create a fraction, use /. When an expression contains several operators, those of highest *precedence* are evaluated first. For arithmetic operators, ^ has highest precedence, * and / have the next highest precedence, and + and - have the lowest precedence.

FriCAS puts implicit parentheses around operations of higher precedence, and groups those of equal precedence from left to right.

```
1 + 2 - 3 / 4 * 3 ^ 2 - 1
```

$$-\frac{19}{4} \tag{1}$$

Fraction (Integer)

The above expression is equivalent to this.

```
((1 + 2) - ((3 / 4) * (3 ^ 2))) - 1
```

$$-\frac{19}{4} \tag{2}$$

Fraction (Integer)

If an expression contains subexpressions enclosed in parentheses, the parenthesized subexpressions are evaluated first (from left to right, from inside out).

```
1 + 2 - 3/ (4 * 3 ^ (2 - 1))
```

$$\frac{11}{4} \tag{3}$$

Fraction (Integer)

1.3.2 Previous Results

Use the percent sign ("%") to refer to the last result. Also, use "%%" to refer to previous results. %%(-1) is equivalent to "%", %%(-2) returns the next to the last result, and so on. %%(1) returns the result from step number 1, %%(2) returns the result from step number 2, and so on. %%(0) is not defined.

This is ten to the tenth power.

```
10 ^ 10
```

$$10000000000 \tag{1}$$

PositiveInteger

This is the last result minus one.

```
% - 1
```

$$9999999999 \tag{2}$$

PositiveInteger

This is the last result.

```
%%(-1)
```

$$9999999999 \tag{3}$$

PositiveInteger

This is the result from step number 1.

```
%%(1)
```

$$10000000000 \tag{4}$$

PositiveInteger

1.3.3 Some Types

Everything in FriCAS has a type. The type determines what operations you can perform on an object and how the object can be used. An entire chapter of this book (Chapter 2) is dedicated to the interactive use of types. Several of the final chapters discuss how types are built and how they are organized in the FriCAS library.

Positive integers are given type **PositiveInteger**.

```
8
```

$$8 \tag{1}$$

PositiveInteger

Negative ones are given type **Integer**. This fine distinction is helpful to the FriCAS interpreter.

```
-8
```

$$-8 \tag{2}$$

Integer

Here a positive integer exponent gives a polynomial result.

```
x^8
```

$$x^8 \tag{3}$$

Polynomial(Integer)

Here a negative integer exponent produces a fraction.

```
x^(-8)
```

$$\frac{1}{x^8} \tag{4}$$

Fraction (Polynomial(Integer))

1.3.4 Symbols, Variables, Assignments, and Declarations

A *symbol* is a literal used for the input of things like the "variables" in polynomials and power series.

We use the three symbols `x`, `y`, and `z` in entering this polynomial.

```
(x - y*z)^2
```

$$y^2\,z^2 - 2\,x\,y\,z + x^2 \tag{1}$$

Polynomial(Integer)

A symbol has a name beginning with an uppercase or lowercase alphabetic character, "`%`", or "`!`". Successive characters (if any) can be any of the above, digits, or "`?`". Case is distinguished: the symbol `points` is different from the symbol `Points`.

A symbol can also be used in FriCAS as a *variable*. A variable refers to a value. To *assign* a value to a variable, the operator "`:=`" is used.[1] A variable initially has no restrictions on the kinds of values to which it can refer.

This assignment gives the value `4` (an integer) to a variable named `x`.

```
x := 4
```

[1] FriCAS actually has two forms of assignment: *immediate* assignment, as discussed here, and *delayed assignment*. See Section 5.1 on page 123 for details.

$$4 \tag{2}$$

PositiveInteger

This gives the value `z + 3/5` (a polynomial) to `x`.

```
x := z + 3/5
```

$$z + \frac{3}{5} \tag{3}$$

Polynomial(Fraction(Integer))

To restrict the types of objects that can be assigned to a variable, use a *declaration*

```
y : Integer
```

After a variable is declared to be of some type, only values of that type can be assigned to that variable.

```
y := 89
```

$$89 \tag{5}$$

Integer

The declaration for `y` forces values assigned to `y` to be converted to integer values.

```
y := sin %pi
```

$$0 \tag{6}$$

Integer

If no such conversion is possible, FriCAS refuses to assign a value to `y`.

```
y := 2/3
```

```
   Cannot convert right-hand side of assignment
   2
   -
   3

      to an object of the type Integer of the left-hand side.
```

A type declaration can also be given together with an assignment. The declaration can assist FriCAS in choosing the correct operations to apply.

```
f : Float := 2/3
```

$$0.66666666666666666667 \tag{7}$$

Float

Any number of expressions can be given on an input line. Just separate them by semicolons. Only the result of evaluating the last expression is displayed.

These two expressions have the same effect as the previous single expression.

```
f : Float; f := 2/3
```

$$0.66666666666666666667 \tag{8}$$

Float

The type of a symbol is either **Symbol** or **Variable**(*name*) where *name* is the name of the symbol.

By default, the interpreter gives this symbol the type **Variable(q)**.

```
q
```

$$q \tag{9}$$

Variable(q)

When multiple symbols are involved, **Symbol** is used.

```
[q, r]
```

$$[q, r] \tag{10}$$

List(OrderedVariableList([q, r]))

What happens when you try to use a symbol that is the name of a variable?

```
f
```

$$0.66666666666666666667 \tag{11}$$

Float

Use a single quote ("'") before the name to get the symbol.

```
'f
```

$$f \tag{12}$$

Variable(f)

Quoting a name creates a symbol by preventing evaluation of the name as a variable. Experience will teach you when you are most likely going to need to use a quote. We try to point out the location of such trouble spots.

1.3.5 Conversion

Objects of one type can usually be "converted" to objects of several other types. To *convert* an object to a new type, use the "::" infix operator.[2] For example, to display an object, it is necessary to convert the object to type **OutputForm**.

This produces a polynomial with rational number coefficients.

```
p := r^2 + 2/3
```

[2]Conversion is discussed in detail in Section 2.7 on page 84.

$$r^2 + \frac{2}{3} \tag{1}$$

Polynomial(Fraction (Integer))

Create a quotient of polynomials with integer coefficients by using "::".

```
p :: Fraction Polynomial Integer
```

$$\frac{3\,r^2 + 2}{3} \tag{2}$$

Fraction (Polynomial(Integer))

Some conversions can be performed automatically when FriCAS tries to evaluate your input. Other conversions must be explicitly requested.

1.3.6 Calling Functions

As we saw earlier, when you want to add or subtract two values, you place the arithmetic operator + or - between the two *arguments* denoting the values. To use most other FriCAS operations, however, you use another syntax: write the name of the operation first, then an open parenthesis, then each of the arguments separated by commas, and, finally, a closing parenthesis. If the operation takes only one argument and the argument is a number or a symbol, you can omit the parentheses.

This calls the operation factor with the single integer argument 120.

```
factor(120)
```

$$2^3\,3\,5 \tag{1}$$

Factored(Integer)

This is a call to divide with the two integer arguments 125 and 7.

```
divide(125,7)
```

$$[quotient = 17,\ remainder = 6] \tag{2}$$

Record(quotient: Integer , remainder: Integer)

This calls quatern with four floating-point arguments.

```
quatern(3.4,5.6,2.9,0.1)
```

$$3.4 + 5.6\,i + 2.9\,j + 0.1\,k \tag{3}$$

Quaternion(Float)

This is the same as `factorial(10)`.

```
factorial 10
```

$$3628800 \tag{4}$$

PositiveInteger

An operation that returns a **Boolean** value (that is, `true` or `false`) frequently has a name suffixed with a question mark ("?"). For example, the even? operation returns `true` if its integer argument is an even number, `false` otherwise.

An operation that can be destructive on one or more arguments usually has a name ending in an exclamation point ("!"). This actually means that it is *allowed* to update its arguments but it is not *required* to do so. For example, the underlying representation of a collection type may not allow the very last element to be removed and so an empty object may be returned instead. Therefore, it is important that you use the object returned by the operation and not rely on a physical change having occurred within the object. Usually, destructive operations are provided for efficiency reasons.

1.3.7 Some Predefined Macros

FriCAS provides several *macros* for your convenience.[3] Macros are names (or forms) that expand to larger expressions for commonly used values.

%i	The square root of -1.
%e	The base of the natural logarithm.
%pi	π.
%infinity	∞.
%plusInfinity	$+\infty$.
%minusInfinity	$-\infty$.

1.3.8 Long Lines

When you enter FriCAS expressions from your keyboard, there will be times when they are too long to fit on one line. FriCAS does not care how long your lines are, so you can let them continue from the right margin to the left side of the next line.

Alternatively, you may want to enter several shorter lines and have FriCAS glue them together. To get this glue, put an underscore (_) at the end of each line you wish to continue.

```
2_
+_
3
```

is the same as if you had entered

```
2+3
```

If you are putting your FriCAS statements in an input file (see Section 4.1 on page 105), you can use indentation to indicate the structure of your program. (see Section 5.2 on page 126).

[3] See Section 6.2 on page 150 for a discussion on how to write your own macros.

1.3.9 Comments

Comment statements begin with two consecutive hyphens or two consecutive plus signs and continue until the end of the line.

The comment beginning with -- is ignored by FriCAS.

```
2 + 3    -- this is rather simple, no?
```

$$5 \tag{1}$$

PositiveInteger

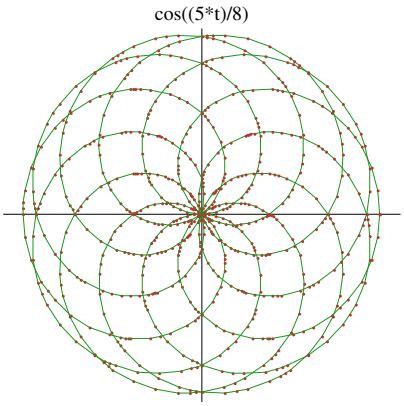
There is no way to write long multi-line comments other than starting each line with "--" or "++".

1.4 Graphics

FriCAS has a two- and three-dimensional drawing and rendering package that allows you to draw, shade, color, rotate, translate, map, clip, scale and combine graphic output of FriCAS computations. The graphics interface is capable of plotting functions of one or more variables and plotting parametric surfaces. Once the graphics figure appears in a window, move your mouse to the window and click. A control panel appears immediately and allows you to interactively transform the object.

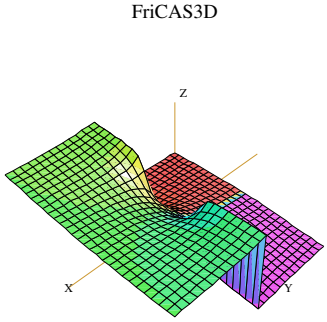
This is an example of FriCAS's two-dimensional plotting. From the 2D Control Panel you can rescale the plot, turn axes and units on and off and save the image, among other things. This PostScript image was produced by clicking on the **PS** 2D Control Panel button.

```
draw(cos(5*t/8), t=0..16*%pi, coordinates==polar)
```



This is an example of FriCAS's three-dimensional plotting. It is a monochrome graph of the complex arctangent function. The image displayed was rotated and had the "shade" and "outline" display options set from the 3D Control Panel. The PostScript output was produced by clicking on the **save** 3D Control Panel button and then clicking on the **PS** button. See Section 8.1 on page 251 for more details and examples of FriCAS's numeric and graphics capabilities.

```
draw((x,y) +-> real atan complex(x,y), -%pi..%pi, -%pi..%pi, colorFunction == (x,y) _
    +-> argument atan complex(x,y))
```



An exhibit of FriCAS Images is given in the center section of this book. For a description of the commands and programs that produced these figures, see Appendix B. PostScript output is available so that FriCAS images can be printed.[4] See Chapter 7 for more examples and details about using FriCAS's graphics facilities.

1.5 Numbers

FriCAS distinguishes very carefully between different kinds of numbers, how they are represented and what their properties are. Here is a sampling of some of these kinds of numbers and some things you can do with them.

Integer arithmetic is always exact.

```
11^13 * 13^11 * 17^7 - 19^5 * 23^3
```

$$25387751112538918594666224484237298 \tag{1}$$

PositiveInteger

Integers can be represented in factored form.

```
factor 643238070748569023720594412551704344145570763243
```

$$11^{13}\,13^{11}\,17^{7}\,19^{5}\,23^{3}\,29^{2} \tag{2}$$

Factored(Integer)

Results stay factored when you do arithmetic. Note that the 12 is automatically factored for you.

```
% * 12
```

$$2^{2}\,3\,11^{13}\,13^{11}\,17^{7}\,19^{5}\,23^{3}\,29^{2} \tag{3}$$

Factored(Integer)

Integers can also be displayed to bases other than 10. This is an integer in base 11.

[4] PostScript is a trademark of Adobe Systems Incorporated, registered in the United States.

```
radix(25937424601,11)
```

$$10000000000 \tag{4}$$

RadixExpansion(11)

Roman numerals are also available for those special occasions.

```
roman(1992)
```

$$MCMXCII \tag{5}$$

RomanNumeral

Rational number arithmetic is also exact.

```
r := 10 + 9/2 + 8/3 + 7/4 + 6/5 + 5/6 + 4/7 + 3/8 + 2/9
```

$$\frac{55739}{2520} \tag{6}$$

Fraction(Integer)

To factor fractions, you have to map factor onto the numerator and denominator.

```
map(factor,r)
```

$$\frac{139\,401}{2^3\,3^2\,5\,7} \tag{7}$$

Fraction(Factored(Integer))

Type **SingleInteger** refers to machine word-length integers. In English, this expression means "11 as a small integer".

```
11@SingleInteger
```

$$11 \tag{8}$$

SingleInteger

Machine double-precision floating-point numbers are also available for numeric and graphical applications.

```
123.21@DoubleFloat
```

$$123.21000000000001 \tag{9}$$

DoubleFloat

The normal floating-point type in FriCAS, **Float**, is a software implementation of floating-point numbers in which the exponent and the mantissa may have any number of digits.[5] The types **Complex(Float)** and **Complex(DoubleFloat)** are the corresponding software implementations of complex floating-point numbers.

[5] See '`Float`' on page 419 and '`DoubleFloat`' on page 395 for additional information on floating-point types.

This is a floating-point approximation to about twenty digits. The "::" is used here to change from one kind of object (here, a rational number) to another (a floating-point number).

```
r :: Float
```

$$22.118650793650793651 \tag{10}$$

Float

Use digits to change the number of digits in the representation. This operation returns the previous value so you can reset it later.

```
digits(22)
```

$$20 \tag{11}$$

PositiveInteger

To 22 digits of precision, the number $e^{\pi\sqrt{163.0}}$ appears to be an integer.

```
exp(%pi * sqrt 163.0)
```

$$262537412640768744.0 \tag{12}$$

Float

Increase the precision to forty digits and try again.

```
digits(40);  exp(%pi * sqrt 163.0)
```

$$262537412640768743.999999999992500725976 \tag{13}$$

Float

Here are complex numbers with rational numbers as real and imaginary parts.

```
(2/3 + %i)^3
```

$$-\frac{46}{27}+\frac{1}{3}\,i \tag{14}$$

Complex(Fraction(Integer))

The standard operations on complex numbers are available.

```
conjugate %
```

$$-\frac{46}{27}-\frac{1}{3}\,i \tag{15}$$

Complex(Fraction(Integer))

You can factor complex integers.

```
factor(89 - 23 * %i)
```

$$-(1+i)\,(2+i)^2\,(3+2\,i)^2 \tag{16}$$

Factored(Complex(Integer))

Complex numbers with floating point parts are also available.

```
exp(%pi/4.0 * %i)
```

$$0.70710678118654752440084436210484903928\,49 + 0.70710678118654752440084436210484903928\,48\,i \tag{17}$$

Complex(Float)

Every rational number has an exact representation as a repeating decimal expansion (see '`DecimalExpansion`' on page 392).

```
decimal(1/352)
```

$$0.0028\overline{409} \tag{18}$$

DecimalExpansion

A rational number can also be expressed as a continued fraction (see '`ContinuedFraction`' on page 375).

```
continuedFraction(6543/210)
```

$$31 + \frac{1|}{|6} + \frac{1|}{|2} + \frac{1|}{|1} + \frac{1|}{|3} \tag{19}$$

ContinuedFraction(Integer)

Also, partial fractions can be used and can be displayed in a compact ...

```
partialFraction(1,factorial(10))
```

$$-\frac{97}{2^8} + \frac{58}{3^4} + \frac{13}{5^2} - \frac{6}{7} \tag{20}$$

PartialFraction (Integer)

or expanded format (see '`PartialFraction`' on page 528).

```
padicFraction(%)
```

$$-\frac{1}{2^2} - \frac{1}{2^3} - \frac{1}{2^8} + \frac{2}{3} + \frac{1}{3^3} + \frac{1}{3^4} + \frac{2}{5} + \frac{3}{5^2} - \frac{6}{7} \tag{21}$$

PartialFraction (Integer)

Like integers, bases (radices) other than ten can be used for rational numbers (see '`RadixExpansion`' on page 541). Here we use base eight.

```
radix(4/7, 8)
```

$$0.\overline{4} \tag{22}$$

RadixExpansion(8)

Of course, there are complex versions of these as well. FriCAS decides to make the result a complex rational number.

```
% + 2/3*%i
```

$$\frac{4}{7} + \frac{2}{3}\,i \tag{23}$$

Complex(Fraction(Integer))

You can also use FriCAS to manipulate fractional powers.

```
(5 + sqrt 63 + sqrt 847)^(1/3)
```

$$\sqrt[3]{14\sqrt{7} + 5} \tag{24}$$

AlgebraicNumber

You can also compute with integers modulo a prime.

```
x : PrimeField 7 := 5
```

$$5 \tag{25}$$

PrimeField(7)

Arithmetic is then done modulo 7.

```
x^3
```

$$6 \tag{26}$$

PrimeField(7)

Since 7 is prime, you can invert nonzero values.

```
1/x
```

$$3 \tag{27}$$

PrimeField(7)

You can also compute modulo an integer that is not a prime.

```
y : IntegerMod 6 := 5
```

$$5 \tag{28}$$

IntegerMod(6)

All of the usual arithmetic operations are available.

```
y^3
```

$$5 \tag{29}$$

IntegerMod(6)

Inversion is not available if the modulus is not a prime number. Modular arithmetic and prime fields are discussed in Section 8.11.1 on page 304.

```
1/y
```

```
   There are 10 exposed and 16 unexposed library operations named / having 2
      argument(s) but none was determined to be applicable. Use HyperDoc Browse,
      or issue
                                    )display op /
      to learn more about the available operations. Perhaps package-calling the
      operation or using coercions on the arguments will allow you to apply the
      operation.
   Cannot find a definition or applicable library operation named / with argument
      type(s)
                                  PositiveInteger
                                   IntegerMod(6)

      Perhaps you should use "@" to indicate the required return type, or "$" to
      specify which version of the function you need.
```

This defines a to be an algebraic number, that is, a root of a polynomial equation.

```
a := rootOf(a^5 + a^3 + a^2 + 3,a)
```

$$a \tag{30}$$

Expression(Integer)

Computations with a are reduced according to the polynomial equation.

```
(a + 1)^10
```

$$-85\,a^4 - 264\,a^3 - 378\,a^2 - 458\,a - 287 \tag{31}$$

Expression(Integer)

Define b to be an algebraic number involving a.

```
b := rootOf(b^4 + a,b)
```

$$b \tag{32}$$

Expression(Integer)

Do some arithmetic.

```
2/(b - 1)
```

$$\frac{2}{b-1} \tag{33}$$

Expression (Integer)

To expand and simplify this, call ratDenom to rationalize the denominator.

```
ratDenom(%)
```

$$\left(a^4-a^3+2\,a^2-a+1\right)\,b^3+\left(a^4-a^3+2\,a^2-a+1\right)\,b^2+\left(a^4-a^3+2\,a^2-a+1\right)\,b+a^4-a^3+2\,a^2-a+1 \tag{34}$$

Expression (Integer)

If we do this, we should get b.

```
2/%+1
```

$$\frac{\left(a^4-a^3+2\,a^2-a+1\right)\,b^3+\left(a^4-a^3+2\,a^2-a+1\right)\,b^2+\left(a^4-a^3+2\,a^2-a+1\right)\,b+a^4-a^3+2\,a^2-a+3}{\left(a^4-a^3+2\,a^2-a+1\right)\,b^3+\left(a^4-a^3+2\,a^2-a+1\right)\,b^2+\left(a^4-a^3+2\,a^2-a+1\right)\,b+a^4-a^3+2\,a^2-a+1} \tag{35}$$

Expression (Integer)

But we need to rationalize the denominator again.

```
ratDenom(%)
```

$$b \tag{36}$$

Expression (Integer)

Types **Quaternion** and **Octonion** are also available. Multiplication of quaternions is non-commutative, as expected.

```
q:=quatern(1,2,3,4)*quatern(5,6,7,8) - quatern(5,6,7,8)*quatern(1,2,3,4)
```

$$-8\,i+16\,j-8\,k \tag{37}$$

Quaternion(Integer)

1.6 Data Structures

FriCAS has a large variety of data structures available. Many data structures are particularly useful for interactive computation and others are useful for building applications. The data structures of FriCAS are organized into *category hierarchies* as shown on the inside back cover.

A *list* is the most commonly used data structure in FriCAS for holding objects all of the same type.[6] The name *list* is short for "linked-list of nodes." Each node consists of a value (first) and a link (rest) that *points* to the next node, or to a distinguished value denoting the empty list. To get to, say, the third element, FriCAS starts at the front of the list, then traverses across two links to the third node.

[6] Lists are discussed in 'List' on page 490 and in Section 5.5 on page 143.

Write a list of elements using square brackets with commas separating the elements.

```
u := [1,-7,11]
```

$$[1, -7, 11] \tag{1}$$

List (Integer)

This is the value at the third node. Alternatively, you can say `u.3`.

```
first rest rest u
```

$$11 \tag{2}$$

PositiveInteger

Many operations are defined on lists, such as: empty?, to test that a list has no elements; cons(x,l), to create a new list with first element x and rest l; reverse, to create a new list with elements in reverse order; and sort, to arrange elements in order.

An important point about lists is that they are "mutable": their constituent elements and links can be changed "in place." To do this, use any of the operations whose names end with the character "!".

The operation concat!(u,v) replaces the last link of the list u to point to some other list v. Since u refers to the original list, this change is seen by u.

```
concat!(u,[9,1,3,-4]); u
```

$$[1, -7, 11, 9, 1, 3, -4] \tag{3}$$

List (Integer)

A *cyclic list* is a list with a "cycle": a link pointing back to an earlier node of the list. To create a cycle, first get a node somewhere down the list.

```
lastnode := rest(u,3)
```

$$[9, 1, 3, -4] \tag{4}$$

List (Integer)

Use setrest! to change the link emanating from that node to point back to an earlier part of the list.

```
setrest!(lastnode,rest(u,2)); u
```

$$[1, -7, \overline{11, 9}] \tag{5}$$

List (Integer)

A *stream* is a structure that (potentially) has an infinite number of distinct elements.[7] Think of a stream as an "infinite list" where elements are computed successively.

Create an infinite stream of factored integers. Only a certain number of initial elements are computed and displayed.

[7] Streams are discussed in '**Stream**' on page 577 and in Section 5.5 on page 143.

```
[factor(i) for i in 2.. by 2]
```

$$[2,\ 2^2,\ 2\,3,\ 2^3,\ 2\,5,\ 2^2\,3,\ 2\,7,\ \ldots] \tag{6}$$

Stream(Factored(Integer))

FriCAS represents streams by a collection of already-computed elements together with a function to compute the next element "on demand." Asking for the n^{th} element causes elements `1` through `n` to be evaluated.

```
%.36
```

$$2^3\,3^2 \tag{7}$$

Factored(Integer)

Streams can also be finite or cyclic. They are implemented by a linked list structure similar to lists and have many of the same operations. For example, first and rest are used to access elements and successive nodes of a stream.

A *one-dimensional array* is another data structure used to hold objects of the same type.[8] Unlike lists, one-dimensional arrays are inflexible—they are implemented using a fixed block of storage. Their advantage is that they give quick and constant access time to any element.

A simple way to create a one-dimensional array is to apply the operation oneDimensionalArray to a list of elements.

```
a := oneDimensionalArray [1, -7, 3, 3/2]
```

$$\left[1,\ -7,\ 3,\ \frac{3}{2}\right] \tag{8}$$

OneDimensionalArray(Fraction(Integer))

One-dimensional arrays are also mutable: you can change their constituent elements "in place."

```
a.3 := 11; a
```

$$\left[1,\ -7,\ 11,\ \frac{3}{2}\right] \tag{9}$$

OneDimensionalArray(Fraction(Integer))

However, one-dimensional arrays are not flexible structures. You cannot destructively concat! them together.

```
concat!(a,oneDimensionalArray [1,-2])
```

```
   There are 5 exposed and 0 unexposed library operations named concat! having 2
      argument(s) but none was determined to be applicable. Use HyperDoc Browse,
      or issue
                                  )display op concat!
      to learn more about the available operations. Perhaps package-calling the
      operation or using coercions on the arguments will allow you to apply the
      operation.
```

[8] See '`OneDimensionalArray`' on page 517 for details.

```
   Cannot find a definition or applicable library operation named concat! with
      argument type(s)
                        OneDimensionalArray(Fraction(Integer))
                             OneDimensionalArray(Integer)

      Perhaps you should use "@" to indicate the required return type, or "$" to
      specify which version of the function you need.
```

Examples of datatypes similar to **OneDimensionalArray** are: **Vector** (vectors are mathematical structures implemented by one-dimensional arrays), **String** (arrays of "characters," represented by byte vectors), and **Bits** (represented by "bit vectors").

A vector of 32 bits, each representing the **Boolean** value `true`.

```
bits(32,true)
```

$$\text{"11111111111111111111111111111111"} \tag{10}$$

Bits

A *flexible array* is a cross between a list and a one-dimensional array.[9] Like a one-dimensional array, a flexible array occupies a fixed block of storage. Its block of storage, however, has room to expand! When it gets full, it grows (a new, larger block of storage is allocated); when it has too much room, it contracts.

Create a flexible array of three elements.

```
f := flexibleArray [2, 7, -5]
```

$$[2,\ 7,\ -5] \tag{11}$$

FlexibleArray (Integer)

Insert some elements between the second and third elements.

```
insert!(flexibleArray [11, -3],f,3)
```

$$[2,\ 7,\ 11,\ -3,\ -5] \tag{12}$$

FlexibleArray (Integer)

Flexible arrays are used to implement "heaps." A *heap* is an example of a data structure called a *priority queue*, where elements are ordered with respect to one another.[10] A heap is organized so as to optimize insertion and extraction of maximum elements. The extract! operation returns the maximum element of the heap, after destructively removing that element and reorganizing the heap so that the next maximum element is ready to be delivered.

An easy way to create a heap is to apply the operation heap to a list of values.

```
h := heap [-4,7,11,3,4,-7]
```

$$[11,\ 7,\ -4,\ 3,\ 4,\ -7] \tag{13}$$

[9] See '`FlexibleArray`' on page 416 for details.

[10] See '`Heap`' on page 439 for more details. Heaps are also examples of data structures called *bags*. Other bag data structures are **Stack**, **Queue**, and **Dequeue**.

Heap(Integer)

This loop extracts elements one-at-a-time from `h` until the heap is exhausted, returning the elements as a list in the order they were extracted.

```
[extract!(h) while not empty?(h)]
```

$$[11,\ 7,\ 4,\ 3,\ -4,\ -7] \tag{14}$$

List (Integer)

A *binary tree* is a "tree" with at most two branches per node: it is either empty, or else is a node consisting of a value, and a left and right subtree (again, binary trees).[11]

A *binary search tree* is a binary tree such that, for each node, the value of the node is greater than all values (if any) in the left subtree, and less than or equal to all values (if any) in the right subtree.

```
binarySearchTree [5,3,2,9,4,7,11]
```

$$[[2,\ 3,\ 4],\ 5,\ [7,\ 9,\ 11]] \tag{15}$$

BinarySearchTree(PositiveInteger)

A *balanced binary tree* is useful for doing modular computations. Given a list `lm` of moduli, modTree`(a, lm)` produces a balanced binary tree with the values a mod m at its leaves.

```
modTree(8,[2,3,5,7])
```

$$[0,\ 2,\ 3,\ 1] \tag{16}$$

List (Integer)

A *set* is a collection of elements where duplication and order is irrelevant.[12] Sets are always finite and have no corresponding structure like streams for infinite collections.

Create sets by using the set function.

```
fs := set[1/3,4/5,-1/3,4/5]
```

$$\left\{-\frac{1}{3},\ \frac{1}{3},\ \frac{4}{5}\right\} \tag{17}$$

Set(Fraction (Integer))

A *multiset* is a set that keeps track of the number of duplicate values.[13] For all the primes `p` between 2 and 1000, find the distribution of p mod 5.

```
multiset [x rem 5 for x in primes(2,1000)]
```

$$\{47\ :\ 2,\ 42\ :\ 3,\ 0,\ 40\ :\ 1,\ 38\ :\ 4\} \tag{18}$$

[11] Example of binary tree types are **BinarySearchTree** (see '`BinarySearchTree`' on page 349, **PendantTree**, **TournamentTree**, and **BalancedBinaryTree** (see '`BalancedBinaryTree`' on page 343).

[12] See '`Set`' on page 565 for more details.

[13] See '`Multiset`' on page 510 for details.

Multiset(Integer)

A *table* is conceptually a set of "key–value" pairs and is a generalization of a multiset.[14] The domain **Table(Key, Entry)** provides a general-purpose type for tables with *values* of type **Entry** indexed by *keys* of type **Key**.

Compute the above distribution of primes using tables. First, let `t` denote an empty table of keys and values, each of type **Integer**.

```
t : Table(Integer,Integer) := empty()
```

$$\texttt{table}() \tag{19}$$

Table(Integer , Integer)

We define a function **howMany** to return the number of values of a given modulus `k` seen so far. It calls search`(k,t)` which returns the number of values stored under the key `k` in table `t`, or `"failed"` if no such value is yet stored in `t` under `k`.

In English, this says "Define `howMany(k)` as follows. First, let n be the value of search(k,t). Then, if n has the value "$failed$", return the value 1; otherwise return $n+1$."

```
howMany(k) == (n:=search(k,t); n case "failed" => 1; n+1)
```

Run through the primes to create the table, then print the table. The expression `t.m := howMany(m)` updates the value in table `t` stored under key `m`.

```
for p in primes(2,1000) repeat (m:= p rem 5; t.m:= howMany(m)); t
```

```
   Compiling function howMany with type Integer -> Integer
```

$$\texttt{table}(4=38, 1=40, 0=1, 3=42, 2=47) \tag{21}$$

Table(Integer , Integer)

A *record* is an example of an inhomogeneous collection of objects.[15] A record consists of a set of named *selectors* that can be used to access its components.

Declare that `daniel` can only be assigned a record with two prescribed fields.

```
daniel : Record(age : Integer, salary : Float)
```

Give `daniel` a value, using square brackets to enclose the values of the fields.

```
daniel := [28, 32005.12]
```

$$[age = 28,\ salary = 32005.12] \tag{23}$$

Record(age: Integer , salary : Float)

Give `daniel` a raise.

```
daniel.salary := 35000; daniel
```

[14] For examples of tables, see **AssociationList** ('AssociationList' on page 341), **HashTable**, **KeyedAccessFile** ('KeyedAccessFile' on page 457), **Library** ('Library' on page 474), **SparseTable** ('SparseTable' on page 570), **StringTable** ('StringTable' on page 583), and **Table** ('Table' on page 587).

[15] See Section 2.4 on page 76 for details.

$$[age = 28,\ salary = 35000.0] \tag{24}$$

Record(age: Integer , salary : Float)

A *union* is a data structure used when objects have multiple types.[16]

Let dog be either an integer or a string value.

```
dog: Union(licenseNumber: Integer, name: String)
```

Give dog a name.

```
dog := "Whisper"
```

$$\texttt{"Whisper"} \tag{26}$$

Union(name: String, ...)

All told, there are over forty different data structures in FriCAS. Using the domain constructors described in Chapter 13, you can add your own data structure or extend an existing one. Choosing the right data structure for your application may be the key to obtaining good performance.

1.7 Expanding to Higher Dimensions

To get higher dimensional aggregates, you can create one-dimensional aggregates with elements that are themselves aggregates, for example, lists of lists, one-dimensional arrays of lists of multisets, and so on. For applications requiring two-dimensional homogeneous aggregates, you will likely find *two-dimensional arrays* and *matrices* most useful.

The entries in **TwoDimensionalArray** and **Matrix** objects are all the same type, except that those for **Matrix** must belong to a **Ring**. You create and access elements in roughly the same way. Since matrices have an understood algebraic structure, certain algebraic operations are available for matrices but not for arrays. Because of this, we limit our discussion here to **Matrix**, that can be regarded as an extension of **TwoDimensionalArray**.[17]

You can create a matrix from a list of lists, where each of the inner lists represents a row of the matrix.

```
m := matrix([[1,2], [3,4]])
```

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \tag{1}$$

Matrix(Integer)

The "collections" construct (see Section 5.5 on page 143) is useful for creating matrices whose entries are given by formulas.

```
matrix([[1/(i + j - x) for i in 1..4] for j in 1..4])
```

[16] See Section 2.5 on page 79 for details.

[17] See 'TwoDimensionalArray' on page 592 for more information about arrays. For more information about FriCAS's linear algebra facilities, see 'Matrix' on page 503, 'Permanent' on page 531, 'SquareMatrix' on page 576, 'Vector' on page 603, Section 8.4 on page 267 (computation of eigenvalues and eigenvectors) , and Section 8.5 on page 270 (solution of linear and polynomial equations) .

$$\begin{bmatrix} -\frac{1}{x-2} & -\frac{1}{x-3} & -\frac{1}{x-4} & -\frac{1}{x-5} \\ -\frac{1}{x-3} & -\frac{1}{x-4} & -\frac{1}{x-5} & -\frac{1}{x-6} \\ -\frac{1}{x-4} & -\frac{1}{x-5} & -\frac{1}{x-6} & -\frac{1}{x-7} \\ -\frac{1}{x-5} & -\frac{1}{x-6} & -\frac{1}{x-7} & -\frac{1}{x-8} \end{bmatrix} \tag{2}$$

Matrix(Fraction (Polynomial(Integer)))

Let vm denote the three by three Vandermonde matrix.

```
vm := matrix [[1,1,1], [x,y,z], [x*x,y*y,z*z]]
```

$$\begin{bmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{bmatrix} \tag{3}$$

Matrix(Polynomial(Integer))

Use this syntax to extract an entry in the matrix.

```
vm(3,3)
```

$$z^2 \tag{4}$$

Polynomial(Integer)

You can also pull out a row or a column.

```
column(vm,2)
```

$$[1,\ y,\ y^2] \tag{5}$$

Vector(Polynomial(Integer))

You can do arithmetic.

```
vm * vm
```

$$\begin{bmatrix} x^2+x+1 & y^2+y+1 & z^2+z+1 \\ x^2 z+x y+x & y^2 z+y^2+x & z^3+y z+x \\ x^2 z^2+x y^2+x^2 & y^2 z^2+y^3+x^2 & z^4+y^2 z+x^2 \end{bmatrix} \tag{6}$$

Matrix(Polynomial(Integer))

You can perform operations such as transpose, trace, and determinant.

```
factor determinant vm
```

$$(y-x)\ (z-y)\ (z-x) \tag{7}$$

Factored(Polynomial(Integer))

1.8 Writing Your Own Functions

FriCAS provides you with a very large library of predefined operations and objects to compute with. You can use the FriCAS library of constructors to create new objects dynamically of quite arbitrary complexity. For example, you can make lists of matrices of fractions of polynomials with complex floating point numbers as coefficients. Moreover, the library provides a wealth of operations that allow you to create and manipulate these objects.

For many applications, you need to interact with the interpreter and write some FriCAS programs to tackle your application. FriCAS allows you to write functions interactively, thereby effectively extending the system library. Here we give a few simple examples, leaving the details to Chapter 6.

We begin by looking at several ways that you can define the "factorial" function in FriCAS. The first way is to give a piece-wise definition of the function. This method is best for a general recurrence relation since the pieces are gathered together and compiled into an efficient iterative function. Furthermore, enough previously computed values are automatically saved so that a subsequent call to the function can pick up from where it left off.

Define the value of **fact** at `0`.

```
fact(0) == 1
```

Define the value of `fact(n)` for general `n`.

```
fact(n) == n*fact(n-1)
```

Ask for the value at `50`. The resulting function created by FriCAS computes the value by iteration.

```
fact(50)
```

```
   Compiling function fact with type Integer -> Integer
   Compiling function fact as a recurrence relation.
```

$$30414093201713378043612608166064768844377641568960512000000000000 \quad (3)$$

PositiveInteger

A second definition uses an `if-then-else` and recursion.

```
fac(n) == if n < 3 then n else n * fac(n - 1)
```

This function is less efficient than the previous version since each iteration involves a recursive function call.

```
fac(50)
```

```
   Compiling function fac with type Integer -> Integer
```

$$30414093201713378043612608166064768844377641568960512000000000000 \quad (5)$$

PositiveInteger

A third version directly uses iteration.

```
fa(n) == (a := 1; for i in 2..n repeat a := a*i; a)
```

This is the least space-consuming version.

```
fa(50)
```

```
   Compiling function fa with type PositiveInteger -> PositiveInteger
```

$$30414093201713378043612608166064768844377641568960512000000000000 \tag{7}$$

PositiveInteger

A final version appears to construct a large list and then reduces over it with multiplication.

```
f(n) == reduce(*,[i for i in 2..n])
```

In fact, the resulting computation is optimized into an efficient iteration loop equivalent to that of the third version.

```
f(50)
```

```
   Compiling function f with type PositiveInteger -> PositiveInteger
```

$$30414093201713378043612608166064768844377641568960512000000000000 \tag{9}$$

PositiveInteger

The library version uses an algorithm that is different from the four above because it highly optimizes the recurrence relation definition of factorial.

```
factorial(50)
```

$$30414093201713378043612608166064768844377641568960512000000000000 \tag{10}$$

PositiveInteger

You are not limited to one-line functions in FriCAS. If you place your function definitions in **.input** files (see Section 4.1 on page 105), you can have multi-line functions that use indentation for grouping.

Given `n` elements, diagonalMatrix creates an `n` by `n` matrix with those elements down the diagonal. This function uses a permutation matrix that interchanges the `i`th and `j`th rows of a matrix by which it is right-multiplied.

This function definition shows a style of definition that can be used in **.input** files. Indentation is used to create *blocks*: sequences of expressions that are evaluated in sequence except as modified by control statements such as `if-then-else` and `return`.

```
permMat(n, i, j) ==
  m := diagonalMatrix
    [(if i = k or j = k then 0 else 1)
      for k in 1..n]
  m(i,j) := 1
  m(j,i) := 1
  m
```

This creates a four by four matrix that interchanges the second and third rows.

```
p := permMat(4,2,3)
```

```
   Compiling function permMat with type (PositiveInteger, PositiveInteger,
      PositiveInteger) -> Matrix(NonNegativeInteger)
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{12}$$

Matrix(NonNegativeInteger)

Create an example matrix to permute.

```
m := matrix [[4*i + j for j in 1..4] for i in 0..3]
```

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \tag{13}$$

Matrix(NonNegativeInteger)

Interchange the second and third rows of m.

```
permMat(4,2,3) * m
```

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 9 & 10 & 11 & 12 \\ 5 & 6 & 7 & 8 \\ 13 & 14 & 15 & 16 \end{bmatrix} \tag{14}$$

Matrix(NonNegativeInteger)

A function can also be passed as an argument to another function, which then applies the function or passes it off to some other function that does. You often have to declare the type of a function that has functional arguments.

This declares **t** to be a two-argument function that returns a **Float**. The first argument is a function that takes one **Float** argument and returns a **Float**.

```
t : (Float -> Float, Float) -> Float
```

This is the definition of **t**.

```
t(fun, x) == fun(x)^2 + sin(x)^2
```

We have not defined a cos in the workspace. The one from the FriCAS library will do.

```
t(cos, 5.2058)
```

```
   Compiling function t with type ((Float -> Float), Float) -> Float
```

$$1.0 \tag{17}$$

Float

Here we define our own (user-defined) function.

```
cosinv(y) == cos(1/y)
```

Pass this function as an argument to **t**.

```
t(cosinv, 5.2058)
```

```
   Compiling function cosinv with type Float -> Float
```

$$1.7392237241800516492541476847729325207 85 \tag{19}$$

Float

FriCAS also has pattern matching capabilities for simplification of expressions and for defining new functions by rules. For example, suppose that you want to apply regularly a transformation that groups together products of radicals:

$$\sqrt{a}\,\sqrt{b} \mapsto \sqrt{ab}, \quad (\forall a)(\forall b)$$

Note that such a transformation is not generally correct. FriCAS never uses it automatically.

Give this rule the name **groupSqrt**.

```
groupSqrt := rule(sqrt(a) * sqrt(b) == sqrt(a*b))
```

$$\%C\,\sqrt{a}\,\sqrt{b} == \%C\,\sqrt{a\,b} \tag{20}$$

RewriteRule(Integer , Integer , Expression (Integer))

Here is a test expression.

```
a := (sqrt(x) + sqrt(y) + sqrt(z))^4
```

$$\begin{aligned}&\left((4\,z+4\,y+12\,x)\,\sqrt{y}+(4\,z+12\,y+4\,x)\,\sqrt{x}\right)\,\sqrt{z}\\&+(12\,z+4\,y+4\,x)\,\sqrt{x}\,\sqrt{y}+z^2+(6\,y+6\,x)\,z+y^2+6\,x\,y+x^2\end{aligned} \tag{21}$$

Expression (Integer)

The rule **groupSqrt** successfully simplifies the expression.

```
groupSqrt a
```

$$(4\,z+4\,y+12\,x)\,\sqrt{y\,z}+(4\,z+12\,y+4\,x)\,\sqrt{x\,z}+(12\,z+4\,y+4\,x)\,\sqrt{x\,y}+z^2+(6\,y+6\,x)\,z+y^2+6\,x\,y+x^2 \tag{22}$$

Expression (Integer)

1.9 Polynomials

Polynomials are the commonly used algebraic types in symbolic computation. Interactive users of FriCAS generally only see one type of polynomial: **Polynomial(R)**. This type represents polynomials in any number of unspecified variables over a particular coefficient domain `R`. This type represents its coefficients *sparsely*: only terms with non-zero coefficients are represented.

In building applications, many other kinds of polynomial representations are useful. Polynomials may have one variable or multiple variables, the variables can be named or unnamed, the coefficients can be stored sparsely or densely. So-called "distributed multivariate polynomials" store polynomials as coefficients paired with vectors of exponents. This type is particularly efficient for use in algorithms for solving systems of non-linear polynomial equations.

The polynomial constructor most familiar to the interactive user is **Polynomial**.

```
(x^2 - x*y^3 +3*y)^2
```

$$x^2\,y^6-6\,x\,y^4-2\,x^3\,y^3+9\,y^2+6\,x^2\,y+x^4 \tag{1}$$

Polynomial(Integer)

If you wish to restrict the variables used, **UnivariatePolynomial** provides polynomials in one variable.

```
p: UP(x,INT) := (3*x-1)^2 * (2*x + 8)
```

$$18\,x^3 + 60\,x^2 - 46\,x + 8 \tag{2}$$

UnivariatePolynomial (x, Integer)

The constructor **MultivariatePolynomial** provides polynomials in one or more specified variables.

```
m: MPOLY([x,y],INT) := (x^2-x*y^3+3*y)^2
```

$$x^4 - 2\,y^3\,x^3 + \left(y^6 + 6\,y\right)\,x^2 - 6\,y^4\,x + 9\,y^2 \tag{3}$$

MultivariatePolynomial ([x, y], Integer)

You can change the way the polynomial appears by modifying the variable ordering in the explicit list.

```
m :: MPOLY([y,x],INT)
```

$$x^2\,y^6 - 6\,x\,y^4 - 2\,x^3\,y^3 + 9\,y^2 + 6\,x^2\,y + x^4 \tag{4}$$

MultivariatePolynomial ([y, x], Integer)

The constructor **DistributedMultivariatePolynomial** provides polynomials in one or more specified variables with the monomials ordered lexicographically.

```
m :: DMP([y,x],INT)
```

$$y^6\,x^2 - 6\,y^4\,x - 2\,y^3\,x^3 + 9\,y^2 + 6\,y\,x^2 + x^4 \tag{5}$$

DistributedMultivariatePolynomial ([y, x], Integer)

The constructor **HomogeneousDistributedMultivariatePolynomial** is similar except that the monomials are ordered by total order refined by reverse lexicographic order.

```
m :: HDMP([y,x],INT)
```

$$y^6\,x^2 - 2\,y^3\,x^3 - 6\,y^4\,x + x^4 + 6\,y\,x^2 + 9\,y^2 \tag{6}$$

HomogeneousDistributedMultivariatePolynomial([y, x], Integer)

More generally, the domain constructor **GeneralDistributedMultivariatePolynomial** allows the user to provide an arbitrary predicate to define their own term ordering. These last three constructors are typically used in Gröbner basis applications and when a flat (that is, non-recursive) display is wanted and the term ordering is critical for controlling the computation.

1.10 Limits

FriCAS's limit function is usually used to evaluate limits of quotients where the numerator and denominator both tend to zero or both tend to infinity. To find the limit of an expression `f` as a real

variable x tends to a limit value a, enter limit(f, x=a). Use complexLimit if the variable is complex. Additional information and examples of limits are in Section 8.6 on page 275.

You can take limits of functions with parameters.

```
g := csc(a*x) / csch(b*x)
```

$$\frac{\csc(a\,x)}{\mathrm{csch}(b\,x)} \tag{1}$$

Expression(Integer)

As you can see, the limit is expressed in terms of the parameters.

```
limit(g,x=0)
```

$$\frac{b}{a} \tag{2}$$

Union(OrderedCompletion(Expression(Integer)), ...)

A variable may also approach plus or minus infinity:

```
h := (1 + k/x)^x
```

$$\left(\frac{x+k}{x}\right)^{x} \tag{3}$$

Expression(Integer)

Use %plusInfinity and %minusInfinity to denote ∞ and $-\infty$.

```
limit(h,x=%plusInfinity)
```

$$e^{k} \tag{4}$$

Union(OrderedCompletion(Expression(Integer)), ...)

A function can be defined on both sides of a particular value, but may tend to different limits as its variable approaches that value from the left and from the right.

```
limit(sqrt(y^2)/y,y = 0)
```

$$[leftHandLimit = -1,\ rightHandLimit = 1] \tag{5}$$

Union(Record(leftHandLimit: Union(OrderedCompletion(Expression(Integer)), "failed"), rightHandLimit: Union(OrderedCompletion(Expression(Integer)), "failed")), ...)

As x approaches 0 along the real axis, exp(-1/x^2) tends to 0.

```
limit(exp(-1/x^2),x = 0)
```

$$0 \tag{6}$$

Union(OrderedCompletion(Expression(Integer)), ...)

However, if `x` is allowed to approach `0` along any path in the complex plane, the limiting value of `exp(-1/x^2)` depends on the path taken because the function has an essential singularity at `x=0`. This is reflected in the error message returned by the function.

```
complexLimit(exp(-1/x^2),x = 0)
```

$$\texttt{"failed"} \tag{7}$$

Union(" failed ", ...)

1.11 Series

FriCAS also provides power series. By default, FriCAS tries to compute and display the first ten elements of a series. Use `)set streams calculate` to change the default value to something else. For the purposes of this book, we have used this system command to display fewer than ten terms. For more information about working with series, see Section 8.9 on page 282.

You can convert a functional expression to a power series by using the operation series. In this example, `sin(a*x)` is expanded in powers of `(x - 0)`, that is, in powers of `x`.

```
series(sin(a*x),x = 0)
```

$$a\,x - \frac{a^3}{6}\,x^3 + \frac{a^5}{120}\,x^5 - \frac{a^7}{5040}\,x^7 + O\left(x^9\right) \tag{1}$$

UnivariatePuiseuxSeries(Expression(Integer), x, 0)

This expression expands `sin(a*x)` in powers of `(x - %pi/4)`.

```
series(sin(a*x),x = %pi/4)
```

$$\begin{aligned}&\sin\left(\frac{a\,\pi}{4}\right) + a\,\cos\left(\frac{a\,\pi}{4}\right)\left(x - \frac{\pi}{4}\right) - \frac{a^2\,\sin\left(\frac{a\,\pi}{4}\right)}{2}\left(x - \frac{\pi}{4}\right)^2 \\ &- \frac{a^3\,\cos\left(\frac{a\,\pi}{4}\right)}{6}\left(x - \frac{\pi}{4}\right)^3 + \frac{a^4\,\sin\left(\frac{a\,\pi}{4}\right)}{24}\left(x - \frac{\pi}{4}\right)^4 + \frac{a^5\,\cos\left(\frac{a\,\pi}{4}\right)}{120}\left(x - \frac{\pi}{4}\right)^5 \\ &- \frac{a^6\,\sin\left(\frac{a\,\pi}{4}\right)}{720}\left(x - \frac{\pi}{4}\right)^6 - \frac{a^7\,\cos\left(\frac{a\,\pi}{4}\right)}{5040}\left(x - \frac{\pi}{4}\right)^7 + O\left(\left(x - \frac{\pi}{4}\right)^8\right)\end{aligned} \tag{2}$$

UnivariatePuiseuxSeries(Expression(Integer), x, %pi/4)

FriCAS provides *Puiseux series:* series with rational number exponents. The first argument to series is an in-place function that computes the n^{th} coefficient. (Recall that the "`+->`" is an infix operator meaning "maps to.")

```
series(n +-> (-1)^((3*n - 4)/6)/factorial(n - 1/3),x = 0,4/3..,2)
```

$$x^{\frac{4}{3}} - \frac{1}{6}\,x^{\frac{10}{3}} + O\left(x^4\right) \tag{3}$$

UnivariatePuiseuxSeries (Expression (Integer), x, 0)

Once you have created a power series, you can perform arithmetic operations on that series. We compute the Taylor expansion of `1/(1-x)`.

```
f := series(1/(1-x),x = 0)
```

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + O(x^8) \tag{4}$$

UnivariatePuiseuxSeries (Expression (Integer), x, 0)

Compute the square of the series.

```
f ^ 2
```

$$1 + 2\,x + 3\,x^2 + 4\,x^3 + 5\,x^4 + 6\,x^5 + 7\,x^6 + 8\,x^7 + O(x^8) \tag{5}$$

UnivariatePuiseuxSeries (Expression (Integer), x, 0)

The usual elementary functions (log, exp, trigonometric functions, and so on) are defined for power series.

```
f := series(1/(1-x),x = 0)
```

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + O(x^8) \tag{6}$$

UnivariatePuiseuxSeries (Expression (Integer), x, 0)

```
g := log(f)
```

$$x + \frac{1}{2}\,x^2 + \frac{1}{3}\,x^3 + \frac{1}{4}\,x^4 + \frac{1}{5}\,x^5 + \frac{1}{6}\,x^6 + \frac{1}{7}\,x^7 + \frac{1}{8}\,x^8 + O(x^9) \tag{7}$$

UnivariatePuiseuxSeries (Expression (Integer), x, 0)

```
exp(g)
```

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + O(x^8) \tag{8}$$

UnivariatePuiseuxSeries (Expression (Integer), x, 0)

Here is a way to obtain numerical approximations of `e` from the Taylor series expansion of `exp(x)`. First create the desired Taylor expansion.

```
f := taylor(exp(x))
```

$$1 + x + \frac{1}{2}\,x^2 + \frac{1}{6}\,x^3 + \frac{1}{24}\,x^4 + \frac{1}{120}\,x^5 + \frac{1}{720}\,x^6 + \frac{1}{5040}\,x^7 + O(x^8) \tag{9}$$

UnivariateTaylorSeries (Expression (Integer), x, 0)

Evaluate the series at the value `1.0`. As you see, you get a sequence of partial sums.

```
eval(f,1.0)
```

$$\begin{aligned}&[1.0,\ 2.0,\ 2.5,\ 2.66666666666666666666666666666666666666666667,\\ &\ 2.70833333333333333333333333333333333333333,\\ &\ 2.71666666666666666666666666666666666666666667,\\ &\ 2.71805555555555555555555555555555555555555556,\ \ldots]\end{aligned} \tag{10}$$

Stream(Expression(Float))

1.12 Derivatives

Use the FriCAS function D to differentiate an expression.

To find the derivative of an expression `f` with respect to a variable `x`, enter `D(f, x)`.

```
f := exp exp x
```

$$e^{e^x} \tag{1}$$

Expression(Integer)

```
D(f, x)
```

$$e^x\, e^{e^x} \tag{2}$$

Expression(Integer)

An optional third argument `n` in D asks FriCAS for the n^{th} derivative of `f`. This finds the fourth derivative of `f` with respect to `x`.

```
D(f, x, 4)
```

$$\left((e^x)^4 + 6\,(e^x)^3 + 7\,(e^x)^2 + e^x\right)\, e^{e^x} \tag{3}$$

Expression(Integer)

You can also compute partial derivatives by specifying the order of differentiation.

```
g := sin(x^2 + y)
```

$$\sin\left(y + x^2\right) \tag{4}$$

Expression(Integer)

```
D(g, y)
```

$$\cos\left(y + x^2\right) \tag{5}$$

Expression(Integer)

```
D(g, [y, y, x, x])
```

$$4\,x^2\,\sin\left(y+x^2\right)-2\,\cos\left(y+x^2\right) \tag{6}$$

Expression(Integer)

FriCAS can manipulate the derivatives (partial and iterated) of expressions involving formal operators. All the dependencies must be explicit. This returns `0` since `F` (so far) does not explicitly depend on `x`.

```
D(F,x)
```

$$0 \tag{7}$$

Polynomial(Integer)

Suppose that we have `F` a function of `x`, `y`, and `z`, where `x` and `y` are themselves functions of `z`. Start by declaring that `F`, `x`, and `y` are operators.

```
F := operator 'F; x := operator 'x; y := operator 'y
```

$$y \tag{8}$$

BasicOperator

You can use `F`, `x`, and `y` in expressions.

```
a := F(x z, y z, z^2) + x y(z+1)
```

$$x(y(z+1))+F\left(x(z),y(z),z^2\right) \tag{9}$$

Expression(Integer)

Differentiate formally with respect to `z`. The formal derivatives appearing in `dadz` are not just formal symbols, but do represent the derivatives of `x`, `y`, and `F`.

```
dadz := D(a, z)
```

$$2\,z\,F_{,3}\left(x(z),y(z),z^2\right)+y'(z)\,F_{,2}\left(x(z),y(z),z^2\right)+x'(z)\,F_{,1}\left(x(z),y(z),z^2\right)+x'(y(z+1))\,y'(z+1) \tag{10}$$

Expression(Integer)

You can evaluate the above for particular functional values of `F`, `x`, and `y`. If `x(z)` is `exp(z)` and `y(z)` is `log(z+1)`, then this evaluates `dadz`.

```
eval(eval(dadz, 'x, z +-> exp z), 'y, z +-> log(z+1))
```

$$\frac{\left(2\,z^2+2\,z\right)\,F_{,3}\left(e^z,\log(z+1),z^2\right)+F_{,2}\left(e^z,\log(z+1),z^2\right)+(z+1)\,e^z\,F_{,1}\left(e^z,\log(z+1),z^2\right)+z+1}{z+1} \tag{11}$$

Expression(Integer)

You obtain the same result by first evaluating `a` and then differentiating.

```
eval(eval(a, 'x, z +-> exp z), 'y, z +-> log(z+1))
```

$$F\left(e^z, \log(z+1), z^2\right) + z + 2 \tag{12}$$

Expression (Integer)

```
D(%, z)
```

$$\frac{\left(2\,z^2+2\,z\right)\ F_{,3}\left(e^z, \log(z+1), z^2\right) + F_{,2}\left(e^z, \log(z+1), z^2\right) + (z+1)\ e^z\ F_{,1}\left(e^z, \log(z+1), z^2\right) + z + 1}{z+1} \tag{13}$$

Expression (Integer)

1.13 Integration

FriCAS has extensive library facilities for integration.

The first example is the integration of a fraction with denominator that factors into a quadratic and a quartic irreducible polynomial. The usual partial fraction approach used by most other computer algebra systems either fails or introduces expensive unneeded algebraic numbers.

We use a factorization-free algorithm.

```
integrate((x^2+2*x+1)/((x+1)^6+1),x)
```

$$\frac{\arctan\left(x^3+3\,x^2+3\,x+1\right)}{3} \tag{1}$$

Union(Expression(Integer), ...)

When real parameters are present, the form of the integral can depend on the signs of some expressions.

Rather than query the user or make sign assumptions, FriCAS returns all possible answers.

```
integrate(1/(x^2 + a),x)
```

$$\left[\frac{\log\left(\frac{\left(x^2-a\right)\sqrt{-a}+2\,a\,x}{x^2+a}\right)}{2\sqrt{-a}},\ \frac{\arctan\left(\frac{x\sqrt{a}}{a}\right)}{\sqrt{a}}\right] \tag{2}$$

Union(List(Expression(Integer)), ...)

The integrate operation generally assumes that all parameters are real. The only exception is when the integrand has complex valued quantities.

If the parameter is complex instead of real, then the notion of sign is undefined and there is a unique answer. You can request this answer by "prepending" the word "complex" to the command name:

```
complexIntegrate(1/(x^2 + a),x)
```

$$\frac{\sqrt{-\frac{1}{a}}\,\log\left(a\sqrt{-\frac{1}{a}}+x\right) - \sqrt{-\frac{1}{a}}\,\log\left(-a\sqrt{-\frac{1}{a}}+x\right)}{2} \tag{3}$$

Expression(Integer)

The following two examples illustrate the limitations of table-based approaches. The two integrands are very similar, but the answer to one of them requires the addition of two new algebraic numbers.

This one is the easy one. The next one looks very similar but the answer is much more complicated.

```
integrate(x^3 / (a+b*x)^(1/3),x)
```

$$\frac{\left(120\,b^3\,x^3-135\,a\,b^2\,x^2+162\,a^2\,b\,x-243\,a^3\right)\,\sqrt[3]{b\,x+a}^{\,2}}{440\,b^4} \tag{4}$$

Union(Expression(Integer), ...)

Only an algorithmic approach is guaranteed to find what new constants must be added in order to find a solution.

```
integrate(1 / (x^3 * (a+b*x)^(1/3)),x)
```

$$\frac{-2\,b^2\,x^2\,\sqrt{3}\,\log\left(\sqrt[3]{a}\,\sqrt[3]{b\,x+a}^{\,2}+\sqrt[3]{a}^{\,2}\,\sqrt[3]{b\,x+a}+a\right)+4\,b^2\,x^2\,\sqrt{3}\,\log\left(\sqrt[3]{a}^{\,2}\,\sqrt[3]{b\,x+a}-a\right)+12\,b^2\,x^2\,\arctan\left(\frac{2\sqrt{3}\,\sqrt[3]{a}^{\,2}\,\sqrt[3]{b\,x+a}+a\sqrt{3}}{3\,a}\right.}{18\,a^2\,x^2\,\sqrt{3}\,\sqrt[3]{a}} \tag{5}$$

Union(Expression(Integer), ...)

Some computer algebra systems use heuristics or table-driven approaches to integration. When these systems cannot determine the answer to an integration problem, they reply "I don't know." FriCAS uses an algorithm for integration that conclusively proves that an integral cannot be expressed in terms of elementary functions.

When FriCAS returns an integral sign, it has proved that no answer exists as an elementary function.

```
integrate(log(1 + sqrt(a*x + b)) / x,x)
```

$$\int^{x}\frac{\log\left(\sqrt{b+\%D\,a}+1\right)}{\%D}\,d\%D \tag{6}$$

Union(Expression(Integer), ...)

FriCAS can handle complicated mixed functions much beyond what you can find in tables. Whenever possible, FriCAS tries to express the answer using the functions present in the integrand.

```
integrate((sinh(1+sqrt(x+b))+2*sqrt(x+b)) / (sqrt(x+b) * (x + cosh(1+sqrt(x + b)))), x)
```

$$2\,\log\left(\frac{-2\,\cosh\left(\sqrt{x+b}+1\right)-2\,x}{\sinh\left(\sqrt{x+b}+1\right)-\cosh\left(\sqrt{x+b}+1\right)}\right)-2\,\sqrt{x+b} \tag{7}$$

Union(Expression(Integer), ...)

A strong structure-checking algorithm in FriCAS finds hidden algebraic relationships between functions.

```
integrate(tan(atan(x)/3),x)
```

$$\frac{8\log\left(3\tan\left(\frac{\arctan(x)}{3}\right)^2-1\right)-3\tan\left(\frac{\arctan(x)}{3}\right)^2+18\,x\,\tan\left(\frac{\arctan(x)}{3}\right)}{18} \tag{8}$$

Union(Expression(Integer), ...)

The discovery of this algebraic relationship is necessary for correct integration of this function. Here are the details:

1. If $x = \tan t$ and $g = \tan(t/3)$ then the following algebraic relation is true:
$$g^3 - 3xg^2 - 3g + x = 0$$
2. Integrate g using this algebraic relation; this produces:
$$\frac{(24g^2-8)\log(3g^2-1)+(81x^2+24)g^2+72xg-27x^2-16}{54g^2-18}$$
3. Rationalize the denominator, producing:
$$\frac{8\log(3g^2-1)-3g^2+18xg+16}{18}$$
Replace g by the initial definition $g = \tan(\arctan(x)/3)$ to produce the final result.

This is an example of a mixed function where the algebraic layer is over the transcendental one.

```
integrate((x + 1) / (x*(x + log x) ^ (3/2)), x)
```

$$-\frac{2\sqrt{\log(x)+x}}{\log(x)+x} \tag{9}$$

Union(Expression(Integer), ...)

While incomplete for non-elementary functions, FriCAS can handle some of them.

```
integrate(exp(-x^2) * erf(x) / (erf(x)^3 - erf(x)^2 - erf(x) + 1),x)
```

$$\frac{(\operatorname{erf}(x)-1)\,\sqrt{\pi}\,\log\left(\frac{\operatorname{erf}(x)-1}{\operatorname{erf}(x)+1}\right)-2\sqrt{\pi}}{8\,\operatorname{erf}(x)-8} \tag{10}$$

Union(Expression(Integer), ...)

More examples of FriCAS's integration capabilities are discussed in Section 8.8 on page 279.

1.14 Differential Equations

The general approach used in integration also carries over to the solution of linear differential equations.

Let's solve some differential equations. Let y be the unknown function in terms of x.

```
y := operator 'y
```

$$y \tag{1}$$

BasicOperator

Here we solve a third order equation with polynomial coefficients.

```
deq := x^3 * D(y x, x, 3) + x^2 * D(y x, x, 2) - 2 * x * D(y x, x) + 2 * y x = 2 * x^4
```

$$x^3\,y'''(x) + x^2\,y''(x) - 2\,x\,y'(x) + 2\,y(x) = 2\,x^4 \tag{2}$$

Equation(Expression(Integer))

```
solve(deq, y, x)
```

$$\left[particular = \frac{x^5 - 10\,x^3 + 20\,x^2 + 4}{15\,x},\ basis = \left[\frac{2\,x^3 - 3\,x^2 + 1}{x},\ \frac{x^3 - 1}{x},\ \frac{x^3 - 3\,x^2 - 1}{x}\right]\right] \tag{3}$$

Union(Record(particular : Expression(Integer), basis : List(Expression(Integer))), ...)

Here we find all the algebraic function solutions of the equation.

```
deq := (x^2 + 1) * D(y x, x, 2) + 3 * x * D(y x, x) + y x = 0
```

$$\left(x^2 + 1\right)\,y''(x) + 3\,x\,y'(x) + y(x) = 0 \tag{4}$$

Equation(Expression(Integer))

```
solve(deq, y, x)
```

$$\left[particular = 0,\ basis = \left[\frac{1}{\sqrt{x^2+1}},\ \frac{\log\left(\sqrt{x^2+1} - x\right)}{\sqrt{x^2+1}}\right]\right] \tag{5}$$

Union(Record(particular : Expression(Integer), basis : List(Expression(Integer))), ...)

Coefficients of differential equations can come from arbitrary constant fields. For example, coefficients can contain algebraic numbers.

This example has solutions whose logarithmic derivative is an algebraic function of degree two.

```
eq := 2*x^3 * D(y x,x,2) + 3*x^2 * D(y x,x) - 2 * y x
```

$$2\,x^3\,y''(x) + 3\,x^2\,y'(x) - 2\,y(x) \tag{6}$$

Expression(Integer)

```
solve(eq,y,x).basis
```

$$\left[e^{-\frac{2}{\sqrt{x}}},\ e^{\frac{2}{\sqrt{x}}}\right] \tag{7}$$

List(Expression(Integer))

Here's another differential equation to solve.

```
deq := D(y x, x) = y(x) / (x + y(x) * log y x)
```

$$y'(x) = \frac{y(x)}{y(x)\,\log(y(x)) + x} \tag{8}$$

Equation(Expression(Integer))

```
solve(deq, y, x)
```

$$\frac{y(x)\log(y(x))^2 - 2\,x}{2\,y(x)} \tag{9}$$

Union(Expression(Integer), ...)

Rather than attempting to get a closed form solution of a differential equation, you instead might want to find an approximate solution in the form of a series.

Let's solve a system of nonlinear first order equations and get a solution in power series. Tell FriCAS that `x` is also an operator.

```
x := operator 'x
```

$$x \tag{10}$$

BasicOperator

Here are the two equations forming our system.

```
eq1 := D(x(t), t) = 1 + x(t)^2
```

$$x'(t) = x(t)^2 + 1 \tag{11}$$

Equation(Expression(Integer))

```
eq2 := D(y(t), t) = x(t) * y(t)
```

$$y'(t) = x(t)\,y(t) \tag{12}$$

Equation(Expression(Integer))

We can solve the system around `t = 0` with the initial conditions `x(0) = 0` and `y(0) = 1`. Notice that since we give the unknowns in the order `[x, y]`, the answer is a list of two series in the order `[series for x(t), series for y(t)]`.

```
seriesSolve([eq2, eq1], [x, y], t = 0, [y(0) = 1, x(0) = 0])
```

```
   Compiling function %JL with type List(UnivariateTaylorSeries(Expression(Integer
      ),t,0)) -> UnivariateTaylorSeries(Expression(Integer),t,0)
   Compiling function %JM with type List(UnivariateTaylorSeries(Expression(Integer
      ),t,0)) -> UnivariateTaylorSeries(Expression(Integer),t,0)
```

$$\left[t + \frac{1}{3}\,t^3 + \frac{2}{15}\,t^5 + \frac{17}{315}\,t^7 + O\left(t^8\right),\ 1 + \frac{1}{2}\,t^2 + \frac{5}{24}\,t^4 + \frac{61}{720}\,t^6 + O\left(t^8\right)\right] \tag{13}$$

List (UnivariateTaylorSeries (Expression(Integer), t, 0))

1.15 Solution of Equations

FriCAS also has state-of-the-art algorithms for the solution of systems of polynomial equations. When the number of equations and unknowns is the same, and you have no symbolic coefficients, you can use solve for real roots and complexSolve for complex roots. In each case, you tell FriCAS how accurate you want your result to be. All operations in the solve family return answers in the form of a list of solution sets, where each solution set is a list of equations.

A system of two equations involving a symbolic parameter t.

```
S(t) == [x^2-2*y^2 - t,x*y-y-5*x + 5]
```

Find the real roots of S(19) with rational arithmetic, correct to within $1/10^{20}$.

```
solve(S(19),1/10^20)
```

```
   Compiling function S with type PositiveInteger -> List(Polynomial(Integer))
```

$$\left[\left[y = 5,\ x = -\frac{80336736493669365924189585}{96714065569170333976494 08}\right],\ \left[y = 5,\ x = \frac{80336736493669365924189585}{96714065569170333976494 08}\right]\right] \tag{2}$$

List (List (Equation(Polynomial(Fraction(Integer)))))

Find the complex roots of S(19) with floating point coefficients to 20 digits accuracy in the mantissa.

```
complexSolve(S(19),10.e-20)
```

$$\begin{aligned}&[[y = 5.0,\ x = 8.3066238629180748525842627449151614865608038331467813256484000739021 67],\\ &\ [y = 5.0,\ x = -8.3066238629180748525842627449151614865 61],\\ &\ [y = -3.0\,i,\ x = 1.0],\ [y = 3.0\,i,\ x = 1.0]]\end{aligned} \tag{3}$$

List (List (Equation(Polynomial(Complex(Float)))))

If a system of equations has symbolic coefficients and you want a solution in radicals, try radicalSolve.

```
radicalSolve(S(a),[x,y])
```

```
   Compiling function S with type Variable(a) -> List(Polynomial(Integer))
```

$$\left[[x = -\sqrt{a+50},\ y = 5],\ [x = \sqrt{a+50},\ y = 5],\ \left[x = 1,\ y = -\sqrt{\frac{-a+1}{2}}\right],\ \left[x = 1,\ y = \sqrt{\frac{-a+1}{2}}\right]\right] \tag{4}$$

List (List (Equation(Expression(Integer))))

For systems of equations with symbolic coefficients, you can apply solve, listing the variables that you want FriCAS to solve for. For polynomial equations, a solution cannot usually be expressed solely in terms of the other variables. Instead, the solution is presented as a "triangular" system of equations, where each polynomial has coefficients involving only the succeeding variables. This is analogous to converting a linear system of equations to "triangular form". A system of three equations in five variables.

```
eqns := [x^2 - y + z,x^2*z + x^4 - b*y, y^2 *z - a - b*x]
```

$$\left[z - y + x^2,\ x^2\,z - b\,y + x^4,\ y^2\,z - b\,x - a\right] \tag{5}$$

List (Polynomial(Integer))

Solve the system for unknowns $[x, y, z]$, reducing the solution to triangular form.

```
solve(eqns,[x,y,z])
```

$$\left[\left[x = -\frac{a}{b},\ y = 0,\ z = -\frac{a^2}{b^2}\right],\ \left[x = \frac{z^3 + 2\,b\,z^2 + b^2\,z - a}{b},\ y = z + b,\right.\right.$$
$$\left.\left. z^6 + 4\,b\,z^5 + 6\,b^2\,z^4 + \left(4\,b^3 - 2\,a\right)\,z^3 + \left(b^4 - 4\,a\,b\right)\,z^2 - 2\,a\,b^2\,z - b^3 + a^2 = 0\right]\right] \tag{6}$$

List (List (Equation(Fraction (Polynomial(Integer)))))

1.16 System Commands

We conclude our tour of FriCAS with a brief discussion of *system commands*. System commands are special statements that start with a closing parenthesis (“)”). They are used to control or display your FriCAS environment, start the HyperDoc system, issue operating system commands and leave FriCAS. For example, `)system` is used to issue commands to the operating system from FriCAS. Here is a brief description of some of these commands. For more information on specific commands, see Appendix A.

Perhaps the most important user command is the `)clear all` command that initializes your environment. Every section and subsection in this book has an invisible `)clear all` that is read prior to the examples given in the section. `)clear all` gives you a fresh, empty environment with no user variables defined and the step number reset to 1. The `)clear` command can also be used to selectively clear values and properties of system variables.

Another useful system command is `)read`. A preferred way to develop an application in FriCAS is to put your interactive commands into a file, say **my.input** file. To get FriCAS to read this file, you use the system command `)read my.input`. If you need to make changes to your approach or definitions, go into your favorite editor, change **my.input**, then `)read my.input` again.

Other system commands include: `)history`, to display previous input and/or output lines; `)display`, to display properties and values of workspace variables; and `)what`.

Issue `)what` to get a list of FriCAS objects that contain a given substring in their name.

```
)what operations integrate
```

```
Operations whose names satisfy the above pattern(s):

HermiteIntegrate       algintegrate           complexIntegrate       expintegrate
fintegrate             infieldIntegrate       integrate              integrateIfCan
integrate_sols         internalIntegrate      internalIntegrate0     lambintegrate
lazyGintegrate         lazyIntegrate          lfintegrate            monomialIntegrate
palgintegrate          pmintegrate            primintegrate

      To get more information about an operation such as integrate , issue the
      command )display op integrate
```

A useful system command is `)undo`. Sometimes while computing interactively with FriCAS, you make a mistake and enter an incorrect definition or assignment. Or perhaps you need to try one of several alternative approaches, one after another, to find the best way to approach an application. For this, you will find the *undo* facility of FriCAS helpful.

System command `)undo n` means "undo back to step `n`"; it restores the values of user variables to those that existed immediately after input expression `n` was evaluated. Similarly, `)undo -n` undoes changes caused by the last `n` input expressions. Once you have done an `)undo`, you can continue on from there, or make a change and **redo** all your input expressions from the point of the `)undo` forward. The `)undo` is completely general: it changes the environment like any user expression. Thus you can `)undo` any previous undo.

Here is a sample dialogue between user and FriCAS. "Let me define two mutually dependent functions `f` and `g` piece-wise."

```
f(0) == 1; g(0) == 1
```

"Here is the general term for `f`."

```
f(n) == e/2*f(n-1) - x*g(n-1)
```

"And here is the general term for `g`."

```
g(n) == -x*f(n-1) + d/3*g(n-1)
```

"What is the value of `f(3)`?"

```
f(3)
```

```
   Compiling function g with type Integer -> Polynomial(Fraction(Integer))
   Compiling function g as a recurrence relation.
   Compiling function g with type Integer -> Polynomial(Fraction(Integer))
   Compiling function g as a recurrence relation.
   Compiling function f with type Integer -> Polynomial(Fraction(Integer))
   Compiling function f as a recurrence relation.
```

$$-x^3+\left(e+\frac{1}{3}\,d\right)x^2+\left(-\frac{1}{4}\,e^2-\frac{1}{6}\,d\,e-\frac{1}{9}\,d^2\right)x+\frac{1}{8}\,e^3 \tag{4}$$

Polynomial(Fraction (Integer))

"Hmm, I think I want to define `f` differently. Undo to the environment right after I defined `f`."

```
)undo 2
```

"Here is how I think I want `f` to be defined instead."

```
f(n) == d/3*f(n-1) - x*g(n-1)
```

```
   1 old definition(s) deleted for function or rule f
```

Redo the computation from expression `3` forward.

```
)undo )redo
```

"I want my old definition of `f` after all. Undo the undo and restore the environment to that immediately after `(4)`."

```
)undo 4
```

"Check that the value of `f(3)` is restored."

```
f(3)
```

```
   Compiling function g with type Integer -> Polynomial(Fraction(Integer))
   Compiling function g as a recurrence relation.
   Compiling function g with type Integer -> Polynomial(Fraction(Integer))
   Compiling function g as a recurrence relation.
   Compiling function f with type Integer -> Polynomial(Fraction(Integer))
   Compiling function f as a recurrence relation.
```

$$-x^3+\left(e+\frac{1}{3}\,d\right)x^2+\left(-\frac{1}{4}\,e^2-\frac{1}{6}\,d\,e-\frac{1}{9}\,d^2\right)x+\frac{1}{8}\,e^3 \tag{6}$$

Polynomial(Fraction (Integer))

After you have gone off on several tangents, then backtracked to previous points in your conversation using `)undo`, you might want to save all the "correct" input commands you issued, disregarding those undone. The system command `)history )write mynew.input` writes a clean straight-line program onto the file **mynew.input** on your disk.

This concludes your tour of FriCAS. To disembark, issue the system command `)quit` to leave FriCAS and return to the operating system.

Chapter 2

Using Types and Modes

In this chapter we look at the key notion of *type* and its generalization *mode*. We show that every FriCAS object has a type that determines what you can do with the object. In particular, we explain how to use types to call specific functions from particular parts of the library and how types and modes can be used to create new objects from old. We also look at **Record** and **Union** types and the special type **Any**. Finally, we give you an idea of how FriCAS manipulates types and modes internally to resolve ambiguities.

2.1 The Basic Idea

The FriCAS world deals with many kinds of objects. There are mathematical objects such as numbers and polynomials, data structure objects such as lists and arrays, and graphics objects such as points and graphic images. Functions are objects too.

FriCAS organizes objects using the notion of *domain of computation*, or simply *domain*. Each domain denotes a class of objects. The class of objects it denotes is usually given by the name of the domain: **Integer** for the integers, **Float** for floating-point numbers, and so on. The convention is that the first letter of a domain name is capitalized. Similarly, the domain **Polynomial(Integer)** denotes "polynomials with integer coefficients." Also, **Matrix(Float)** denotes "matrices with floating-point entries."

Every basic FriCAS object belongs to a unique domain. The integer `3` belongs to the domain **Integer** and the polynomial `x + 3` belongs to the domain **Polynomial(Integer)**. The domain of an object is also called its *type*. Thus we speak of "the type **Integer**" and "the type **Polynomial(Integer)**."

After an FriCAS computation, the type is displayed toward the right-hand side of the page (or screen).

```
-3
```

$$-3 \tag{1}$$

Integer

Here we create a rational number but it looks like the last result. The type, however, tells you it is different. You cannot identify the type of an object by how FriCAS displays the object.

```
-3/1
```

$$-3 \tag{2}$$

Fraction (Integer)

When a computation produces a result of a simpler type, FriCAS leaves the type unsimplified. Thus no information is lost.

```
x + 3 - x
```

$$3 \tag{3}$$

Polynomial(Integer)

This seldom matters since FriCAS retracts the answer to the simpler type if it is necessary.

```
factorial(%)
```

$$6 \tag{4}$$

Expression (Integer)

When you issue a positive number, the type **PositiveInteger** is printed. Surely, `3` also has type **Integer**! The curious reader may now have two questions. First, is the type of an object not unique? Second, how is **PositiveInteger** related to **Integer**? Read on!

```
3
```

$$3 \tag{5}$$

PositiveInteger

Any domain can be refined to a *subdomain* by a membership *predicate*.[1] For example, the domain **Integer** can be refined to the subdomain **PositiveInteger**, the set of integers `x` such that `x > 0`, by giving the FriCAS predicate `x` $\mapsto$ `x > 0`. Similarly, FriCAS can define subdomains such as "the subdomain of diagonal matrices," "the subdomain of lists of length two," "the subdomain of monic irreducible polynomials in `x`," and so on. Trivially, any domain is a subdomain of itself.

While an object belongs to a unique domain, it can belong to any number of subdomains. Any subdomain of the domain of an object can be used as the *type* of that object. The type of `3` is indeed both **Integer** and **PositiveInteger** as well as any other subdomain of Integer whose predicate is satisfied, such as "the prime integers," "the odd positive integers between 3 and 17," and so on.

2.1.1 Domain Constructors

In FriCAS, domains are objects. You can create them, pass them to functions, and, as we'll see later, test them for certain properties.

In FriCAS, you ask for a value of a function by applying its name to a set of arguments.

To ask for "the factorial of 7" you enter this expression to FriCAS. This applies the function `factorial` to the value `7` to compute the result.

[1] A predicate is a function that, when applied to an object of the domain, returns either `true` or `false`.

```
factorial(7)
```

$$5040 \tag{1}$$

PositiveInteger

Enter the type **Polynomial (Integer)** as an expression to FriCAS. This looks much like a function call as well. It is! The result is stated to be of type **Type**, which according to our usual convention, denotes the class of all domains.

```
Polynomial(Integer)
```

$$\text{Polynomial}(Integer) \tag{2}$$

Type

The most basic operation involving domains is that of building a new domain from a given one. To create the domain of "polynomials over the integers," FriCAS applies the function **Polynomial** to the domain **Integer**. A function like **Polynomial** is called a *domain constructor* or, more simply, a *constructor*. A domain constructor is a function that creates a domain. An argument to a domain constructor can be another domain or, in general, an arbitrary kind of object. **Polynomial** takes a single domain argument while **SquareMatrix** takes a positive integer as an argument to give its dimension and a domain argument to give the type of its components.

What kinds of domains can you use as the argument to **Polynomial** or **SquareMatrix** or **List**? Well, the first two are mathematical in nature. You want to be able to perform algebraic operations like + and * on polynomials and square matrices, and operations such as determinant on square matrices. So you want to allow polynomials of integers *and* polynomials of square matrices with complex number coefficients and, in general, anything that "makes sense." At the same time, you don't want FriCAS to be able to build nonsense domains such as "polynomials of strings!"

In contrast to algebraic structures, data structures can hold any kind of object. Operations on lists such as insert, delete, and concat just manipulate the list itself without changing or operating on its elements. Thus you can build **List** over almost any datatype, including itself. Create a complicated algebraic domain.

```
List (List (Matrix (Polynomial (Complex (Fraction (Integer))))))
```

$$\text{List}(\text{List}(\text{Matrix}(\text{Polynomial}(\text{Complex}(\text{Fraction}(Integer)))))) \tag{3}$$

Type

Try to create a meaningless domain.

```
Polynomial(String)
```

```
   Polynomial(String) is not a valid type.
```

Evidently from our last example, FriCAS has some mechanism that tells what a constructor can use as an argument. This brings us to the notion of *category*. As domains are objects, they too have a domain. The domain of a domain is a category. A category is simply a type whose members are domains.

A common algebraic category is **Ring**, the class of all domains that are "rings." A ring is an algebraic structure with constants 0 and 1 and operations +, -, and *. These operations are assumed "closed" with respect to the domain, meaning that they take two objects of the domain and produce a result object also in the domain. The operations are understood to satisfy certain "axioms," certain mathematical

principles providing the algebraic foundation for rings. For example, the *additive inverse axiom* for rings states:

Every element `x` has an additive inverse `y` such that `x + y = 0`.

The prototypical example of a domain that is a ring is the integers. Keep them in mind whenever we mention **Ring**.

Many algebraic domain constructors such as **Complex**, **Polynomial**, **Fraction**, take rings as arguments and return rings as values. You can use the infix operator "`has`" to ask a domain if it belongs to a particular category.

All numerical types are rings. Domain constructor **Polynomial** builds "the ring of polynomials over any other ring."

```
Polynomial(Integer) has Ring
```

$$\texttt{true} \tag{4}$$

Boolean

Constructor **List** never produces a ring.

```
List(Integer) has Ring
```

$$\texttt{false} \tag{5}$$

Boolean

The constructor **Matrix(R)** builds "the domain of all matrices over the ring `R`." This domain is never a ring since the operations "`+`", "`-`", and "`*`" on matrices of arbitrary shapes are undefined.

```
Matrix(Integer) has Ring
```

$$\texttt{false} \tag{6}$$

Boolean

Thus you can never build polynomials over matrices.

```
Polynomial(Matrix(Integer))
```

```
   Polynomial(Matrix(Integer)) is not a valid type.
```

Use **SquareMatrix(n,R)** instead. For any positive integer `n`, it builds "the ring of `n` by `n` matrices over `R`."

```
Polynomial(SquareMatrix(7,Complex(Integer)))
```

$$\text{Polynomial}(\text{SquareMatrix}(7, \text{Complex}(\mathit{Integer}))) \tag{7}$$

Type

Another common category is **Field**, the class of all fields. A field is a ring with additional operations. For example, a field has commutative multiplication and a closed operation `/` for the division of two elements. **Integer** is not a field since, for example, `3/2` does not have an integer result. The prototypical example of a field is the rational numbers, that is, the domain **Fraction(Integer)**. In general, the

constructor **Fraction** takes a ring as an argument and returns a field.[2] Other domain constructors, such as **Complex**, build fields only if their argument domain is a field.

The complex integers (often called the "Gaussian integers") do not form a field.

```
Complex(Integer) has Field
```

```
                                    false                                    (8)
                                                                        Boolean
```

But fractions of complex integers do.

```
Fraction(Complex(Integer)) has Field
```

```
                                    true                                     (9)
                                                                        Boolean
```

The algebraically equivalent domain of complex rational numbers is a field since domain constructor **Complex** produces a field whenever its argument is a field.

```
Complex(Fraction(Integer)) has Field
```

```
                                    true                                     (10)
                                                                        Boolean
```

The most basic category is **Type**. It denotes the class of all domains and subdomains.[3] Domain constructor **List** is able to build "lists of elements from domain `D`" for arbitrary `D` simply by requiring that `D` belong to category **Type**.

Now, you may ask, what exactly is a category? Like domains, categories can be defined in the FriCAS language. A category is defined by three components:

1. a name (for example, **Ring**), used to refer to the class of domains that the category represents;
2. a set of operations, used to refer to the operations that the domains of this class support (for example, `+`, `-`, and `*` for rings); and
3. an optional list of other categories that this category extends.

This last component is a new idea. And it is key to the design of FriCAS! Because categories can extend one another, they form hierarchies. Detailed charts showing the category hierarchies in FriCAS are displayed in the endpages of this book. There you see that all categories are extensions of **Type** and that **Field** is an extension of **Ring**.

The operations supported by the domains of a category are called the *exports* of that category because these are the operations made available for system-wide use. The exports of a domain of a given category are not only the ones explicitly mentioned by the category. Since a category extends other categories, the operations of these other categories—and all categories these other categories extend—are also exported by the domains.

[2] Actually, the argument domain must have some additional properties so as to belong to category **IntegralDomain**.
[3] **Type** does not denote the class of all types. The type of all categories is **Category**. The type of **Type** itself is undefined.

For example, polynomial domains belong to **PolynomialCategory**. This category explicitly mentions some twenty-nine operations on polynomials, but it extends eleven other categories (including **Ring**). As a result, the current system has over one hundred operations on polynomials.

If a domain belongs to a category that extends, say, **Ring**, it is convenient to say that the domain exports **Ring**. The name of the category thus provides a convenient shorthand for the list of operations exported by the category. Rather than listing operations such as + and * of **Ring** each time they are needed, the definition of a type simply asserts that it exports category **Ring**.

The category name, however, is more than a shorthand. The name **Ring**, in fact, implies that the operations exported by rings are required to satisfy a set of "axioms" associated with the name **Ring**.[4]

Why is it not correct to assume that some type is a ring if it exports all of the operations of **Ring**? Here is why. Some languages such as **APL** denote the **Boolean** constants `true` and `false` by the integers `1` and `0` respectively, then use + and * to denote the logical operators or and and. But with these definitions **Boolean** is not a ring since the additive inverse axiom is violated.[5] This alternative definition of **Boolean** can be easily and correctly implemented in FriCAS, since **Boolean** simply does not assert that it is of category **Ring**. This prevents the system from building meaningless domains such as **Polynomial(Boolean)** and then wrongfully applying algorithms that presume that the ring axioms hold.

Enough on categories. To learn more about them, see Chapter 12. We now return to our discussion of domains.

Domains *export* a set of operations to make them available for system-wide use. **Integer**, for example, exports the operations + and = given by the *signatures* `+: (Integer,Integer)→Integer` and `=: (Integer,Integer)→Boolean`, respectively. Each of these operations takes two **Integer** arguments. The + operation also returns an **Integer** but = returns a **Boolean**: `true` or `false`. The operations exported by a domain usually manipulate objects of the domain—but not always.

The operations of a domain may actually take as arguments, and return as values, objects from any domain. For example, **Fraction (Integer)** exports the operations `/: (Integer,Integer)→Fraction (Integer)` and characteristic: `→NonNegativeInteger`.

Suppose all operations of a domain take as arguments and return as values, only objects from *other* domains. This kind of domain is what FriCAS calls a *package*.

A package does not designate a class of objects at all. Rather, a package is just a collection of operations. Actually the bulk of the FriCAS library of algorithms consists of packages. The facilities for factorization; integration; solution of linear, polynomial, and differential equations; computation of limits; and so on, are all defined in packages. Domains needed by algorithms can be passed to a package as arguments or used by name if they are not "variable." Packages are useful for defining operations that convert objects of one type to another, particularly when these types have different parameterizations. As an example, the package **PolynomialFunction2(R,S)** defines operations that convert polynomials over a domain `R` to polynomials over `S`. To convert an object from **Polynomial(Integer)** to **Polynomial(Float)**, FriCAS builds the package **PolynomialFunctions2(Integer,Float)** in order to create the required conversion function. (This happens "behind the scenes" for you: see Section 2.7 on page 84 for details on how to convert objects.)

FriCAS categories, domains and packages and all their contained functions are written in the FriCAS programming language and have been compiled into machine code. This is what comprises the FriCAS *library*. In the rest of this book we show you how to use these domains and their functions and how to

[4] This subtle but important feature distinguishes FriCAS from other abstract datatype designs.

[5] There is no inverse element `a` such that `1 + a = 0`, or, in the usual terms: `true or a = false`.

write your own functions.

2.2 Writing Types and Modes

We have already seen in the last section several examples of types. Most of these examples had either no arguments (for example, **Integer**) or one argument (for example, **Polynomial (Integer)**). In this section we give details about writing arbitrary types. We then define *modes* and discuss how to write them. We conclude the section with a discussion on constructor abbreviations.

When might you need to write a type or mode? You need to do so when you declare variables.

```
a : PositiveInteger
```

You need to do so when you declare functions (Section 2.3 on page 74),

```
f : Integer -> String
```

You need to do so when you convert an object from one type to another (Section 2.7 on page 84).

```
factor(2 :: Complex(Integer))
```

$$-i\,(1+i)^2 \tag{3}$$

Factored(Complex(Integer))

```
(2 = 3)$Integer
```

```
false                                                    (4)
```

Boolean

You need to do so when you give computation target type information (Section 2.9 on page 89).

```
(2 = 3)@Boolean
```

```
false                                                    (5)
```

Boolean

2.2.1 Types with No Arguments

A constructor with no arguments can be written either with or without trailing opening and closing parentheses ("`()`").

Boolean() is the same as **Boolean** **Integer()** is the same as **Integer**
String() is the same as **String** **Void()** is the same as **Void**

and so on. It is customary to omit the parentheses.

2.2.2 Types with One Argument

A constructor with one argument can frequently be written with no parentheses. Types nest from right to left so that **Complex Fraction Polynomial Integer** is the same as **Complex (Fraction (Polynomial (Integer)))**. You need to use parentheses to force the application of a constructor to the correct argument, but you need not use any more than is necessary to remove ambiguities.

Here are some guidelines for using parentheses (they are possibly slightly more restrictive than they need to be). If the argument is an expression like `2 + 3` then you must enclose the argument in parentheses.

```
e : PrimeField(2 + 3)
```

If the type is to be used with package calling then you must enclose the argument in parentheses.

```
content(2)$Polynomial(Integer)
```

$$2 \tag{2}$$

Integer

Alternatively, you can write the type without parentheses then enclose the whole type expression with parentheses.

```
content(2)$(Polynomial Complex Fraction Integer)
```

$$2 \tag{3}$$

Complex(Fraction(Integer))

If you supply computation target type information (Section 2.9 on page 89) then you should enclose the argument in parentheses.

```
(2/3)@Fraction(Polynomial(Integer))
```

$$\frac{2}{3} \tag{4}$$

Fraction(Polynomial(Integer))

If the type itself has parentheses around it and we are not in the case of the first example above, then the parentheses can usually be omitted.

```
(2/3)@Fraction(Polynomial Integer)
```

$$\frac{2}{3} \tag{5}$$

Fraction(Polynomial(Integer))

If the type is used in a declaration and the argument is a single-word type, integer or symbol, then the parentheses can usually be omitted.

```
(d,f,g) : Complex Polynomial Integer
```

2.2.3 Types with More Than One Argument

If a constructor has more than one argument, you must use parentheses. Some examples are

UnivariatePolynomial(x, Float)
MultivariatePolynomial([z,w,r], Complex Float)
SquareMatrix(3, Integer)
FactoredFunctions2(Integer,Fraction Integer)

2.2.4 Modes

A *mode* is a type that possibly is a question mark ("?") or contains one in an argument position. For example, the following are all modes.

?	**Polynomial ?**
Matrix Polynomial ?	**SquareMatrix(3,?)**
Integer	**OneDimensionalArray(Float)**

As is evident from these examples, a mode is a type with a part that is not specified (indicated by a question mark). Only one "?" is allowed per mode and it must appear in the most deeply nested argument that is a type. Thus **?(Integer)**, **Matrix(? (Polynomial))**, **SquareMatrix(?, Integer)** and **SquareMatrix(?, ?)** are all invalid. The question mark must take the place of a domain, not data (for example, the integer that is the dimension of a square matrix). This rules out, for example, the two **SquareMatrix** expressions.

Modes can be used for declarations (Section 2.3 on page 74) and conversions (Section 2.7 on page 84). However, you cannot use a mode for package calling or giving target type information.

2.2.5 Abbreviations

Every constructor has an abbreviation that you can freely substitute for the constructor name. In some cases, the abbreviation is nothing more than the capitalized version of the constructor name.

Aside from allowing types to be written more concisely, abbreviations are used by FriCAS to name various system files for constructors (such as library filenames, test input files and example files). Here are some common abbreviations.

COMPLEX abbreviates **Complex**	**DFLOAT** abbreviates **DoubleFloat**
EXPR abbreviates **Expression**	**FLOAT** abbreviates **Float**
FRAC abbreviates **Fraction**	**INT** abbreviates **Integer**
MATRIX abbreviates **Matrix**	**NNI** abbreviates **NonNegativeInteger**
PI abbreviates **PositiveInteger**	**POLY** abbreviates **Polynomial**
STRING abbreviates **String**	**UP** abbreviates **UnivariatePolynomial**

You can combine both full constructor names and abbreviations in a type expression. Here are some types using abbreviations.

POLY INT is the same as **Polynomial(INT)**
POLY(Integer) is the same as **Polynomial(Integer)**
POLY(Integer) is the same as **Polynomial(INT)**
FRAC(COMPLEX(INT)) is the same as **Fraction Complex Integer**
FRAC(COMPLEX(INT)) is the same as **FRAC(Complex Integer)**

There are several ways of finding the names of constructors and their abbreviations. For a specific constructor, use `)abbreviation query`. You can also use the `)what` system command to see the names and abbreviations of constructors. For more information about `)what`, see Section A.27 on page 761. `)abbreviation query` can be abbreviated (no pun intended) to `)abb q`.

```
)abb q Integer
```

```
   INT abbreviates domain Integer
```

The `)abbreviation query` command lists the constructor name if you give the abbreviation. Issue `)abb q` if you want to see the names and abbreviations of all FriCAS constructors.

```
)abb q DMP
```

```
   DMP abbreviates domain DistributedMultivariatePolynomial
```

Issue this to see all packages whose names contain the string "ode".

```
)what packages ode
```

```
------------------------------------ Packages -------------------------------------

Packages with names matching patterns:
     ode

 COMPCODE compCode                              EXPRODE  ExpressionSpaceODESolver
 FCPAK1   FortranCodePackage1                   FCTOOL   FortranCodeTools
 GRAY     GrayCode                              LODEEF   ElementaryFunctionLODESolver
 NODE1    NonLinearFirstOrderODESolver          ODECONST ConstantLODE
 ODEEF    ElementaryFunctionODESolver           ODEINT   ODEIntegration
 ODEPAL   PureAlgebraicLODE                     ODERAT   RationalLODE
 ODERED   ReduceLODE                            ODESYS   SystemODESolver
 ODETOOLS ODETools                              UTSODE   UnivariateTaylorSeriesODESolver
 UTSODETL UTSodetools
```

2.3 Declarations

A *declaration* is an expression used to restrict the type of values that can be assigned to variables. A colon (":") is always used after a variable or list of variables to be declared.

For a single variable, the syntax for declaration is

$$variableName\ :\ typeOrMode$$

For multiple variables, the syntax is

$$(variableName_1,\ variableName_2,\ \ldots variableName_N):\quad typeOrMode$$

You can always combine a declaration with an assignment. When you do, it is equivalent to first giving a declaration statement, then giving an assignment. For more information on assignment, see Section

1.3.4 on page 25 and Section 5.1 on page 123. To see how to declare your own functions, see Section 6.4 on page 153.

This declares one variable to have a type.

```
a : Integer
```

This declares several variables to have a type.

```
(b,c) : Integer
```

a, b and c can only hold integer values.

```
a := 45
```

$$45 \tag{3}$$

Integer

If a value cannot be converted to a declared type, an error message is displayed.

```
b := 4/5
```

```
   Cannot convert right-hand side of assignment
   4
   -
   5

      to an object of the type Integer of the left-hand side.
```

This declares a variable with a mode.

```
n : Complex ?
```

This declares several variables with a mode.

```
(p,q,r) : Matrix Polynomial ?
```

This complex object has integer real and imaginary parts.

```
n := -36 + 9 * %i
```

$$-36+9\,i \tag{6}$$

Complex(Integer)

This complex object has fractional symbolic real and imaginary parts.

```
n := complex(4/(x + y),y/x)
```

$$\frac{4}{y+x}+\frac{y}{x}\,i \tag{7}$$

Complex(Fraction(Polynomial(Integer)))

This matrix has entries that are polynomials with integer coefficients.

```
p := [[1,2],[3,4],[5,6]]
```

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \tag{8}$$

Matrix(Polynomial(Integer))

This matrix has a single entry that is a polynomial with rational number coefficients.

```
q := [[x - 2/3]]
```

$$\left[x - \tfrac{2}{3}\right] \tag{9}$$

Matrix(Polynomial(Fraction (Integer)))

This matrix has entries that are polynomials with complex integer coefficients.

```
r := [[1-%i*x,7*y+4*%i]]
```

$$\begin{bmatrix} -i\,x+1 & 7\,y+4\,i \end{bmatrix} \tag{10}$$

Matrix(Polynomial(Complex(Integer)))

Note the difference between this and the next example. This is a complex object with polynomial real and imaginary parts.

```
f : COMPLEX POLY ? := (x + y*%i)^2
```

$$-y^2+x^2+2\,x\,y\,i \tag{11}$$

Complex(Polynomial(Integer))

This is a polynomial with complex integer coefficients. The objects are convertible from one to the other. See Section 2.7 on page 84 for more information.

```
g : POLY COMPLEX ? := (x + y*%i)^2
```

$$-y^2+2\,i\,x\,y+x^2 \tag{12}$$

Polynomial(Complex(Integer))

2.4 Records

A **Record** is an object composed of one or more other objects, each of which is referenced with a *selector*. Components can all belong to the same type or each can have a different type.

The syntax for writing a **Record** type is

`Record(`$selector_1$`:`$type_1$`,` $selector_2$`:`$type_2$`, ..., `$selector_N$`:`$type_N$`)`

You must be careful if a selector has the same name as a variable in the workspace. If this occurs, precede the selector name by a single quote.

Record components are implicitly ordered. All the components of a record can be set at once by assigning the record a bracketed *tuple* of values of the proper length (for example, `r : Record(a : Integer, b: String):= [1, "two"]`). To access a component of a record `r`, write the name `r`, followed by a period, followed by a selector.

The object returned by this computation is a record with two components: a `quotient` part and a `remainder` part.

```
u := divide(5,2)
```

$$[quotient = 2,\ remainder = 1] \tag{1}$$

Record(quotient: Integer, remainder: Integer)

This is the quotient part.

```
u.quotient
```

$$2 \tag{2}$$

PositiveInteger

This is the remainder part.

```
u.remainder
```

$$1 \tag{3}$$

PositiveInteger

You can use selector expressions on the left-hand side of an assignment to change destructively the components of a record.

```
u.quotient := 8978
```

$$8978 \tag{4}$$

PositiveInteger

The selected component `quotient` has the value `8978`, which is what is returned by the assignment. Check that the value of `u` was modified.

```
u
```

$$[quotient = 8978,\ remainder = 1] \tag{5}$$

Record(quotient: Integer, remainder: Integer)

Selectors are evaluated. Thus you can use variables that evaluate to selectors instead of the selectors themselves.

```
s := 'quotient
```

$$quotient \tag{6}$$

Variable (quotient)

Be careful! A selector could have the same name as a variable in the workspace. If this occurs, precede the selector name by a single quote, as in `u.'quotient`.

```
divide(5,2).s
```

$$2 \tag{7}$$

PositiveInteger

Here we declare that the value of `bd` has two components: a string, to be accessed via `name`, and an integer, to be accessed via `birthdayMonth`.

```
bd : Record(name : String, birthdayMonth : Integer)
```

You must initially set the value of the entire **Record** at once.

```
bd := ["Judith", 3]
```

$$[name = \texttt{"Judith"},\ birthdayMonth = 3] \tag{9}$$

Record(name: String, birthdayMonth: Integer)

Once set, you can change any of the individual components.

```
bd.name := "Katie"
```

$$\texttt{"Katie"} \tag{10}$$

String

Records may be nested and the selector names can be shared at different levels.

```
r : Record(a : Record(b: Integer, c: Integer), b: Integer)
```

The record `r` has a `b` selector at two different levels. Here is an initial value for `r`.

```
r := [[1,2],3]
```

$$[a = [b = 1,\ c = 2],\ b = 3] \tag{12}$$

Record(a: Record(b: Integer , c: Integer), b: Integer)

This extracts the `b` component from the `a` component of `r`.

```
r.a.b
```

$$1 \tag{13}$$

PositiveInteger

This extracts the `b` component from `r`.

```
r.b
```

$$3 \tag{14}$$

PositiveInteger

You can also use spaces or parentheses to refer to **Record** components. This is the same as `r.a`.

```
r(a)
```

$$[b = 1,\ c = 2] \tag{15}$$

Record(b: Integer , c: Integer)

This is the same as `r.b`.

```
r b
```

$$3 \tag{16}$$

PositiveInteger

This is the same as `r.b := 10`.

```
r(b) := 10
```

$$10 \tag{17}$$

PositiveInteger

Look at `r` to make sure it was modified.

```
r
```

$$[a = [b = 1,\ c = 2]\ ,\ b = 10] \tag{18}$$

Record(a: Record(b: Integer , c: Integer), b: Integer)

2.5 Unions

Type **Union** is used for objects that can be of any of a specific finite set of types. Two versions of unions are available, one with selectors (like records) and one without.

2.5.1 Unions Without Selectors

The declaration `x : Union(Integer, String, Float)` states that `x` can have values that are integers, strings or "big" floats. If, for example, the **Union** object is an integer, the object is said to belong to the **Integer** *branch* of the **Union**.[6]

[6] Note that we are being a bit careless with the language here. Technically, the type of `x` is always **Union(Integer, String, Float)**. If it belongs to the **Integer** branch, `x` may be converted to an object of type **Integer**.

The syntax for writing a **Union** type without selectors is

$$\texttt{Union(}type_1\texttt{, }type_2\texttt{, ..., }type_N\texttt{)}$$

The types in a union without selectors must be distinct.

It is possible to create unions like **Union(Integer, PositiveInteger)** but they are difficult to work with because of the overlap in the branch types. See below for the rules FriCAS uses for converting something into a union object.

The `case` infix operator returns a **Boolean** and can be used to determine the branch in which an object lies.

This function displays a message stating in which branch of the **Union** the object (defined as `x` above) lies.

```
sayBranch(x : Union(Integer,String,Float)) : Void  ==
  output
    x case Integer => "Integer branch"
    x case String  => "String branch"
    "Float branch"
```

```
   Function declaration sayBranch : Union(Integer,String,Float) -> Void has been
      added to workspace.
```

This tries **sayBranch** with an integer.

```
sayBranch 1
```

```
   Compiling function sayBranch with type Union(Integer,String,Float) -> Void
   Integer branch
```

This tries **sayBranch** with a string.

```
sayBranch "hello"
```

```
   String branch
```

This tries **sayBranch** with a floating-point number.

```
sayBranch 2.718281828
```

```
   Float branch
```

There are two things of interest about this particular example to which we would like to draw your attention.

1. FriCAS normally converts a result to the target value before passing it to the function. If we left the declaration information out of this function definition then the `sayBranch` call would have been attempted with an **Integer** rather than a **Union**, and an error would have resulted.

2. The types in a **Union** are searched in the order given. So if the type were given as

   ```
   sayBranch(x: Union(String,Integer,Float,Any)): Void
   ```

   then the result would have been "String branch" because there is a conversion from **Integer** to **String**.

Sometimes **Union** types can have extremely long names. FriCAS therefore abbreviates the names of unions by printing the type of the branch first within the **Union** and then eliding the remaining types with an ellipsis ("...").

Here the **Integer** branch is displayed first. Use "::" to create a **Union** object from an object.

```
78 :: Union(Integer,String)
```

$$78 \tag{5}$$

Union(Integer, ...)

Here the **String** branch is displayed first.

```
s := "string" :: Union(Integer,String)
```

$$\text{"string"} \tag{6}$$

Union(String, ...)

Use `typeOf` to see the full and actual **Union** type.

```
typeOf s
```

$$\text{Union}(Integer, String) \tag{7}$$

Type

A common operation that returns a union is exquo which returns the "exact quotient" if the quotient is exact,...

```
three := exquo(6,2)
```

$$3 \tag{8}$$

Union(Integer, ...)

and `"failed"` if the quotient is not exact.

```
exquo(5,2)
```

$$\text{"failed"} \tag{9}$$

Union("failed", ...)

A union with a `"failed"` is frequently used to indicate the failure or lack of applicability of an object. As another example, assign an integer to a variable `r` declared to be a rational number.

```
r: FRAC INT := 3
```

$$3 \tag{10}$$

Fraction(Integer)

The operation retractIfCan tries to retract the fraction to the underlying domain **Integer**. It produces a union object. Here it succeeds.

```
retractIfCan(r)
```

$$3 \tag{11}$$

Union(Integer , ...)

Assign it a rational number.

```
r := 3/2
```

$$\frac{3}{2} \tag{12}$$

Fraction (Integer)

Here the retraction fails.

```
retractIfCan(r)
```

$$\texttt{"failed"} \tag{13}$$

Union(" failed ", ...)

2.5.2 Unions With Selectors

Like records (Section 2.4 on page 76), you can write **Union** types with selectors.

> The syntax for writing a **Union** type with selectors is
>
> $$\texttt{Union(}selector_1\texttt{:}type_1\texttt{,}\ selector_2\texttt{:}type_2\texttt{,}\ \ldots\texttt{,}\ selector_N\texttt{:}type_N\texttt{)}$$
>
> You must be careful if a selector has the same name as a variable in the workspace. If this occurs, precede the selector name by a single quote. It is an error to use a selector that does not correspond to the branch of the **Union** in which the element actually lies.

Be sure to understand the difference between records and unions with selectors. Records can have more than one component and the selectors are used to refer to the components. Unions always have one component but the type of that one component can vary. An object of type **Record(a: Integer, b: Float, c: String)** contains an integer *and* a float *and* a string. An object of type **Union(a: Integer, b: Float, c: String)** contains an integer *or* a float *or* a string.

Here is a version of the **sayBranch** function (cf. Section 2.5.1 on page 79) that works with a union with selectors. It displays a message stating in which branch of the **Union** the object lies.

```
sayBranch(x:Union(i:Integer,s:String,f:Float)):Void==
  output
    x case i => "Integer branch"
    x case s  => "String branch"
    "Float branch"
```

Note that `case` uses the selector name as its right-hand argument.

Declare variable `u` to have a union type with selectors.

```
u : Union(i : Integer, s : String)
```

Give an initial value to u.

```
u := "good morning"
```

"good morning" (2)

Union(s: String, ...)

Use case to determine in which branch of a **Union** an object lies.

```
u case i
```

false (3)

Boolean

```
u case s
```

true (4)

Boolean

To access the element in a particular branch, use the selector.

```
u.s
```

"good morning" (5)

String

2.6 The "Any" Domain

With the exception of objects of type **Record**, all FriCAS data structures are homogeneous, that is, they hold objects all of the same type. If you need to get around this, you can use type **Any**. Using **Any**, for example, you can create lists whose elements are integers, rational numbers, strings, and even other lists.

Declare u to have type **Any**.

```
u: Any
```

Assign a list of mixed type values to u

```
u := [1, 7.2, 3/2, x^2, "wally"]
```

$$\left[1,\ 7.2,\ \frac{3}{2},\ x^2,\ \texttt{"wally"}\right] \quad (2)$$

List (Any)

When we ask for the elements, FriCAS displays these types.

```
u.1
```

$$1 \tag{3}$$

PositiveInteger

Actually, these objects belong to **Any** but FriCAS automatically converts them to their natural types for you.

```
u.3
```

$$\frac{3}{2} \tag{4}$$

Fraction (Integer)

Since type **Any** can be anything, it can only belong to type **Type**. Therefore it cannot be used in algebraic domains.

```
v : Matrix(Any)
```

```
   Matrix(Any) is not a valid type.
```

Perhaps you are wondering how FriCAS internally represents objects of type **Any**. An object of type **Any** consists not only a data part representing its normal value, but also a type part (a *badge*) giving its type. For example, the value `1` of type **PositiveInteger** as an object of type **Any** internally looks like `[1,PositiveInteger()]`.

2.7 Conversion

Conversion is the process of changing an object of one type into an object of another type. The syntax for conversion is:

object `::` *newType*

By default, `3` has the type **PositiveInteger**.

```
3
```

$$3 \tag{1}$$

PositiveInteger

We can change this into an object of type **Fraction Integer** by using "`::`".

```
3 :: Fraction Integer
```

$$3 \tag{2}$$

Fraction (Integer)

A *coercion* is a special kind of conversion that FriCAS is allowed to do automatically when you enter an expression. Coercions are usually somewhat safer than more general conversions. The FriCAS library

contains operations called coerce and convert. Only the coerce operations can be used by the interpreter to change an object into an object of another type unless you explicitly use a "::".

By now you will be quite familiar with what types and modes look like. It is useful to think of a type or mode as a pattern for what you want the result to be. Let's start with a square matrix of polynomials with complex rational number coefficients.

```
m : SquareMatrix(2,POLY COMPLEX FRAC INT)
```

```
m := matrix [[x-3/4*%i,z*y^2+1/2],[3/7*%i*y^4 - x,12-%i*9/5]]
```

$$\begin{bmatrix} x-\frac{3}{4}\,i & y^2\,z+\frac{1}{2} \\ \frac{3}{7}\,i\,y^4-x & 12-\frac{9}{5}\,i \end{bmatrix} \tag{4}$$

SquareMatrix(2, Polynomial(Complex(Fraction(Integer))))

We first want to interchange the **Complex** and **Fraction** layers. We do the conversion by doing the interchange in the type expression.

```
m1 := m :: SquareMatrix(2,POLY FRAC COMPLEX INT)
```

$$\begin{bmatrix} x-\frac{3\,i}{4} & y^2\,z+\frac{1}{2} \\ \frac{3\,i}{7}\,y^4-x & \frac{60-9\,i}{5} \end{bmatrix} \tag{5}$$

SquareMatrix(2, Polynomial(Fraction(Complex(Integer))))

Interchange the **Polynomial** and the **Fraction** levels.

```
m2 := m1 :: SquareMatrix(2,FRAC POLY COMPLEX INT)
```

$$\begin{bmatrix} \frac{4\,x-3\,i}{4} & \frac{2\,y^2\,z+1}{2} \\ \frac{3\,i\,y^4-7\,x}{7} & \frac{60-9\,i}{5} \end{bmatrix} \tag{6}$$

SquareMatrix(2, Fraction(Polynomial(Complex(Integer))))

Interchange the **Polynomial** and the **Complex** levels.

```
m3 := m2 :: SquareMatrix(2,FRAC COMPLEX POLY INT)
```

$$\begin{bmatrix} \frac{4\,x-3\,i}{4} & \frac{2\,y^2\,z+1}{2} \\ \frac{-7\,x+3\,y^4\,i}{7} & \frac{60-9\,i}{5} \end{bmatrix} \tag{7}$$

SquareMatrix(2, Fraction(Complex(Polynomial(Integer))))

All the entries have changed types, although in comparing the last two results only the entry in the lower left corner looks different. We did all the intermediate steps to show you what FriCAS can do.

In fact, we could have combined all these into one conversion.

```
m :: SquareMatrix(2,FRAC COMPLEX POLY INT)
```

$$\begin{bmatrix} \frac{4\,x-3\,i}{4} & \frac{2\,y^2\,z+1}{2} \\ \frac{-7\,x+3\,y^4\,i}{7} & \frac{60-9\,i}{5} \end{bmatrix} \tag{8}$$

SquareMatrix(2, Fraction(Complex(Polynomial(Integer))))

There are times when FriCAS is not able to do the conversion in one step. You may need to break up the transformation into several conversions in order to get an object of the desired type.

We cannot move either **Fraction** or **Complex** above (or to the left of, depending on how you look at it) **SquareMatrix** because each of these levels requires that its argument type have commutative multiplication, whereas **SquareMatrix** does not.[7] The **Integer** level did not move anywhere because it does not allow any arguments. We also did not move the **SquareMatrix** part anywhere, but we could have. Recall that `m` looks like this.

```
m
```

$$\begin{bmatrix} x - \frac{3}{4}\,i & y^2\,z + \frac{1}{2} \\ \frac{3}{7}\,i\,y^4 - x & 12 - \frac{9}{5}\,i \end{bmatrix} \tag{9}$$

SquareMatrix(2, Polynomial(Complex(Fraction(Integer))))

If we want a polynomial with matrix coefficients rather than a matrix with polynomial entries, we can just do the conversion.

```
m :: POLY SquareMatrix(2,COMPLEX FRAC INT)
```

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} y^2\,z + \begin{bmatrix} 0 & 0 \\ \frac{3}{7}\,i & 0 \end{bmatrix} y^4 + \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} -\frac{3}{4}\,i & \frac{1}{2} \\ 0 & 12 - \frac{9}{5}\,i \end{bmatrix} \tag{10}$$

Polynomial(SquareMatrix(2, Complex(Fraction(Integer))))

We have not yet used modes for any conversions. Modes are a great shorthand for indicating the type of the object you want. Instead of using the long type expression in the last example, we could have simply said this.

```
m :: POLY ?
```

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} y^2\,z + \begin{bmatrix} 0 & 0 \\ \frac{3}{7}\,i & 0 \end{bmatrix} y^4 + \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} -\frac{3}{4}\,i & \frac{1}{2} \\ 0 & 12 - \frac{9}{5}\,i \end{bmatrix} \tag{11}$$

Polynomial(SquareMatrix(2, Complex(Fraction(Integer))))

We can also indicate more structure if we want the entries of the matrices to be fractions.

```
m :: POLY SquareMatrix(2,FRAC ?)
```

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} y^2\,z + \begin{bmatrix} 0 & 0 \\ \frac{3\,i}{7} & 0 \end{bmatrix} y^4 + \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} -\frac{3\,i}{4} & \frac{1}{2} \\ 0 & \frac{60-9\,i}{5} \end{bmatrix} \tag{12}$$

Polynomial(SquareMatrix(2, Fraction(Complex(Integer))))

2.8 Subdomains Again

A *subdomain* `S` of a domain `D` is a domain consisting of

[7] **Fraction** requires that its argument belong to the category **IntegralDomain** and **Complex** requires that its argument belong to **CommutativeRing**. See Section 2.1 on page 65 for a brief discussion of categories.

1. those elements of `D` that satisfy some *predicate* (that is, a test that returns `true` or `false`) and
2. a subset of the operations of `D`.

Every domain is a subdomain of itself, trivially satisfying the membership test: `true`.

Currently, there are only two system-defined subdomains in FriCAS that receive substantial use. **PositiveInteger** and **NonNegativeInteger** are subdomains of **Integer**. An element `x` of **NonNegativeInteger** is an integer that is greater than or equal to zero, that is, satisfies `x >= 0`. An element `x` of **PositiveInteger** is a nonnegative integer that is, in fact, greater than zero, that is, satisfies `x > 0`. Not all operations from **Integer** are available for these subdomains. For example, negation and subtraction are not provided since the subdomains are not closed under those operations. When you use an integer in an expression, FriCAS assigns to it the type that is the most specific subdomain whose predicate is satisfied. This is a positive integer.

```
5
```

$$5 \tag{1}$$

PositiveInteger

This is a nonnegative integer.

```
0
```

$$0 \tag{2}$$

NonNegativeInteger

This is neither of the above.

```
-5
```

$$-5 \tag{3}$$

Integer

Furthermore, unless you are assigning an integer to a declared variable or using a conversion, any integer result has as type the most specific subdomain.

```
(-2) - (-3)
```

$$1 \tag{4}$$

PositiveInteger

```
0 :: Integer
```

$$0 \tag{5}$$

Integer

```
x : NonNegativeInteger := 5
```

$$5 \tag{6}$$

NonNegativeInteger

When necessary, FriCAS converts an integer object into one belonging to a less specific subdomain. For example, in `3-2`, the arguments to `-` are both elements of **PositiveInteger**, but this type does not provide a subtraction operation. Neither does **NonNegativeInteger**, so `3` and `2` are viewed as elements of **Integer**, where their difference can be calculated. The result is `1`, which FriCAS then automatically assigns the type **PositiveInteger**.

Certain operations are very sensitive to the subdomains to which their arguments belong. This is an element of **PositiveInteger**.

```
2 ^ 2
```

$$4 \tag{7}$$

PositiveInteger

This is an element of **Fraction Integer**.

```
2 ^ (-2)
```

$$\frac{1}{4} \tag{8}$$

Fraction (Integer)

It makes sense then that this is a list of elements of **PositiveInteger**.

```
[10^i for i in 2..5]
```

$$[100,\ 1000,\ 10000,\ 100000] \tag{9}$$

List (PositiveInteger)

What should the type of `[10^(i-1)for i in 2..5]` be? On one hand, `i-1` is always an integer greater than zero as `i` ranges from `2` to `5` and so `10^i` is also always a positive integer. On the other, `i-1` is a very simple function of `i`. FriCAS does not try to analyze every such function over the index's range of values to determine whether it is always positive or nowhere negative. For an arbitrary FriCAS function, this analysis is not possible.

So, to be consistent, no such analysis is done and we get this.

```
[10^(i-1) for i in 2..5]
```

$$[10,\ 100,\ 1000,\ 10000] \tag{10}$$

List (Fraction (Integer))

To get a list of elements of **PositiveInteger** instead, you have two choices. You can use a conversion.

```
[10^((i-1) :: PI) for i in 2..5]
```

```
   Compiling function G297 with type Integer -> Boolean
   Compiling function G299 with type NonNegativeInteger -> Boolean
```

$$[10,\ 100,\ 1000,\ 10000] \tag{11}$$

List (PositiveInteger)

Or you can use `pretend`.

```
[10^((i-1) pretend PI) for i in 2..5]
```

$$[10,\ 100,\ 1000,\ 10000] \tag{12}$$

List (PositiveInteger)

The operation `pretend` is used to defeat the FriCAS type system. The expression `object pretend D` means "make a new object (without copying) of type `D` from `object`." If `object` were an integer and you told FriCAS to pretend it was a list, you would probably see a message about a fatal error being caught and memory possibly being damaged. Lists do not have the same internal representation as integers!

You use `pretend` at your peril.

Use `pretend` with great care! FriCAS trusts you that the value is of the specified type.

```
(2/3) pretend Complex Integer
```

$$2 + 3\,i \tag{13}$$

Complex(Integer)

2.9 Package Calling and Target Types

FriCAS works hard to figure out what you mean by an expression without your having to qualify it with type information. Nevertheless, there are times when you need to help it along by providing hints (or even orders!) to get FriCAS to do what you want.

We saw in Section 2.3 on page 74 that declarations using types and modes control the type of the results produced. For example, we can either produce a complex object with polynomial real and imaginary parts or a polynomial with complex integer coefficients, depending on the declaration.

Package calling is how you tell FriCAS to use a particular function from a particular part of the library.

Use the / from **Fraction Integer** to create a fraction of two integers.

```
2/3
```

$$\frac{2}{3} \tag{1}$$

Fraction (Integer)

If we wanted a floating point number, we can say "use the / in **Float**."

```
(2/3)$Float
```

$$0.66666666666666666667 \tag{2}$$

Float

Perhaps we actually wanted a fraction of complex integers.

```
(2/3)$Fraction(Complex Integer)
```

$$\frac{2}{3} \tag{3}$$

Fraction(Complex(Integer))

In each case, FriCAS used the indicated operations, sometimes first needing to convert the two integers into objects of an appropriate type. In these examples, / is written as an infix operator.

> To use package calling with an infix operator, use the following syntax:
>
> (arg_1 *op* arg_1)$*type*

We used, for example, `(2/3)$Float`. The expression `2 + 3 + 4` is equivalent to `(2+3) + 4`. Therefore in the expression `(2 + 3 + 4)$Float` the second `+` comes from the **Float** domain. Can you guess whether the first `+` comes from **Integer** or **Float**?[8]

> For an operator written before its arguments, you must use parentheses around the arguments (even if there is only one), and follow the closing parenthesis by a "$" and then the type.
>
> *fun* (arg_1, arg_1, ..., arg_N)$*type*

For example, to call the "minimum" function from **DoubleFloat** on two integers, you could write `min(4,89)$DoubleFloat`. Another use of package calling is to tell FriCAS to use a library function rather than a function you defined. We discuss this in Section 6.9 on page 158.

Sometimes rather than specifying where an operation comes from, you just want to say what type the result should be. We say that you provide a *target type* for the expression. Instead of using a "$", use a "@" to specify the requested target type. Otherwise, the syntax is the same. Note that giving a target type is not the same as explicitly doing a conversion. The first says "try to pick operations so that the result has such-and-such a type." The second says "compute the result and then convert to an object of such-and-such a type."

Sometimes it makes sense, as in this expression, to say "choose the operations in this expression so that the final result is a **Float**."

```
(2/3)@Float
```

$$0.66666666666666666667 \tag{4}$$

[8] **Float**, because the package call causes FriCAS to convert `(2 + 3)` and `4` to type **Float**. Before the sum is converted, it is given a target type (see below) of **Float** by FriCAS and then evaluated. The target type causes the `+` from **Float** to be used.

Float

Here we used "@" to say that the target type of the left-hand side was **Float**. In this simple case, there was no real difference between using "$" and "@". You can see the difference if you try the following. This says to try to choose + so that the result is a string. FriCAS cannot do this.

```
(2 + 3)@String
```

```
   An expression involving @ String actually evaluated to one of type
      PositiveInteger . Perhaps you should use :: String .
```

This says to get the + from **String** and apply it to the two integers. FriCAS also cannot do this because there is no + exported by **String**.

```
(2 + 3)$String
```

```
   The function + is not implemented in String
```

(By the way, the operation concat or juxtaposition is used to concatenate two strings.)

When we have more than one operation in an expression, the difference is even more evident. The following two expressions show that FriCAS uses the target type to create different objects. The +, * and ^ operations are all chosen so that an object of the correct final type is created.

This says that the operations should be chosen so that the result is a **Complex** object.

```
((x + y * %i)^2)@(Complex Polynomial Integer)
```

$$-y^2+x^2+2\,x\,y\,i \tag{5}$$

Complex(Polynomial(Integer))

This says that the operations should be chosen so that the result is a **Polynomial** object.

```
((x + y * %i)^2)@(Polynomial Complex Integer)
```

$$-y^2+2\,i\,x\,y+x^2 \tag{6}$$

Polynomial(Complex(Integer))

What do you think might happen if we left off all target type and package call information in this last example?

```
(x + y * %i)^2
```

$$-y^2+2\,i\,x\,y+x^2 \tag{7}$$

Polynomial(Complex(Integer))

We can convert it to **Complex** as an afterthought. But this is more work than just saying making what we want in the first place.

```
% :: Complex ?
```

$$-y^2+x^2+2\,x\,y\,i \tag{8}$$

Complex(Polynomial(Integer))

Finally, another use of package calling is to qualify fully an operation that is passed as an argument to a function.

Start with a small matrix of integers.

```
h := matrix [[8,6],[-4,9]]
```

$$\begin{bmatrix} 8 & 6 \\ -4 & 9 \end{bmatrix} \tag{9}$$

Matrix(Integer)

We want to produce a new matrix that has for entries the multiplicative inverses of the entries of `h`. One way to do this is by calling map with the inv function from **Fraction (Integer)**.

```
map(inv$Fraction(Integer),h)
```

$$\begin{bmatrix} \frac{1}{8} & \frac{1}{6} \\ -\frac{1}{4} & \frac{1}{9} \end{bmatrix} \tag{10}$$

Matrix(Fraction (Integer))

We could have been a bit less verbose and used abbreviations.

```
map(inv$FRAC(INT),h)
```

$$\begin{bmatrix} \frac{1}{8} & \frac{1}{6} \\ -\frac{1}{4} & \frac{1}{9} \end{bmatrix} \tag{11}$$

Matrix(Fraction (Integer))

As it turns out, FriCAS is smart enough to know what we mean anyway. We can just say this.

```
map(inv,h)
```

$$\begin{bmatrix} \frac{1}{8} & \frac{1}{6} \\ -\frac{1}{4} & \frac{1}{9} \end{bmatrix} \tag{12}$$

Matrix(Fraction (Integer))

2.10 Resolving Types

In this section we briefly describe an internal process by which FriCAS determines a type to which two objects of possibly different types can be converted. We do this to give you further insight into how FriCAS takes your input, analyzes it, and produces a result.

What happens when you enter `x + 1` to FriCAS? Let's look at what you get from the two terms of this expression.

This is a symbolic object whose type indicates the name.

```
x
```

$$x \tag{1}$$

Variable (x)

This is a positive integer.

```
1
```

$$1 \tag{2}$$

PositiveInteger

There are no operations in **PositiveInteger** that add positive integers to objects of type **Variable(x)** nor are there any in **Variable(x)**. Before it can add the two parts, FriCAS must come up with a common type to which both `x` and `1` can be converted. We say that FriCAS must *resolve* the two types into a common type. In this example, the common type is **Polynomial(Integer)**.

Once this is determined, both parts are converted into polynomials, and the addition operation from **Polynomial(Integer)** is used to get the answer.

```
x + 1
```

$$x + 1 \tag{3}$$

Polynomial(Integer)

FriCAS can always resolve two types: if nothing resembling the original types can be found, then **Any** is used. This is fine and useful in some cases.

```
["string",3.14159]
```

$$\texttt{["string"}, 3.14159\texttt{]} \tag{4}$$

List (Any)

In other cases objects of type **Any** can't be used by the operations you specified.

```
"string" + 3.14159
```

```
   There are 13 exposed and 13 unexposed library operations named + having 2
      argument(s) but none was determined to be applicable. Use HyperDoc Browse,
      or issue
                                    )display op +
      to learn more about the available operations. Perhaps package-calling the
      operation or using coercions on the arguments will allow you to apply the
      operation.
   Cannot find a definition or applicable library operation named + with argument
      type(s)
                                          String
                                          Float

      Perhaps you should use "@" to indicate the required return type, or "$" to
      specify which version of the function you need.
```

Although this example was contrived, your expressions may need to be qualified slightly to help FriCAS resolve the types involved. You may need to declare a few variables, do some package calling, provide some target type information or do some explicit conversions.

We suggest that you just enter the expression you want evaluated and see what FriCAS does. We think you will be impressed with its ability to "do what I mean." If FriCAS is still being obtuse, give it some hints. As you work with FriCAS, you will learn where it needs a little help to analyze quickly and perform your computations.

2.11 Exposing Domains and Packages

In this section we discuss how FriCAS makes some operations available to you while hiding others that are meant to be used by developers or only in rare cases. If you are a new user of FriCAS, it is likely that everything you need is available by default and you may want to skip over this section on first reading.

Every domain and package in the FriCAS library is either *exposed* (meaning that you can use its operations without doing anything special) or it is *hidden* (meaning you have to either package call (see Section 2.9 on page 89) the operations it contains or explicitly expose it to use the operations). The initial exposure status for a constructor is set in the file **exposed.lsp** (see the *Installer's Note* for FriCAS if you need to know the location of this file). Constructors are collected together in *exposure groups*. Categories are all in the exposure group "categories" and the bulk of the basic set of packages and domains that are exposed are in the exposure group "basic." Here is an abbreviated sample of the file (without the Lisp parentheses):

```
basic
        AlgebraicNumber                          AN
        AlgebraGivenByStructuralConstants        ALGSC
        Any                                      ANY
        AnyFunctions1                            ANY1
        BinaryExpansion                          BINARY
        Boolean                                  BOOLEAN
        CardinalNumber                           CARD
        CartesianTensor                          CARTEN
        Character                                CHAR
        CharacterClass                           CCLASS
        CliffordAlgebra                          CLIF
        Color                                    COLOR
        Complex                                  COMPLEX
        ContinuedFraction                        CONTFRAC
        DecimalExpansion                         DECIMAL
        ...

categories
        AbelianGroup                             ABELGRP
        AbelianMonoid                            ABELMON
        AbelianMonoidRing                        AMR
        AbelianSemiGroup                         ABELSG
        Aggregate                                AGG
        Algebra                                  ALGEBRA
        AlgebraicallyClosedField                 ACF
        AlgebraicallyClosedFunctionSpace         ACFS
```

```
ArcHyperbolicFunctionCategory              AHYP
...
```

For each constructor in a group, the full name and the abbreviation is given. There are other groups in **exposed.lsp** but initially only the constructors in exposure groups "basic" and "categories" are exposed.

As an interactive user of FriCAS, you do not need to modify this file. Instead, use `)set expose` to expose, hide, or query the exposure status of an individual constructor or exposure group. The reason for having exposure groups is to be able to expose or hide multiple constructors with a single command. For example, you might group together into exposure group "quantum" a number of domains and packages useful for quantum mechanical computations. These probably should not be available to every user, but you want an easy way to make the whole collection visible to FriCAS when it is looking for operations to apply.

If you wanted to hide all the basic constructors available by default, you would issue `)set expose drop group basic`. We do not recommend that you do this. If, however, you discover that you have hidden all the basic constructors, you should issue `)set expose add group basic` to restore your default environment.

It is more likely that you would want to expose or hide individual constructors. In Section 6.19 on page 184 we use several operations from **OutputForm**, a domain usually hidden. To avoid package calling every operation from **OutputForm**, we expose the domain and let FriCAS conclude that those operations should be used. Use `)set expose add constructor` and `)set expose drop constructor` to expose and hide a constructor, respectively. You should use the constructor name, not the abbreviation. The `)set expose` command guides you through these options.

If you expose a previously hidden constructor, FriCAS exhibits new behavior (that was your intention) though you might not expect the results that you get. **OutputForm** is, in fact, one of the worst offenders in this regard. This domain is meant to be used by other domains for creating a structure that FriCAS knows how to display. It has functions like + that form output representations rather than do mathematical calculations. Because of the order in which FriCAS looks at constructors when it is deciding what operation to apply, **OutputForm** might be used instead of what you expect. This is a polynomial.

```
x + x
```

$$2\,x \tag{1}$$

Polynomial(Integer)

Expose **OutputForm**.

```
)set expose add constructor OutputForm
```

```
   OutputForm is now explicitly exposed in frame initial
```

This is what we get when **OutputForm** is automatically available.

```
x + x
```

$$x + x \tag{2}$$

OutputForm

Hide **OutputForm** so we don't run into problems with any later examples!

```
)set expose drop constructor OutputForm
```

```
   OutputForm is now explicitly hidden in frame initial
```

Finally, exposure is done on a frame-by-frame basis. A *frame* (see Section A.11 on page 745) is one of possibly several logical FriCAS workspaces within a physical one, each having its own environment (for example, variables and function definitions). If you have several FriCAS workspace windows on your screen, they are all different frames, automatically created for you by HyperDoc. Frames can be manually created, made active and destroyed by the `)frame` system command. They do not share exposure information, so you need to use `)set expose` in each one to add or drop constructors from view.

2.12 Commands for Snooping

To conclude this chapter, we introduce you to some system commands that you can use for getting more information about domains, packages, categories, and operations. The most powerful FriCAS facility for getting information about constructors and operations is the Browse component of HyperDoc. This is discussed in Chapter 14.

Use the `)what` system command to see lists of system objects whose names contain a particular substring (uppercase or lowercase is not significant).

Issue this to see a list of all operations with "`complex`" in their names.

```
)what operation complex
```

```
Operations whose names satisfy the above pattern(s):

chainComplex                      coChainComplex
complex                           complex?
complexEigenvalues                complexEigenvectors
complexElementary                 complexExpand
complexForm                       complexIntegrate
complexLimit                      complexNormalize
complexNumeric                    complexNumericIfCan
complexRoots                      complexSolve
complexZeros                      complex_curve
complex_roots                     createLowComplexityNormalBasis
createLowComplexityTable          cubicalComplex
deltaComplex                      doubleComplex?
drawComplex                       drawComplexVectorField
fortranComplex                    fortranDoubleComplex
simplicialComplex                 simplicialComplexIfCan
testComplexEquals                 testComplexEqualsAux
xftestComplexEquals               xftestComplexEqualsAux

      To get more information about an operation such as complexForm , issue the
      command )display op complexForm
```

If you want to see all domains with "`matrix`" in their names, issue this.

```
)what domain matrix
```

```
------------------------------------ Domains ------------------------------------
```

```
Domains with names matching patterns:
     matrix

 CDFMAT    ComplexDoubleFloatMatrix           DFMAT     DoubleFloatMatrix
 DHMATRIX  DenavitHartenbergMatrix            DPMM      DirectProductMatrixModule
 I16MAT    I16Matrix                          I32MAT    I32Matrix
 I8MAT     I8Matrix                           IMATRIX   IndexedMatrix
 LINPEN    LinearMultivariateMatrixPencil     LSQM      LieSquareMatrix
 M3D       ThreeDimensionalMatrix             MATCAT-   MatrixCategory&
 MATRIX    Matrix                             RMATCAT-  RectangularMatrixCategory&
 RMATRIX   RectangularMatrix                  SEM       SparseEchelonMatrix
 SMATCAT-  SquareMatrixCategory&              SQMATRIX  SquareMatrix
 U16MAT    U16Matrix                          U32MAT    U32Matrix
 U8MAT     U8Matrix
```

Similarly, if you wish to see all packages whose names contain "gauss", enter this.

```
)what package gauss
```

```
------------------------------------ Packages ------------------------------------

Packages with names matching patterns:
     gauss

 FFFG      FractionFreeFastGaussian           FFFGF     FractionFreeFastGaussianFractions
 GAUSSFAC  GaussianFactorizationPackage       UGAUSS    UnitGaussianElimination
```

This command shows all the operations that **Any** provides. Wherever "%" appears, it means "**Any**".

```
)show Any
```

```
 Any is a domain constructor.
 Abbreviation for Any is ANY
 This constructor is exposed in this frame.
 10 names for 10 operations in this domain.
------------------------------------ Operations ------------------------------------

 ?=? : (%, %) -> Boolean                    any : (SExpression, None) -> %
 coerce : % -> OutputForm                   dom : % -> SExpression
 domainOf : % -> OutputForm                 latex : % -> String
 obj : % -> None                            objectOf : % -> OutputForm
 showTypeInOutput : Boolean -> String       ?~=? : (%, %) -> Boolean
```

This displays all operations with the name complex.

```
)display operation complex
```

```
There is one exposed function called complex :
   [1] (D1,D1) -> D from D if D has COMPCAT(D1) and D1 has COMRING
```

Let's analyze this output. First we find out what some of the abbreviations mean.

```
)abbreviation query COMPCAT
```

```
   COMPCAT abbreviates category ComplexCategory
```

```
)abbreviation query COMRING
```

```
   COMRING abbreviates category CommutativeRing
```

So if D1 is a commutative ring (such as the integers or floats) and D belongs to **ComplexCategory D1**, then there is an operation called complex that takes two elements of D1 and creates an element of D. The primary example of a constructor implementing domains belonging to **ComplexCategory** is **Complex**. See 'Complex' on page 372 for more information on that and see Section 6.4 on page 153 for more information on function types.

Chapter 3

Using HyperDoc

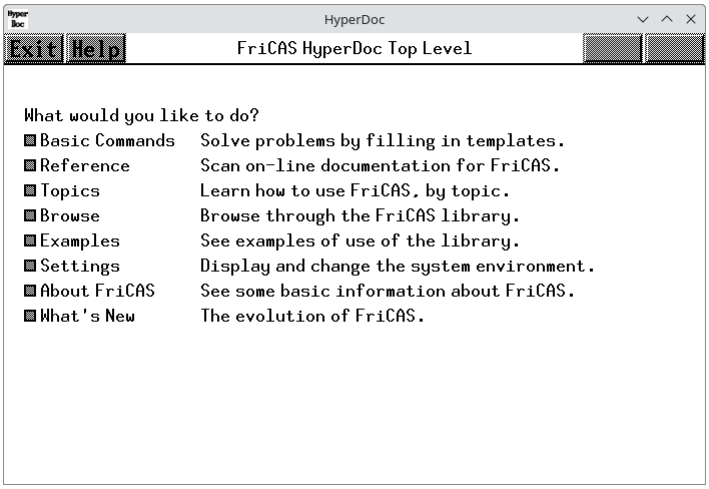


Figure 3.1: The HyperDoc root window page.

HyperDoc is the gateway to FriCAS. It's both an on-line tutorial and an on-line reference manual. It also enables you to use FriCAS simply by using the mouse and filling in templates. HyperDoc is available to you if you are running FriCAS under the X Window System.

Pages usually have active areas, marked in **this font** (bold face). As you move the mouse pointer to an active area, the pointer changes from a filled dot to an open circle. The active areas are usually linked to other pages. When you click on an active area, you move to the linked page.

3.1 Headings

Most pages have a standard set of buttons at the top of the page. These are what they mean:

Help Click on this to get help. The button only appears if there is specific help for the page you are viewing. You can get *general* help for HyperDoc by clicking the help button on the home page.

Click here to go back one page. By clicking on this button repeatedly, you can go back several pages and then take off in a new direction.

Home Go back to the home page, that is, the page on which you started. Use HyperDoc to explore, to make forays into new topics. Don't worry about how to get back. HyperDoc remembers where you came from. Just click on this button to return.

Exit From the root window (the one that is displayed when you start the system) this button leaves the HyperDoc program, and it must be restarted if you want to use it again. From any other HyperDoc window, it just makes that one window go away. You *must* use this button to get rid of a window. If you use the window manager "Close" button, then all of HyperDoc goes away.

The buttons are not displayed if they are not applicable to the page you are viewing. For example, there is no Home button on the top-level menu.

3.2 Key Definitions

The following keyboard definitions are in effect throughout HyperDoc. See Section 3.3 on page 100 and Section 3.4 on page 101 for some contextual key definitions.

F1 Display the main help page.

F3 Same as Exit, makes the window go away if you are not at the top-level window or quits the HyperDoc facility if you are at the top-level.

F5 Rereads the HyperDoc database, if necessary (for system developers).

F9 Displays this information about key definitions.

F12 Same as **F3**.

Up Arrow Scroll up one line.

Down Arrow Scroll down one line.

Page Up Scroll up one page.

Page Down Scroll down one page.

3.3 Scroll Bars

Whenever there is too much text to fit on a page, a *scroll bar* automatically appears along the right side.

With a scroll bar, your page becomes an aperture, that is, a window into a larger amount of text than can be displayed at one time. The scroll bar lets you move up and down in the text to see different parts. It also shows where the aperture is relative to the whole text. The aperture is indicated by a strip on the scroll bar.

Move the cursor with the mouse to the "down-arrow" at the bottom of the scroll bar and click. See that the aperture moves down one line. Do it several times. Each time you click, the aperture moves down one line. Move the mouse to the "up-arrow" at the top of the scroll bar and click. The aperture moves up one line each time you click.

Next move the mouse to any position along the middle of the scroll bar and click. HyperDoc attempts to move the top of the aperture to this point in the text.

You cannot make the aperture go off the bottom edge. When the aperture is about half the size of text, the lowest you can move the aperture is halfway down.

To move up or down one screen at a time, use the **PageUp** and **PageDown** keys on your keyboard. They move the visible part of the region up and down one page each time you press them.

If the HyperDoc page does not contain an input area (see Section 3.4 on page 101), you can also use the **Home** and ↑ and ↓ arrow keys to navigate. When you press the **Home** key, the screen is positioned at the very top of the page. Use the ↑ and ↓ arrow keys to move the screen up and down one line at a time, respectively.

3.4 Input Areas

Input areas are boxes where you can put data.

To enter characters, first move your mouse cursor to somewhere within the HyperDoc page. Characters that you type are inserted in front of the underscore. This means that when you type characters at your keyboard, they go into this first input area.

The input area grows to accommodate as many characters as you type. Use the **Backspace** key to erase characters to the left. To modify what you type, use the right-arrow → and left-arrow keys ← and the keys **Insert**, **Delete**, **Home** and **End**. These keys are found immediately on the right of the standard IBM keyboard.

If you press the **Home** key, the cursor moves to the beginning of the line and if you press the **End** key, the cursor moves to the end of the line. Pressing **Ctrl**–**End** deletes all the text from the cursor to the end of the line.

A page may have more than one input area. Only one input area has an underscore cursor. When you first see a page, the top-most input area contains the cursor. To type information into another input area, use the **Enter** or **Tab** key to move from one input area to another. To move in the reverse order, use **Shift**–**Tab**.

You can also move from one input area to another using your mouse. Notice that each input area is active. Click on one of the areas. As you can see, the underscore cursor moves to that window.

3.5 Radio Buttons and Toggles

Some pages have *radio buttons* and *toggles*. Radio buttons are a group of buttons like those on car radios: you can select only one at a time. Once you have selected a button, it appears to be inverted and contains a checkmark. To change the selection, move the cursor with the mouse to a different radio

button and click.

A toggle is an independent button that displays some on/off state. When "on", the button appears to be inverted and contains a checkmark. When "off", the button is raised. Unlike radio buttons, you can set a group of them any way you like. To change toggle the selection, move the cursor with the mouse to the button and click.

3.6 Search Strings

A *search string* is used for searching some database. To learn about search strings, we suggest that you bring up the HyperDoc glossary. To do this from the top-level page of HyperDoc:

1. Click on **Reference**, bringing up the FriCAS Reference page.
2. Click on **Glossary**, bringing up the glossary.

The glossary has an input area at its bottom. We review the various kinds of search strings you can enter to search the glossary.

The simplest search string is a word, for example, `operation`. A word only matches an entry having exactly that spelling. Enter the word `operation` into the input area above then click on **Search**. As you can see, `operation` matches only one entry, namely with `operation` itself.

Normally matching is insensitive to whether the alphabetic characters of your search string are in uppercase or lowercase. Thus `operation` and `OperAtion` both have the same effect.

You will very often want to use the wildcard "`*`" in your search string so as to match multiple entries in the list. The search key "`*`" matches every entry in the list. You can also use "`*`" anywhere within a search string to match an arbitrary substring. Try `cat*` for example: enter `cat*` into the input area and click on **Search**. This matches several entries.

You can use any number of wildcards in a search string as long as they are not adjacent. Try search strings such as `*dom*`. As you see, this search string matches `domain`, `domain constructor`, `subdomain`, and so on.

3.6.1 Logical Searches

For more complicated searches, you can use "`and`", "`or`", and "`not`" with basic search strings; write logical expressions using these three operators just as in the FriCAS language. For example, `domain or package` matches the two entries `domain` and `package`. Similarly, `dom* and *con*` matches `domain constructor` and others. Also `not *a*` matches every entry that does not contain the letter `a` somewhere.

Use parentheses for grouping. For example, `dom* and (not *con*)` matches `domain` but not `domain constructor`.

There is no limit to how complex your logical expression can be. For example,

```
a* or b* or c* or d* or e* and (not *a*)
```

is a valid expression.

3.7 Example Pages

Many pages have FriCAS example commands. Each command has an active "button" along the left margin. When you click on this button, the output for the command is "pasted-in." Click again on the button and you see that the pasted-in output disappears.

Maybe you would like to run an example? To do so, just click on any part of its text! When you do, the example line is copied into a new interactive FriCAS buffer for this HyperDoc page.

Sometimes one example line cannot be run before you run an earlier one. Don't worry—HyperDoc automatically runs all the necessary lines in the right order!

The new interactive FriCAS buffer disappears when you leave HyperDoc. If you want to get rid of it beforehand, use the **Cancel** button of the X Window manager or issue the FriCAS system command `)close`.

3.8 X Window Resources for HyperDoc

You can control the appearance of HyperDoc while running under Version 11 of the X Window System by placing the following resources in the file **.Xdefaults** in your home directory. In what follows, *font* is any valid X11 font name (for example, `Rom14`) and *color* is any valid X11 color specification (for example, `NavyBlue`). For more information about fonts and colors, refer to the X Window documentation for your system.

`FriCAS.hyperdoc.RmFont:` *font*
: This is the standard text font. The default value is `"Rom14"`.

`FriCAS.hyperdoc.RmColor:` *color*
: This is the standard text color. The default value is `"black"`.

`FriCAS.hyperdoc.ActiveFont:` *font*
: This is the font used for HyperDoc link buttons. The default value is `"Bld14"`.

`FriCAS.hyperdoc.ActiveColor:` *color*
: This is the color used for HyperDoc link buttons. The default value is `"black"`.

`FriCAS.hyperdoc.FriCASFont:` *font*
: This is the font used for active FriCAS commands. The default value is `"Bld14"`.

`FriCAS.hyperdoc.FriCASColor:` *color*
: This is the color used for active FriCAS commands. The default value is `"black"`.

`FriCAS.hyperdoc.BoldFont:` *font*
: This is the font used for bold face. The default value is `"Bld14"`.

`FriCAS.hyperdoc.BoldColor:` *color*
: This is the color used for bold face. The default value is `"black"`.

`FriCAS.hyperdoc.TtFont:` *font*
: This is the font used for FriCAS output in HyperDoc. This font must be fixed-width. The default value is `"Rom14"`.

`FriCAS.hyperdoc.TtColor:` *color*
This is the color used for FriCAS output in HyperDoc. The default value is `"black"`.

`FriCAS.hyperdoc.EmphasizeFont:` *font*
This is the font used for italics. The default value is `"Itl14"`.

`FriCAS.hyperdoc.EmphasizeColor:` *color*
This is the color used for italics. The default value is `"black"`.

`FriCAS.hyperdoc.InputBackground:` *color*
This is the color used as the background for input areas. The default value is `"black"`.

`FriCAS.hyperdoc.InputForeground:` *color*
This is the color used as the foreground for input areas. The default value is `"white"`.

`FriCAS.hyperdoc.BorderColor:` *color*
This is the color used for drawing border lines. The default value is `"black"`.

`FriCAS.hyperdoc.Background:` *color*
This is the color used for the background of all windows. The default value is `"white"`.

Note: In the past, resource names used the word Axiom instead of FriCAS.

Chapter 4

Input Files and Output Styles

In this chapter we discuss how to collect FriCAS statements and commands into files and then read the contents into the workspace. We also show how to display the results of your computations in several different styles including TEX, FORTRAN and monospace two-dimensional format.[1]

The printed version of this book uses the FriCAS TEX output formatter. When we demonstrate a particular output style, we will need to turn TEX formatting off and the output style on so that the correct output is shown in the text.

4.1 Input Files

In this section we explain what an *input file* is and why you would want to know about it. We discuss where FriCAS looks for input files and how you can direct it to look elsewhere. We also show how to read the contents of an input file into the *workspace* and how to use the *history* facility to generate an input file from the statements you have entered directly into the workspace.

An *input* file contains FriCAS expressions and system commands. Anything that you can enter directly to FriCAS can be put into an input file. This is how you save input functions and expressions that you wish to read into FriCAS more than one time.

To read an input file into FriCAS, use the `)read` system command. For example, you can read a file in a particular directory by issuing

```
)read /spad/src/input/matrix.input
```

The "**.input**" is optional; this also works:

```
)read /spad/src/input/matrix
```

What happens if you just enter `)read matrix.input` or even `)read matrix`? FriCAS looks in your current working directory for input files that are not qualified by a directory name. Typically, this directory is the directory from which you invoked FriCAS. To change the current working directory, use

[1] TEX is a trademark of the American Mathematical Society.

the `)cd` system command. The command `)cd` by itself shows the current working directory. To change it to the `src/input` subdirectory for user "babar", issue

```
)cd /u/babar/src/input
```

FriCAS looks first in this directory for an input file. If it is not found, it looks in the system's directories, assuming you meant some input file that was provided with FriCAS.

> If you have the FriCAS history facility turned on (which it is by default), you can save all the lines you have entered into the workspace by entering
>
> `)history )write`
>
> FriCAS tells you what input file to edit to see your statements. The file is in your home directory or in the directory you specified with `)cd`.

In Section 5.2 on page 126 we discuss using indentation in input files to group statements into *blocks*.

4.2 The .fricas.input File

When FriCAS starts up, it tries to read the input file **.fricas.input** from your home directory. If there is no **.fricas.input** in your home directory, it reads the copy located in its own **src/input** directory. The file usually contains system commands to personalize your FriCAS environment. In the remainder of this section we mention a few things that users frequently place in their **.fricas.input** files.

In order to have FORTRAN output always produced from your computations, place the system command `)set output fortran on` in **.fricas.input**. If you do want to be prompted for confirmation when you issue the `)quit` system command, place `)set quit protected` in **.fricas.input**. If you then decide that you do not want to be prompted, issue `)set quit unprotected`. This is the default setting [2]

To see the other system variables you can set, issue `)set` or use the HyperDoc **Settings** facility to view and change FriCAS system variables.

4.3 Common Features of Using Output Formats

In this section we discuss how to start and stop the display of the different output formats and how to send the output to the screen or to a file. To fix ideas, we use FORTRAN output format for most of the examples.

You can use the `)set output` system command to toggle or redirect the different kinds of output. The name of the kind of output follows "output" in the command. The names are:

[2] The system command `)pquit` always prompts you for confirmation.

fortran for FORTRAN output.
algebra for monospace two-dimensional mathematical output.
tex for TEX output.
mathml for MathML output.
texmacs for TeXmacs output.

For example, issue `)set output fortran on` to turn on FORTRAN format and issue `)set output fortran off` to turn it off. By default, `algebra` is `on` and all others are `off`. When output is started, it is sent to the screen. To send the output to a file, give the file name without directory or extension. FriCAS appends a file extension depending on the kind of output being produced. Issue this to redirect FORTRAN output to, for example, the file **linalg.sfort**.

```
)set output fortran linalg
```

```
   FORTRAN output will be written to file linalg.sfort .
```

You must *also* turn on the creation of FORTRAN output. The above just says where it goes if it is created.

```
)set output fortran on
```

In what directory is this output placed? It goes into the directory from which you started FriCAS, or if you have used the `)cd` system command, the one that you specified with `)cd`. You should use `)cd` before you send the output to the file.

You can always direct output back to the screen by issuing this.

```
)set output fortran console
```

Let's make sure FORTRAN formatting is off so that nothing we do from now on produces FORTRAN output.

```
)set output fortran off
```

We also delete the demonstrated output file we created.

```
)system rm linalg.sfort
```

You can abbreviate the words "`on`," "`off`" and "`console`" to the minimal number of characters needed to distinguish them. Because of this, you cannot send output to files called **on.sfort, off.sfort, of.sfort, console.sfort, consol.sfort** and so on.

The width of the output on the page is set by `)set output length` for all formats except FORTRAN. Use `)set fortran fortlength` to change the FORTRAN line length from its default value of `72`.

4.4 Monospace Two-Dimensional Mathematical Format

This is the default output format for FriCAS. It is usually on when you start the system.

If it is not, issue this.

```
)set output algebra on
```

Since the printed version of this book (as opposed to the HyperDoc version) shows output produced by the TEX output formatter, let us temporarily turn off TEX output.

```
)set output tex off
```

Here is an example of what it looks like.

```
matrix [[i*x^i + j*%i*y^j for i in 1..2] for j in 3..4]
```

```
+     3           3     2+
|3%i y  + x  3%i y  + 2x |
|                        |
|     4           4     2|
+4%i y  + x  4%i y  + 2x +
```

Issue this to turn off this kind of formatting.

```
)set output algebra off
```

Turn TEX output on again.

```
)set output tex on
```

The characters used for the matrix brackets above are rather ugly. You get this character set when you issue `)set output characters plain`. This character set should be used when your machine or your version of FriCAS does not support Unicode character set. If your machine and your version of FriCAS support Unicode, issue `)set output characters default` to get better looking output.

4.5 TeX Format

FriCAS can produce TEX output for your expressions. The output is produced using macros from the LATEX document preparation system by Leslie Lamport.[3] The printed version of this book was produced using this formatter.

To turn on TEX output formatting, issue this.

```
)set output tex on
```

Here is an example of its output.

```
matrix [[i*x^i + j*%i*y^j for i in 1..2] for j in 3..4]
```

```
\begin{fricasmath}{1}
\begin{MATRIX}{2}3\TIMES \ImaginaryI \TIMES \SUPER{\SYMBOL{y}}{3}+\SYMBOL{x}&%
3\TIMES \ImaginaryI \TIMES \SUPER{\SYMBOL{y}}{3}+2\TIMES \SUPER{\SYMBOL{x}}{2%
}\\4\TIMES \ImaginaryI \TIMES \SUPER{\SYMBOL{y}}{4}+\SYMBOL{x}&4\TIMES %
\ImaginaryI \TIMES \SUPER{\SYMBOL{y}}{4}+2\TIMES \SUPER{\SYMBOL{x}}{2}%
\end{MATRIX}%
\end{fricasmath}
```

With the definition of the fricasmath environment as defined in fricasmath.sty this formats as

$$\begin{bmatrix} 3\,i\,y^3+x & 3\,i\,y^3+2\,x^2 \\ 4\,i\,y^4+x & 4\,i\,y^4+2\,x^2 \end{bmatrix} \tag{1}$$

[3] See Leslie Lamport, *LaTeX: A Document Preparation System,* Reading, Massachusetts: Addison-Wesley Publishing Company, Inc., 1986.

To turn TEX output formatting off, issue `)set output tex off`. The LATEX macros in the output generated by FriCAS are generic. See the source file of **TexFormat** for appropriate definitions of these commands.

4.6 MathML Format

FriCAS can produce MathML format output for your expressions.

To turn MathML Format on, issue this.

```
)set output mathml on
```

Here is an example of its output.

```
x+sqrt(2)

<math xmlns="http://www.w3.org/1998/Math/MathML" mathsize="big" display="block">
<mrow><mi>x</mi><mo>+</mo><msqrt><mrow><mn>2</mn></mrow></msqrt></mrow>
</math>
```

To turn MathML Format output formatting off, issue this.

```
)set output mathml off
```

4.7 TeXmacs Format

FriCAS can produce TeXmacs Scheme format output for your expressions. This is mostly useful for interfacing with TeXmacs.

To turn TeXmacs Format on, issue this.

```
)set output texmacs on
```

Here is an example of its output.

```
x+sqrt(2)

scheme: (with "mode" "math"
(concat (concat  "x" ) "+" (sqrt (concat  "2" )))
)
```

To turn TeXmacs Format output formatting off, issue this.

```
)set output texmacs off
```

4.8 FORTRAN Format

In addition to turning FORTRAN output on and off and stating where the output should be placed, there are many options that control the appearance of the generated code. In this section we describe some of the basic options. Issue `)set fortran` to see a full list with their current settings.

The output FORTRAN expression usually begins in column 7. If the expression needs more than one line, the ampersand character "&" is used in column 6. Since some versions of FORTRAN have restrictions on the number of lines per statement, FriCAS breaks long expressions into segments with a maximum of 1320 characters (20 lines of 66 characters) per segment. If you want to change this, say, to 660 characters, issue the system command `)set fortran explength 660`. You can turn off the line breaking by issuing `)set fortran segment off`. Various code optimization levels are available. FORTRAN output is produced after you issue this.

```
)set output fortran on
```

For the initial examples, we set the optimization level to 0, which is the lowest level.

```
)set fortran optlevel 0
```

The output is usually in columns 7 through 72, although fewer columns are used in the following examples so that the output fits nicely on the page.

```
)set fortran fortlength 60
```

By default, the output goes to the screen and is displayed before the standard FriCAS two-dimensional output. In this example, an assignment to the variable R1 was generated because this is the result of step 1.

```
(x+y)^3
```

$$y^3 + 3\,x\,y^2 + 3\,x^2\,y + x^3 \tag{1}$$

Polynomial(Integer)

Here is an example that illustrates the line breaking.

```
(x+y+z)^3
```

$$z^3 + (3\,y + 3\,x)\,z^2 + \left(3\,y^2 + 6\,x\,y + 3\,x^2\right)\,z + y^3 + 3\,x\,y^2 + 3\,x^2\,y + x^3 \tag{2}$$

Polynomial(Integer)

Note in the above examples that integers are generally converted to floating point numbers, except in exponents. This is the default behavior but can be turned off by issuing `)set fortran ints2floats off`. The rules governing when the conversion is done are:

1. If an integer is an exponent, convert it to a floating point number if it is greater than 32767 in absolute value, otherwise leave it as an integer.
2. Convert all other integers in an expression to floating point numbers.

These rules only govern integers in expressions. Numbers generated by FriCAS for DIMENSION statements are also integers.

To set the type of generated FORTRAN data, use one of the following:

```
)set fortran defaulttype REAL
)set fortran defaulttype INTEGER
)set fortran defaulttype COMPLEX
)set fortran defaulttype LOGICAL
)set fortran defaulttype CHARACTER
```

When temporaries are created, they are given a default type of `REAL`. Also, the `REAL` versions of functions are used by default.

```
sin(x)
```

$$\sin(x) \tag{3}$$

Expression(Integer)

At optimization level 1, FriCAS removes common subexpressions.

```
)set fortran optlevel 1
```

```
(x+y+z)^3
```

$$z^3 + (3\,y + 3\,x)\,z^2 + \left(3\,y^2 + 6\,x\,y + 3\,x^2\right)\,z + y^3 + 3\,x\,y^2 + 3\,x^2\,y + x^3 \tag{4}$$

Polynomial(Integer)

This changes the precision to `DOUBLE`. Substitute `single` for `double` to return to single precision.

```
)set fortran precision double
```

Complex constants display the precision.

```
2.3 + 5.6*%i
```

$$2.3 + 5.6\,i \tag{5}$$

Complex(Float)

The function names that FriCAS generates depend on the chosen precision.

```
sin %e
```

$$\sin(e) \tag{6}$$

Expression(Integer)

Reset the precision to `single` and look at these two examples again.

```
)set fortran precision single
```

```
2.3 + 5.6*%i
```

$$2.3 + 5.6\,i \tag{7}$$

Complex(Float)

```
sin %e
```

$$\sin(e) \tag{8}$$

Expression(Integer)

Expressions that look like lists, streams, sets or matrices cause array code to be generated.

```
[x+1,y+1,z+1]
```

$$[x+1,\ y+1,\ z+1] \tag{9}$$

List(Polynomial(Integer))

A temporary variable is generated to be the name of the array. This may have to be changed in your particular application.

```
set[2,3,4,3,5]
```

$$\{2,\ 3,\ 4,\ 5\} \tag{10}$$

Set(PositiveInteger)

By default, the starting index for generated FORTRAN arrays is 0.

```
matrix [[2.3,9.7],[0.0,18.778]]
```

$$\begin{bmatrix} 2.3 & 9.7 \\ 0.0 & 18.778 \end{bmatrix} \tag{11}$$

Matrix(Float)

To change the starting index for generated FORTRAN arrays to be 1, issue this. This value can only be 0 or 1.

```
)set fortran startindex 1
```

Look at the code generated for the matrix again.

```
matrix [[2.3,9.7],[0.0,18.778]]
```

$$\begin{bmatrix} 2.3 & 9.7 \\ 0.0 & 18.778 \end{bmatrix} \tag{12}$$

Matrix(Float)

4.9 General Fortran-generation utilities in FriCAS

This section describes more advanced facilities which are available to users who wish to generate Fortran code from within FriCAS. There are facilities to manipulate templates, store type information, and generate code fragments or complete programs.

4.9.1 Template Manipulation

A template is a skeletal program which is "fleshed out" with data when it is processed. It is a sequence of *active* and *passive* parts: active parts are sequences of FriCAS commands which are processed as if they had been typed into the interpreter; passive parts are simply echoed verbatim on the Fortran output stream.

Suppose, for example, that we have the following template, stored in the file "test.tem":

```
-- A simple template
beginVerbatim
      DOUBLE PRECISION FUNCTION F(X)
      DOUBLE PRECISION X
endVerbatim
outputAsFortran("F",f)
beginVerbatim
      RETURN
      END
endVerbatim
```

The passive parts lie between the two tokens `beginVerbatim` and `endVerbatim`. There are two active statements: one which is simply a FriCAS (`--`) comment, and one which produces an assignment to the current value of `f`. We could use it as follows:

```
(4) ->f := 4.0/(1+X^2)

            4
   (4)   ------
                2
               X  + 1

(5) ->processTemplate "test.tem"
      DOUBLE PRECISION FUNCTION F(X)
      DOUBLE PRECISION X
      F=4.0D0/(X*X+1.0D0)
      RETURN
      END

   (5)  "CONSOLE"
```

(A more reliable method of specifying the filename will be introduced below.) Note that the Fortran assignment `F=4.0D0/(X*X+1.0D0)` automatically converted 4.0 and 1 into DOUBLE PRECISION numbers; in general, the FriCAS Fortran generation facility will convert anything which should be a floating point object into either a Fortran REAL or DOUBLE PRECISION object. Which alternative is used is determined by the command

```
)set fortran precision
```

It is sometimes useful to end a template before the file itself ends (e.g. to allow the template to be tested incrementally or so that a piece of text describing how the template works can be included). It is of course possible to "comment-out" the remainder of the file. Alternatively, the single token `endInput` as part of an active portion of the template will cause processing to be ended prematurely at that point.

The processTemplate command comes in two flavours. In the first case, illustrated above, it takes one argument of domain **FileName**, the name of the template to be processed, and writes its output on the current Fortran output stream. In general, a filename can be generated from *directory*, *name* and *extension* components, using the operation filename, as in

```
processTemplate filename("","test","tem")
```

There is an alternative version of processTemplate, which takes two arguments (both of domain **FileName**). In this case the first argument is the name of the template to be processed, and the second is the file in which to write the results. Both versions return the location of the generated Fortran code as their result (`"CONSOLE"` in the above example).

It is sometimes useful to be able to mix active and passive parts of a line or statement. For example you might want to generate a Fortran Comment describing your data set. For this kind of application we provide three functions as follows:

fortranLiteral	writes a string on the Fortran output stream
fortranCarriageReturn	writes a carriage return on the Fortran output stream
fortranLiteralLine	writes a string followed by a return on the Fortran output stream

So we could create our comment as follows:

```
m := matrix [[1,2,3],[4,5,6]]
```

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \tag{1}$$

Matrix(Integer)

```
fortranLiteralLine(concat ["C\ \ \ \ \ \ The\ Matrix\ has\ ", nrows(m)::String, "\ -
    rows\ and\ ", ncols(m)::String, "\ columns"])$FortranTemplate
```

or, alternatively:

```
fortranLiteral("C\ \ \ \ \ \ The\ Matrix\ has\ ")$FortranTemplate

fortranLiteral(nrows(m)::String)$FortranTemplate

fortranLiteral("\ rows\ and\ ")$FortranTemplate

fortranLiteral(ncols(m)::String)$FortranTemplate

fortranLiteral("\ columns")$FortranTemplate

fortranCarriageReturn()$FortranTemplate
```

We should stress that these functions, together with the outputAsFortran function are the *only* sure ways of getting output to appear on the Fortran output stream. Attempts to use FriCAS commands such as output or writeline! may appear to give the required result when displayed on the console, but will give the wrong result when Fortran and algebraic output are sent to differing locations. On the other hand, these functions can be used to send helpful messages to the user, without interfering with the generated Fortran.

4.9.2 Manipulating the Fortran Output Stream

Sometimes it is useful to manipulate the Fortran output stream in a program, possibly without being aware of its current value. The main use of this is for gathering type declarations (see "Fortran Types" below) but it can be useful in other contexts as well. Thus we provide a set of commands to manipulate a stack of (open) output streams. Only one stream can be written to at any given time. The stack is never empty—its initial value is the console or the current value of the Fortran output stream, and can be determined using

```
topFortranOutputStack()$FortranOutputStackPackage
```

```
                                    "linalg.sfort"                                    (1)
                                                                                String
```

(see below). The commands available to manipulate the stack are:

clearFortranOutputStack	resets the stack to the console
pushFortranOutputStack	pushes a **FileName** onto the stack
popFortranOutputStack	pops the stack
showFortranOutputStack	returns the current stack
topFortranOutputStack	returns the top element of the stack

These commands are all part of **FortranOutputStackPackage**.

4.9.3 Fortran Types

When generating code it is important to keep track of the Fortran types of the objects which we are generating. This is useful for a number of reasons, not least to ensure that we are actually generating legal Fortran code. The current type system is built up in several layers, and we shall describe each in turn.

4.9.4 FortranScalarType

This domain represents the simple Fortran datatypes: REAL, DOUBLE PRECISION, COMPLEX, LOGICAL, INTEGER, and CHARACTER. It is possible to coerce a **String** or **Symbol** into the domain, test whether two objects are equal, and also apply the predicate functions real? etc.

4.9.5 FortranType

This domain represents "full" types: i.e., datatype plus array dimensions (where appropriate) plus whether or not the parameter is an external subprogram. It is possible to coerce an object of **FortranScalarType** into the domain or construct one from an element of **FortranScalarType**, a list of **Polynomial Integer**s (which can of course be simple integers or symbols) representing its

dimensions, and a **Boolean** declaring whether it is external or not. The list of dimensions must be empty if the **Boolean** is `true`. The functions scalarTypeOf, dimensionsOf and external? return the appropriate parts, and it is possible to get the various basic Fortran Types via functions like fortranReal. For example:

```
type:=construct(real,[i,10],false)$FortranType
```

or

```
type:=[real,[i,10],false]$FortranType
```

```
scalarTypeOf type
```

$$REAL \tag{1}$$

Union(fst: FortranScalarType, ...)

```
dimensionsOf type
```

$$[i,\ 10] \tag{2}$$

List(Polynomial(Integer))

```
external? type
```

$$\texttt{false} \tag{3}$$

Boolean

```
fortranLogical()$FortranType
```

$$LOGICAL \tag{4}$$

FortranType

```
construct(integer,[],true)$FortranType
```

4.9.6 SymbolTable

This domain creates and manipulates a symbol table for generated Fortran code. This is used by **FortranProgram** to represent the types of objects in a subprogram. The commands available are:

empty	creates a new **SymbolTable**
declare!	creates a new entry in a table
fortranTypeOf	returns the type of an object in a table
parametersOf	returns a list of all the symbols in the table
typeList	returns a list of all objects of a given type
typeLists	returns a list of lists of all objects sorted by type
externalList	returns a list of all `EXTERNAL` objects
printTypes	produces Fortran type declarations from a table

```
symbols := empty()$SymbolTable
```

$$\texttt{table}() \tag{1}$$

SymbolTable

```
declare!(X, fortranReal()$FortranType, symbols)
```

$$REAL \tag{2}$$

FortranType

```
declare!(M,construct(real,[i,j],false)$FortranType,symbols)
```

```
declare!([i,j], fortranInteger()$FortranType, symbols)
```

$$INTEGER \tag{3}$$

FortranType

```
symbols
```

```
fortranTypeOf(i,symbols)
```

$$INTEGER \tag{4}$$

FortranType

```
typeList(real,symbols)
```

$$[X,\ [M,\ i,\ j]] \tag{5}$$

List(Union(name: Symbol, bounds: List(Union(S: Symbol, P: Polynomial(Integer)))))

```
printTypes symbols
```

4.9.7 TheSymbolTable

This domain creates and manipulates one global symbol table to be used, for example, during template processing. It is also used when linking to external Fortran routines. The information stored for each subprogram (and the main program segment, where relevant) is:

- its name;
- its return type;
- its argument list;
- and its argument types.

Initially, any information provided is deemed to be for the main program segment. Issuing the following command indicates that from now on all information refers to the subprogram F.

```
newSubProgram(F)$TheSymbolTable
```

It is possible to return to processing the main program segment by issuing the command:

```
endSubProgram()$TheSymbolTable
```

$$MAIN \tag{2}$$

Symbol

The following commands exist:

returnType!	declares the return type of the current subprogram
returnTypeOf	returns the return type of a subprogram
argumentList!	declares the argument list of the current subprogram
argumentListOf	returns the argument list of a subprogram
declare!	provides type declarations for parameters of the current subprogram
symbolTableOf	returns the symbol table of a subprogram
printHeader	produces the Fortran header for the current subprogram

In addition there are versions of these commands which are parameterised by the name of a subprogram, and others parameterised by both the name of a subprogram and by an instance of **TheSymbolTable**.

```
newSubProgram(F)$TheSymbolTable
```

```
argumentList!(F, [X])$TheSymbolTable
```

```
returnType!(F,real)$TheSymbolTable
```

```
declare!(X,fortranReal(),F)$TheSymbolTable
```

$$REAL \tag{6}$$

FortranType

```
printHeader(F)$TheSymbolTable
```

4.9.8 Advanced Fortran Code Generation

This section describes facilities for representing Fortran statements, and building up complete subprograms from them.

4.9.9 Switch

This domain is used to represent statements like `x < y`. Although these can be represented directly in FriCAS, it is a little cumbersome.

Instead we have a set of operations, such as LT to represent <, to let us build such statements. The available constructors are:

LT	$<$
GT	$>$
LE	$\leq$
GE	$\geq$
EQ	$=$
AND	and
OR	or
NOT	not

So for example:

```
LT(x,y)$Switch
```

$$x < y \tag{1}$$

Switch

4.9.10 FortranCode

This domain represents code segments or operations: currently assignments, conditionals, blocks, comments, gotos, continues, various kinds of loops, and return statements. For example we can create quite a complicated conditional statement using assignments, and then turn it into Fortran code:

```
c := _
    cond(LT(X,Y),assign(F,X),cond(GT(Y,Z),assign(F,Y),assign(F,Z))$FortranCode)$FortranCode
```

```
"conditional"                                                    (1)
```

FortranCode

```
printCode c
```

The Fortran code is printed on the current Fortran output stream.

4.9.11 FortranProgram

This domain is used to construct complete Fortran subprograms out of elements of **FortranCode**. It is parameterised by the name of the target subprogram (a **Symbol**), its return type (from **Union**(**FortranScalarType**,"void")), its arguments (from **List Symbol**), and its symbol table (from **SymbolTable**). One can coerce elements of either **FortranCode** or **Expression** into it.

First of all we create a symbol table:

```
symbols := empty()$SymbolTable
```

```
table()                                                          (1)
```

SymbolTable

Now put some type declarations into it:

```
declare!([X,Y],fortranReal()$FortranType,symbols)
```

$$REAL \tag{2}$$

FortranType

Then (for convenience) we set up the particular instantiation of **FortranProgram**

```
FP := FortranProgram(F,real,[X,Y],symbols)
```

$$\text{FortranProgram}(F, REAL, [X,\ Y]\,, \texttt{table}(Y = REAL, X = REAL)) \tag{3}$$

Type

Create an object of type **Expression(Integer)**:

```
asp := X*sin(Y)
```

$$X\ \sin(Y) \tag{4}$$

Expression (Integer)

Now coerce it into **FP**, and print its Fortran form:

```
outputAsFortran(asp::FP)
```

We can generate a **FortranProgram** using **FortranCode**. For example: Augment our symbol table:

```
declare!(Z,fortranReal()$FortranType,symbols)
```

$$REAL \tag{6}$$

FortranType

prepare conditional:

```
cc := cond(LT(X,Y),assign(F,X),assign(F,Y))$FortranCode
```

$$\texttt{"conditional"} \tag{7}$$

FortranCode

and transform the conditional expression we prepared earlier:

```
outputAsFortran([cc,returns()$FortranCode]::FP)
```

Chapter 5

Introduction to the FriCAS Interactive Language

In this chapter we look at some of the basic components of the FriCAS language that you can use interactively. We show how to create a *block* of expressions, how to form loops and list iterations, how to modify the sequential evaluation of a block and how to use `if-then-else` to evaluate parts of your program conditionally. We suggest you first read the boxed material in each section and then proceed to a more thorough reading of the chapter.

5.1 Immediate and Delayed Assignments

A *variable* in FriCAS refers to a value. A variable has a name beginning with an uppercase or lowercase alphabetic character, "`%`", or "`!`". Successive characters (if any) can be any of the above, digits, or "`?`". Case is distinguished. The following are all examples of valid, distinct variable names:

```
a             tooBig?     a1B2c3%!?
A             %j          numberOfPoints
beta6         %J          numberofpoints
```

The "`:=`" operator is the immediate *assignment* operator. Use it to associate a value with a variable.

> The syntax for immediate assignment for a single variable is
>
> *variable* `:=` *expression*
>
> The value returned by an immediate assignment is the value of *expression*.

The right-hand side of the expression is evaluated, yielding `1`. This value is then assigned to `a`.

```
a := 1
```

$$1 \tag{1}$$

PositiveInteger

The right-hand side of the expression is evaluated, yielding 1. This value is then assigned to b. Thus a and b both have the value 1 after the sequence of assignments.

```
b := a
```

$$1 \tag{2}$$

PositiveInteger

What is the value of b if a is assigned the value 2?

```
a := 2
```

$$2 \tag{3}$$

PositiveInteger

As you see, the value of b is left unchanged.

```
b
```

$$1 \tag{4}$$

PositiveInteger

This is what we mean when we say this kind of assignment is *immediate*; b has no dependency on a after the initial assignment. This is the usual notion of assignment found in programming languages such as C, PASCAL and FORTRAN.

FriCAS provides delayed assignment with "==". This implements a delayed evaluation of the right-hand side and dependency checking.

> The syntax for delayed assignment is
>
> *variable* == *expression*
>
> The value returned by a delayed assignment is the unique value of **Void**.

Using a and b as above, these are the corresponding delayed assignments.

```
a == 1
```

```
b == a
```

The right-hand side of each delayed assignment is left unevaluated until the variables on the left-hand sides are evaluated. Therefore this evaluation and ...

```
a
```

```
   Compiling body of rule a to compute value of type PositiveInteger
```

$$1 \tag{7}$$

PositiveInteger

this evaluation seem the same as before.

```
b
```

```
   Compiling body of rule b to compute value of type PositiveInteger
```

$$1 \tag{8}$$

PositiveInteger

If we change `a` to `2`

```
a == 2
```

```
   Compiled code for a has been cleared.
   Compiled code for b has been cleared.
   1 old definition(s) deleted for function or rule a
```

then `a` evaluates to `2`, as expected, but

```
a
```

```
   Compiling body of rule a to compute value of type PositiveInteger
```

$$2 \tag{10}$$

PositiveInteger

the value of `b` reflects the change to `a`.

```
b
```

```
   Compiling body of rule b to compute value of type PositiveInteger
```

$$2 \tag{11}$$

PositiveInteger

It is possible to set several variables at the same time by using a *tuple* of variables and a tuple of expressions.[1]

> The syntax for multiple immediate assignments is
>
> $$\texttt{(}\ var_1\texttt{,}\ var_2\texttt{,}\ \texttt{...,}\ var_N\ \texttt{)}\ \texttt{:=}\ \texttt{(}\ expr_1\texttt{,}\ expr_2\texttt{,}\ \texttt{...,}\ expr_N\ \texttt{)}$$
>
> The value returned by an immediate assignment is the value of $expr_N$.

This sets `x` to `1` and `y` to `2`.

```
(x,y) := (1,2)
```

[1] A *tuple* is a collection of things separated by commas, often surrounded by parentheses.

$$2 \tag{12}$$

PositiveInteger

Multiple immediate assignments are parallel in the sense that the expressions on the right are all evaluated before any assignments on the left are made. However, the order of evaluation of these expressions is undefined. You can use multiple immediate assignment to swap the values held by variables.

```
(x,y) := (y,x)
```

$$1 \tag{13}$$

PositiveInteger

`x` has the previous value of `y`.

```
x
```

$$2 \tag{14}$$

PositiveInteger

`y` has the previous value of `x`.

```
y
```

$$1 \tag{15}$$

PositiveInteger

There is no syntactic form for multiple delayed assignments. See the discussion in Section 6.8 on page 158 about how FriCAS differentiates between delayed assignments and user functions of no arguments.

5.2 Blocks

A *block* is a sequence of expressions evaluated in the order that they appear, except as modified by control expressions such as `break`, `return`, `iterate` and `if-then-else` constructions. The value of a block is the value of the expression last evaluated in the block.

To leave a block early, use "`=>`". For example, `i < 0 => x`. The expression before the "`=>`" must evaluate to `true` or `false`. The expression following the "`=>`" is the return value for the block.

A block can be constructed in two ways:

1. the expressions can be separated by semicolons and the resulting expression surrounded by parentheses, and
2. the expressions can be written on succeeding lines with each line indented the same number of spaces (which must be greater than zero). A block entered in this form is called a *pile*.

Only the first form is available if you are entering expressions directly to FriCAS. Both forms are available in **.input** files.

The syntax for a simple block of expressions entered interactively is

$$(\ expression_1 ; \ expression_2 ; \ \ldots ; \ expression_N \ )$$

The value returned by a block is the value of an "=>" expression, or $expression_N$ if no "=>" is encountered.

In **.input** files, blocks can also be written using *piles*. The examples throughout this book are assumed to come from **.input** files.

In this example, we assign a rational number to a using a block consisting of three expressions. This block is written as a pile. Each expression in the pile has the same indentation, in this case two spaces to the right of the first line.

```
a :=
  i := gcd(234,672)
  i := 3*i^5 - i + 1
  1 / i
```

$$\frac{1}{23323} \tag{1}$$

Fraction (Integer)

Here is the same block written on one line. This is how you are required to enter it at the input prompt.

```
a := (i := gcd(234,672); i := 3*i^5 - i + 1; 1 / i)
```

$$\frac{1}{23323} \tag{2}$$

Fraction (Integer)

Blocks can be used to put several expressions on one line. The value returned is that of the last expression.

```
(a := 1; b := 2; c := 3; [a,b,c])
```

$$[1,\ 2,\ 3] \tag{3}$$

List (PositiveInteger)

FriCAS gives you two ways of writing a block and the preferred way in an **.input** file is to use a pile. Roughly speaking, a pile is a block whose constituent expressions are indented the same amount. You begin a pile by starting a new line for the first expression, indenting it to the right of the previous line. You then enter the second expression on a new line, vertically aligning it with the first line. And so on. If you need to enter an inner pile, further indent its lines to the right of the outer pile. FriCAS knows where a pile ends. It ends when a subsequent line is indented to the left of the pile or the end of the file.

Blocks can be used to perform several steps before an assignment (immediate or delayed) is made.

```
d :=
   c := a^2 + b^2
   sqrt(c * 1.3)
```

$$2.549509756796392415 \tag{4}$$

Float

Blocks can be used in the arguments to functions. (Here `h` is assigned `2.1 + 3.5`.)

```
h := 2.1 +
   1.0
   3.5
```

$$5.6 \tag{5}$$

Float

Here the second argument to eval is `x = z`, where the value of `z` is computed in the first line of the block starting on the second line.

```
eval(x^2 - x*y^2,
     z := %pi/2.0 - exp(4.1)
     x = z
   )
```

$$58.769491270567072878\,y^2 + 3453.853104201259382 \tag{6}$$

Polynomial(Float)

Blocks can be used in the clauses of `if-then-else` expressions (see Section 5.3 on page 129).

```
if h > 3.1 then 1.0 else (z := cos(h); max(z,0.5))
```

$$1.0 \tag{7}$$

Float

This is the pile version of the last block.

```
if h > 3.1 then
    1.0
  else
    z := cos(h)
    max(z,0.5)
```

$$1.0 \tag{8}$$

Float

Blocks can be nested.

```
a := (b := factorial(12); c := (d := eulerPhi(22); factorial(d));b+c)
```

$$482630400 \tag{9}$$

PositiveInteger

This is the pile version of the last block.

```
a :=
  b := factorial(12)
```

```
c :=
  d := eulerPhi(22)
  factorial(d)
b+c
```

$$482630400 \tag{10}$$

PositiveInteger

Since `c + d` does equal `3628855`, `a` has the value of `c` and the last line is never evaluated.

```
a :=
  c := factorial 10
  d := fibonacci 10
  c + d = 3628855 => c
  d
```

$$3628800 \tag{11}$$

PositiveInteger

5.3 if-then-else

Like many other programming languages, FriCAS uses the three keywords `if, then` and `else` to form conditional expressions. The `else` part of the conditional is optional. The expression between the `if` and `then` keywords is a *predicate*: an expression that evaluates to or is convertible to either `true` or `false`, that is, a **Boolean**.

The syntax for conditional expressions is

`if` *predicate* `then` $expression_1$ `else` $expression_2$

where the `else` $expression_2$ part is optional. The value returned from a conditional expression is $expression_1$ if the predicate evaluates to `true` and $expression_2$ otherwise. If no `else` clause is given, the value is always the unique value of **Void**.

An `if-then-else` expression always returns a value. If the `else` clause is missing then the entire expression returns the unique value of **Void**. If both clauses are present, the type of the value returned by `if` is obtained by resolving the types of the values of the two clauses. See Section 2.10 on page 92 for more information.

The predicate must evaluate to, or be convertible to, an object of type **Boolean**: `true` or `false`. By default, the equal sign `=` creates an equation.

This is an equation. In particular, it is an object of type **Equation Polynomial Integer**.

```
x + 1 = y
```

$$x + 1 = y \tag{1}$$

Equation(Polynomial(Integer))

However, for predicates in `if` expressions, FriCAS places a default target type of **Boolean** on the predicate and equality testing is performed. Thus you need not qualify the "`=`" in any way. In other contexts you may need to tell FriCAS that you want to test for equality rather than create an equation. In those cases, use "`@`" and a target type of **Boolean**. See Section 2.9 on page 89 for more information.

The compound symbol meaning "not equal" in FriCAS is `~=`. This can be used directly without a package call or a target specification. The expression `a ~= b` is directly translated into `not (a = b)`.

Many other functions have return values of type **Boolean**. These include `<`, `<=`, `>`, `>=`, `~=` and `member?`. By convention, operations with names ending in "`?`" return **Boolean** values.

The usual rules for piles are suspended for conditional expressions. In **.input** files, the `then` and `else` keywords can begin in the same column as the corresponding `if` but may also appear to the right. Each of the following styles of writing `if-then-else` expressions is acceptable:

```
if i>0 then output("positive") else output("nonpositive")

if i > 0 then output("positive")
  else output("nonpositive")

if i > 0 then output("positive")
else output("nonpositive")

if i > 0
then output("positive")
else output("nonpositive")

if i > 0
  then output("positive")
  else output("nonpositive")
```

A block can follow the `then` or `else` keywords. In the following two assignments to `a`, the `then` and `else` clauses each are followed by two-line piles. The value returned in each is the value of the second line.

```
a :=
  if i > 0 then
    j := sin(i * pi())
    exp(j + 1/j)
  else
    j := cos(i * 0.5 * pi())
    log(abs(j)^5 + 1)

a :=
  if i > 0
    then
      j := sin(i * pi())
      exp(j + 1/j)
    else
```

```
    j := cos(i * 0.5 * pi())
    log(abs(j)^5 + 1)
```

These are both equivalent to the following:

```
a :=
  if i > 0 then (j := sin(i * pi()); exp(j + 1/j))
  else (j := cos(i * 0.5 * pi()); log(abs(j)^5 + 1))
```

5.4 Loops

A *loop* is an expression that contains another expression, called the *loop body*, which is to be evaluated zero or more times. All loops contain the `repeat` keyword and return the unique value of **Void**. Loops can contain inner loops to any depth.

The most basic loop is of the form

`repeat` *loopBody*

Unless *loopBody* contains a `break` or `return` expression, the loop repeats forever. The value returned by the loop is the unique value of **Void**.

5.4.1 Compiling vs. Interpreting Loops

FriCAS tries to determine completely the type of every object in a loop and then to translate the loop body to LISP or even to machine code. This translation is called *compilation*.

If FriCAS decides that it cannot compile the loop, it issues a message stating the problem and then the following message:

We will attempt to step through and interpret the code.

It is still possible that FriCAS can evaluate the loop but in *interpret-code mode*. See Section 6.10 on page 160 where this is discussed in terms of compiling versus interpreting functions.

5.4.2 return in Loops

A `return` expression is used to exit a function with a particular value. In particular, if a `return` is in a loop within the function, the loop is terminated whenever the `return` is evaluated. Suppose we start with this.

```
f() ==
  i := 1
  repeat
    if factorial(i) > 1000 then return i
    i := i + 1
```

When `factorial(i)` is big enough, control passes from inside the loop all the way outside the function, returning the value of `i` (or so we think).

```
f()
```

```
   Compiling function f with type () -> Void
```

What went wrong? Isn't it obvious that this function should return an integer? Well, FriCAS makes no attempt to analyze the structure of a loop to determine if it always returns a value because, in general, this is impossible. So FriCAS has this simple rule: the type of the function is determined by the type of its body, in this case a block. The normal value of a block is the value of its last expression, in this case, a loop. And the value of every loop is the unique value of **Void**! So the return type of **f** is **Void**.

There are two ways to fix this. The best way is for you to tell FriCAS what the return type of `f` is. You do this by giving `f` a declaration `f: () → Integer` prior to calling for its value. This tells FriCAS: "trust me—an integer is returned." We'll explain more about this in the next chapter. Another clumsy way is to add a dummy expression as follows.

Since we want an integer, let's stick in a dummy final expression that is an integer and will never be evaluated.

```
f() ==
  i := 1
  repeat
    if factorial(i) > 1000 then return i
    i := i + 1
  0
```

```
   Compiled code for f has been cleared.
   1 old definition(s) deleted for function or rule f
```

When we try **f** again we get what we wanted. See Section 6.15 on page 172 for more information.

```
f()
```

```
   Compiling function f with type () -> NonNegativeInteger
```

$$7 \qquad (4)$$

PositiveInteger

5.4.3 break in Loops

The `break` keyword is often more useful in terminating a loop. A `break` causes control to transfer to the expression immediately following the loop. As loops always return the unique value of **Void**, you cannot return a value with `break`. That is, `break` takes no argument.

This example is a modification of the last example in the previous section. Instead of using `return`, we'll use `break`.

```
f() ==
  i := 1
  repeat
    if factorial(i) > 1000 then break
    i := i + 1
  i
```

The loop terminates when `factorial(i)` gets big enough, the last line of the function evaluates to the corresponding "good" value of `i`, and the function terminates, returning that value.

```
f()
```

```
   Compiling function f with type () -> PositiveInteger
```

$$7 \tag{2}$$

PositiveInteger

You can only use `break` to terminate the evaluation of one loop. Let's consider a loop within a loop, that is, a loop with a nested loop. First, we initialize two counter variables.

```
(i,j) := (1, 1)
```

$$1 \tag{3}$$

PositiveInteger

Nested loops must have multiple `break` expressions at the appropriate nesting level. How would you rewrite this so `(i + j)> 10` is only evaluated once?

```
repeat
  repeat
    if (i + j) > 10 then break
    j := j + 1
  if (i + j) > 10 then break
  i := i + 1
```

5.4.4 break vs. => in Loop Bodies

Compare the following two loops:

```
i := 1
repeat
  i := i + 1
  i > 3 => i
  output(i)
```

```
i := 1
repeat
  i := i + 1
  if i > 3 then break
  output(i)
```

In the example on the left, the values 2 and 3 for `i` are displayed but then the "`=>`" does not allow control to reach the call to `output` again. The loop will not terminate until you run out of space or interrupt the execution. The variable `i` will continue to be incremented because the "`=>`" only means to leave the *block,* not the loop.

In the example on the right, upon reaching 4, the `break` will be executed, and both the block and the loop will terminate. This is one of the reasons why both "`=>`" and `break` are provided. Using a `while` clause (see below) with the "`=>`" lets you simulate the action of `break`.

5.4.5 More Examples of break

Here we give four examples of `repeat` loops that terminate when a value exceeds a given bound.

First, initialize `i` as the loop counter.

```
i := 0
```

$$0 \tag{1}$$

NonNegativeInteger

Here is the first loop. When the square of `i` exceeds `100`, the loop terminates.

```
repeat
  i := i + 1
  if i^2 > 100 then break
```

Upon completion, `i` should have the value `11`.

```
i
```

$$11 \tag{3}$$

NonNegativeInteger

Do the same thing except use "`=>`" instead of an `if-then` expression.

```
i := 0
```

$$0 \tag{4}$$

NonNegativeInteger

```
repeat
  i := i + 1
  i^2 > 100 => break
```

```
i
```

$$11 \tag{6}$$

NonNegativeInteger

As a third example, we use a simple loop to compute `n!`.

```
(n, i, f) := (100, 1, 1)
```

$$1 \tag{7}$$

PositiveInteger

Use `i` as the iteration variable and `f` to compute the factorial.

```
repeat
  if i > n then break
  f := f * i
  i := i + 1
```

Look at the value of `f`.

```
f
```

933262154439441526816992388562667004907159682643816214685929638952175999932299156089414639761(9)65182862536979208

PositiveInteger

Finally, we show an example of nested loops. First define a four by four matrix.

```
m := matrix [[21,37,53,14], [8,-24,22,-16], [2,10,15,14], [26,33,55,-13]]
```

$$\begin{bmatrix} 21 & 37 & 53 & 14 \\ 8 & -24 & 22 & -16 \\ 2 & 10 & 15 & 14 \\ 26 & 33 & 55 & -13 \end{bmatrix} \tag{10}$$

Matrix(Integer)

Next, set row counter `r` and column counter `c` to 1. Note: if we were writing a function, these would all be local variables rather than global workspace variables.

```
(r, c) := (1, 1)
```

$$1 \tag{11}$$

PositiveInteger

Also, let `lastrow` and `lastcol` be the final row and column index.

```
(lastrow, lastcol) := (nrows(m), ncols(m))
```

$$4 \tag{12}$$

PositiveInteger

Scan the rows looking for the first negative element. We remark that you can reformulate this example in a better, more concise form by using a `for` clause with `repeat`. See Section 5.4.8 on page 138 for more information.

```
repeat
  if r > lastrow then break
  c := 1
  repeat
    if c > lastcol then break
    if elt(m,r,c) < 0 then
      output [r, c, elt(m,r,c)]
      r := lastrow
      break      -- don't look any further
    c := c + 1
  r := r + 1
```

```
  [2, 2, - 24]
```

5.4.6 iterate in Loops

FriCAS provides an `iterate` expression that skips over the remainder of a loop body and starts the next loop iteration. We first initialize a counter.

```
i := 0
```

$$0 \tag{1}$$

NonNegativeInteger

Display the even integers from 2 to 5.

```
repeat
  i := i + 1
  if i > 5 then break
  if odd?(i) then iterate
  output(i)
```

```
   2
   4
```

5.4.7 while Loops

The `repeat` in a loop can be modified by adding one or more `while` clauses. Each clause contains a *predicate* immediately following the `while` keyword. The predicate is tested *before* the evaluation of the body of the loop. The loop body is evaluated whenever the predicates in a `while` clause are all `true`.

> The syntax for a simple loop using `while` is
>
> `while` *predicate* `repeat` *loopBody*
>
> The *predicate* is evaluated before *loopBody* is evaluated. A `while` loop terminates immediately when *predicate* evaluates to `false` or when a `break` or `return` expression is evaluated in *loopBody*. The value returned by the loop is the unique value of **Void**.

Here is a simple example of using `while` in a loop. We first initialize the counter.

```
i := 1
```

$$1 \tag{1}$$

PositiveInteger

The steps involved in computing this example are (1) set `i` to `1`, (2) test the condition `i < 1` and determine that it is not true, and (3) do not evaluate the loop body and therefore do not display `"hello"`.

```
while i < 1 repeat
  output "hello"
  i := i + 1
```

If you have multiple predicates to be tested use the logical `and` operation to separate them. FriCAS evaluates these predicates from left to right.

```
(x, y) := (1, 1)
```

$$1 \tag{3}$$

PositiveInteger

```
while x < 4 and y < 10 repeat
  output [x,y]
  x := x + 1
  y := y + 2
```

```
  [1, 1]
  [2, 3]
  [3, 5]
```

A `break` expression can be included in a loop body to terminate a loop even if the predicate in any `while` clauses are not `false`.

```
(x, y) := (1, 1)
```

$$1 \tag{5}$$

PositiveInteger

This loop has multiple `while` clauses and the loop terminates before any one of their conditions evaluates to `false`.

```
while x < 4 while y < 10 repeat
  if x + y > 7 then break
  output [x,y]
  x := x + 1
  y := y + 2
```

```
  [1, 1]
  [2, 3]
```

Here's a different version of the nested loops that looked for the first negative element in a matrix.

```
m := matrix [[21,37,53,14], [8,-24,22,-16], [2,10,15,14], [26,33,55,-13]]
```

$$\begin{bmatrix} 21 & 37 & 53 & 14 \\ 8 & -24 & 22 & -16 \\ 2 & 10 & 15 & 14 \\ 26 & 33 & 55 & -13 \end{bmatrix} \tag{7}$$

Matrix(Integer)

Initialize the row index to `1` and get the number of rows and columns. If we were writing a function, these would all be local variables.

```
r := 1
```

$$1 \tag{8}$$

PositiveInteger

```
(lastrow, lastcol) := (nrows(m), ncols(m))
```

$$4 \tag{9}$$

PositiveInteger

Scan the rows looking for the first negative element.

```
while r <= lastrow repeat
  c := 1  -- index of first column
  while c <= lastcol repeat
    if elt(m,r,c) < 0 then
      output [r, c, elt(m,r,c)]
      r := lastrow
      break      -- don't look any further
    c := c + 1
  r := r + 1
```

```
   [2, 2, - 24]
```

5.4.8 for Loops

FriCAS provides the `for` and `in` keywords in `repeat` loops, allowing you to iterate across all elements of a list, or to have a variable take on integral values from a lower bound to an upper bound. We shall refer to these modifying clauses of `repeat` loops as `for` clauses. These clauses can be present in addition to `while` clauses. As with all other types of `repeat` loops, `break` can be used to prematurely terminate the evaluation of the loop.

The syntax for a simple loop using `for` is

`for` *iterator* `repeat` *loopBody*

The *iterator* has several forms. Each form has an end test which is evaluated before *loopBody* is evaluated. A `for` loop terminates immediately when the end test succeeds (evaluates to `true`) or when a `break` or `return` expression is evaluated in *loopBody*. The value returned by the loop is the unique value of **Void**.

5.4.9 for i in n..m repeat

If `for` is followed by a variable name, the `in` keyword and then an integer segment of the form `n..m`, the end test for this loop is the predicate `i > m`. The body of the loop is evaluated `m-n+1` times if this number is greater than 0. If this number is less than or equal to 0, the loop body is not evaluated at all.

The variable `i` has the value `n, n+1, ..., m` for successive iterations of the loop body. The loop variable is a *local variable* within the loop body: its value is not available outside the loop body and its value and type within the loop body completely mask any outer definition of a variable with the same name.

This loop prints the values of 10^3, 11^3, and 12^3:

```
for i in 10..12 repeat output(i^3)
```

```
   1000
   1331
   1728
```

Here is a sample list.

```
a := [1,2,3]
```

$$[1,\ 2,\ 3] \tag{2}$$

List (PositiveInteger)

Iterate across this list, using "." to access the elements of a list and the # operation to count its elements.

```
for i in 1..#a repeat output(a.i)
```

```
   1
   2
   3
```

This type of iteration is applicable to anything that uses ".". You can also use it with functions that use indices to extract elements. Define m to be a matrix.

```
m := matrix [[1,2],[4,3],[9,0]]
```

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 9 & 0 \end{bmatrix} \tag{4}$$

Matrix(NonNegativeInteger)

Display the rows of m.

```
for i in 1..nrows(m) repeat output row(m,i)
```

```
   [1, 2]
   [4, 3]
   [9, 0]
```

You can use iterate with for-loops. Display the even integers in a segment.

```
for i in 1..5 repeat
  if odd?(i) then iterate
  output(i)
```

```
   2
   4
```

See 'Segment' on page 562 for more information about segments.

5.4.10 for i in n..m by s repeat

By default, the difference between values taken on by a variable in loops such as `for i in n..m repeat ...` is 1. It is possible to supply another, possibly negative, step value by using the by keyword along with for and in. Like the upper and lower bounds, the step value following the by keyword must be an integer. Note that the loop `for i in 1..2 by 0 repeat output(i)` will not terminate by itself, as the step value does not change the index from its initial value of 1.

This expression displays the odd integers between two bounds.

```
for i in 1..5 by 2 repeat output(i)
```

```
   1
   3
   5
```

Use this to display the numbers in reverse order.

```
for i in 5..1 by -2 repeat output(i)
```

```
   5
   3
   1
```

5.4.11 for i in n.. repeat

If the value after the ".." is omitted, the loop has no end test. A potentially infinite loop is thus created. The variable is given the successive values `n, n+1, n+2, ...` and the loop is terminated only if a `break` or `return` expression is evaluated in the loop body. However you may also add some other modifying clause on the `repeat` (for example, a `while` clause) to stop the loop.

This loop displays the integers greater than or equal to `15` and less than the first prime greater than `15`.

```
for i in 15.. while not prime?(i) repeat output(i)
```

```
   15
   16
```

5.4.12 for x in l repeat

Another variant of the `for` loop has the form:

for x in list repeat loopBody

This form is used when you want to iterate directly over the elements of a list. In this form of the `for` loop, the variable `x` takes on the value of each successive element in `l`. The end test is most simply stated in English: "are there no more `x` in `l`?"

If `l` is this list,

```
l := [0,-5,3]
```

$$[0, -5, 3] \tag{1}$$

List (Integer)

display all elements of `l`, one per line.

```
for x in l repeat output(x)
```

```
   0
   - 5
   3
```

Since the list constructing expression `expand [n..m]` creates the list `[n, n+1, ..., m]`[2], you might be tempted to think that the loops

```
for i in n..m repeat output(i)
```

and

```
for x in expand [n..m] repeat output(x)
```

are equivalent. The second form first creates the list `expand [n..m]` (no matter how large it might be) and then does the iteration. The first form potentially runs in much less space, as the index variable `i` is simply incremented once per loop and the list is not actually created. Using the first form is much more efficient. Of course, sometimes you really want to iterate across a specific list. This displays each of the factors of `2400000`.

[2]This list is empty if `n > m`.

```
for f in factors(factor(2400000)) repeat output(f)
```

```
   [factor = 2, exponent = 8]
   [factor = 3, exponent = 1]
   [factor = 5, exponent = 5]
```

5.4.13 "Such that" Predicates

A `for` loop can be followed by a "`|`" and then a predicate. The predicate qualifies the use of the values from the iterator following the `for`. Think of the vertical bar "`|`" as the phrase "such that." This loop expression prints out the integers `n` in the given segment such that `n` is odd.

```
for n in 0..4 | odd? n repeat output n
```

```
   1
   3
```

A `for` loop can also be written

`for` *iterator* `|` *predicate* `repeat` *loopBody*

which is equivalent to:

`for` *iterator* `repeat if` *predicate* `then` *loopBody* `else iterate`

The predicate need not refer only to the variable in the `for` clause: any variable in an outer scope can be part of the predicate. In this example, the predicate on the inner `for` loop uses `i` from the outer loop and the `j` from the `for` clause that it directly modifies.

```
for i in 1..50 repeat
  for j in 1..50 | factorial(i+j) < 25 repeat
    output [i,j]
```

```
   [1, 1]
   [1, 2]
   [1, 3]
   [2, 1]
   [2, 2]
   [3, 1]
```

5.4.14 Parallel Iteration

The last example of the previous section gives an example of *nested iteration*: a loop is contained in another loop. Sometimes you want to iterate across two lists in parallel, or perhaps you want to traverse a list while incrementing a variable.

The general syntax of a repeat loop is

$$iterator_1 \ \ iterator_2 \ \ \ldots iterator_N \ \texttt{repeat} \ \ loopBody$$

where each *iterator* is either a `for` or a `while` clause. The loop terminates immediately when the end test of any *iterator* succeeds or when a `break` or `return` expression is evaluated in *loopBody*. The value returned by the loop is the unique value of **Void**.

Here we write a loop to iterate across two lists, computing the sum of the pairwise product of elements. Here is the first list.

```
l := [1,3,5,7]
```

$$[1,\ 3,\ 5,\ 7] \tag{1}$$

List (PositiveInteger)

And the second.

```
m := [100,200]
```

$$[100,\ 200] \tag{2}$$

List (PositiveInteger)

The initial value of the sum counter.

```
sum := 0
```

$$0 \tag{3}$$

NonNegativeInteger

The last two elements of `l` are not used in the calculation because `m` has two fewer elements than `l`.

```
for x in l for y in m repeat
    sum := sum + x*y
```

Display the "dot product."

```
sum
```

$$700 \tag{5}$$

NonNegativeInteger

Next, we write a loop to compute the sum of the products of the loop elements with their positions in the loop.

```
l := [2,3,5,7,11,13,17,19,23,29,31,37]
```

$$[2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ 23,\ 29,\ 31,\ 37] \tag{6}$$

List (PositiveInteger)

The initial sum.

```
sum := 0
```

$$0 \tag{7}$$

NonNegativeInteger

Here looping stops when the list `l` is exhausted, even though the `for i in 0..` specifies no terminating condition.

```
for i in 0.. for x in l repeat sum := i * x
```

Display this weighted sum.

```
sum
```

```
                                407                                (9)
                                                    NonNegativeInteger
```

When "|" is used to qualify any of the `for` clauses in a parallel iteration, the variables in the predicates can be from an outer scope or from a `for` clause in or to the left of a modified clause.

This is correct:

```
for i in 1..10 repeat
  for j in 200..300 | odd? (i+j) repeat
    output [i,j]
```

This is not correct since the variable `j` has not been defined outside the inner loop.

```
for i in 1..10 | odd? (i+j) repeat  -- wrong, j not defined
  for j in 200..300 repeat
    output [i,j]
```

This example shows that it is possible to mix several of the forms of `repeat` modifying clauses on a loop.

```
for i in 1..10
    for j in 151..160 | odd? j
      while i + j < 160 repeat
        output [i,j]
```

```
   [1, 151]
   [3, 153]
```

Here are useful rules for composing loop expressions:

1. `while` predicates can only refer to variables that are global (or in an outer scope) or that are defined in `for` clauses to the left of the predicate.
2. A "such that" predicate (something following "|") must directly follow a `for` clause and can only refer to variables that are global (or in an outer scope) or defined in the modified `for` clause or any `for` clause to the left.

5.5 Creating Lists and Streams with Iterators

All of what we did for loops in Section 5.4 on page 131 can be transformed into expressions that create lists and streams. The `repeat, break` or `iterate` words are not used but all the other ideas carry over. Before we give you the general rule, here are some examples which give you the idea.

This creates a simple list of the integers from `1` to `10`.

```
mylist := [i for i in 1..10]
```

$$[1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10] \tag{1}$$

List (PositiveInteger)

Create a stream of the integers greater than or equal to 1.

```
mystream := [i for i in 1..]
```

$$[1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ \ldots] \tag{2}$$

Stream(PositiveInteger)

This is a list of the prime integers between 1 and 10, inclusive.

```
[i for i in 1..10 | prime? i]
```

$$[2,\ 3,\ 5,\ 7] \tag{3}$$

List (PositiveInteger)

This is a stream of the prime integers greater than or equal to 1.

```
[i for i in 1..   | prime? i]
```

$$[2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ \ldots] \tag{4}$$

Stream(PositiveInteger)

This is a list of the integers between 1 and 10, inclusive, whose squares are less than 700.

```
[i for i in 1..10 while i*i < 700]
```

$$[1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10] \tag{5}$$

List (PositiveInteger)

This is a stream of the integers greater than or equal to 1 whose squares are less than 700.

```
[i for i in 1..   while i*i < 700]
```

$$[1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ \ldots] \tag{6}$$

Stream(PositiveInteger)

Got the idea? Here is the general rule.

The general syntax of a collection is

$$\texttt{[}\ collectExpression\ \ iterator_1\ \ iterator_2\ \ \ldots iterator_N\ \texttt{]}$$

where each $iterator_i$ is either a `for` or a `while` clause. The loop terminates immediately when the end test of any $iterator_i$ succeeds or when a `return` expression is evaluated in *collectExpression*. The value returned by the collection is either a list or a stream of elements, one for each iteration of the *collectExpression*.

Be careful when you use `while` to create a stream. By default, FriCAS tries to compute and display the first ten elements of a stream. If the `while` condition is not satisfied quickly, FriCAS can spend a long (possibly infinite) time trying to compute the elements. Use `)set streams calculate` to change the default to something else. This also affects the number of terms computed and displayed for power series. For the purposes of this book, we have used this system command to display fewer than ten terms. Use nested iterators to create lists of lists which can then be given as an argument to matrix.

```
matrix [[x^i+j for i in 1..3] for j in 10..12]
```

$$\begin{bmatrix} x+10 & x^2+10 & x^3+10 \\ x+11 & x^2+11 & x^3+11 \\ x+12 & x^2+12 & x^3+12 \end{bmatrix} \tag{7}$$

Matrix(Polynomial(Integer))

You can also create lists of streams, streams of lists and streams of streams. Here is a stream of streams.

```
[[i/j for i in j+1..] for j in 1..]
```

$$\left[[2,\,3,\,4,\,5,\,6,\,7,\,8,\,\ldots],\left[\frac{3}{2},\,2,\,\frac{5}{2},\,3,\,\frac{7}{2},\,4,\,\frac{9}{2},\,\ldots\right],\left[\frac{4}{3},\,\frac{5}{3},\,2,\,\frac{7}{3},\,\frac{8}{3},\,3,\,\frac{10}{3},\,\ldots\right],\left[\frac{5}{4},\,\frac{3}{2},\,\frac{7}{4},\,2,\,\frac{9}{4},\,\frac{5}{2},\,\frac{11}{4},\,\ldots\right],\left[\frac{6}{5},\,\frac{7}{5},\,\frac{8}{5},\,\frac{9}{5},\,2,\,\frac{11}{5},\,\frac{12}{5},\,\ldots\right],\left[\frac{7}{6},\,\frac{4}{3},\,\frac{3}{2},\,\frac{5}{3},\,\frac{11}{6},\,2,\,\frac{13}{6},\,\ldots\right],\left[\frac{8}{7},\,\frac{9}{7},\,\frac{10}{7},\,\frac{11}{7},\,\frac{12}{7},\,\frac{13}{7},\,2,\,\ldots\right],\ldots\right] \tag{8}$$

Stream(Stream(Fraction(Integer)))

You can use parallel iteration across lists and streams to create new lists.

```
[i/j for i in 3.. by 10 for j in 2..]
```

$$\left[\frac{3}{2},\,\frac{13}{3},\,\frac{23}{4},\,\frac{33}{5},\,\frac{43}{6},\,\frac{53}{7},\,\frac{63}{8},\,\ldots\right] \tag{9}$$

Stream(Fraction(Integer))

Iteration stops if the end of a list or stream is reached.

```
[i^j for i in 1..7 for j in 2.. ]
```

$$[1,\,8,\,81,\,1024,\,15625,\,279936,\,5764801] \tag{10}$$

Stream(Integer)

As with loops, you can combine these modifiers to make very complicated conditions.

```
[[i,j] for i in 10..15 | prime? i] for j in 17..22 | j = squareFreePart j]
```

$$[[[11,\,17],\,[13,\,17]],\,[[11,\,19],\,[13,\,19]],\,[[11,\,21],\,[13,\,21]],\,[[11,\,22],\,[13,\,22]]] \tag{11}$$

List (List (List (PositiveInteger)))

See '`List`' on page 490 and '`Stream`' on page 577 for more information on creating and manipulating lists and streams, respectively.

5.6 An Example: Streams of Primes

We conclude this chapter with an example of the creation and manipulation of infinite streams of prime integers. This might be useful for experiments with numbers or other applications where you are using sequences of primes over and over again. As for all streams, the stream of primes is only computed as far out as you need. Once computed, however, all the primes up to that point are saved for future reference.

Two useful operations provided by the FriCAS library are prime? and nextPrime. A straight-forward way to create a stream of prime numbers is to start with the stream of positive integers `[2,..]` and filter out those that are prime. Create a stream of primes.

```
primes : Stream Integer := [i for i in 2.. | prime? i]
```

$$[2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ \ldots] \tag{1}$$

Stream(Integer)

A more elegant way, however, is to use the stream operation from **Stream**. Given an initial value `a` and a function `f`, stream constructs the stream `[a, f(a), f(f(a)), ...]`. This function gives you the quickest method of getting the stream of primes. This is how you use stream to generate an infinite stream of primes.

```
primes := stream(nextPrime,2)
```

$$[2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ \ldots] \tag{2}$$

Stream(Integer)

Once the stream is generated, you might only be interested in primes starting at a particular value.

```
smallPrimes := [p for p in primes | p > 1000]
```

$$[1009,\ 1013,\ 1019,\ 1021,\ 1031,\ 1033,\ 1039,\ \ldots] \tag{3}$$

Stream(Integer)

Here are the first 11 primes greater than 1000.

```
[p for p in smallPrimes for i in 1..11]
```

$$[1009,\ 1013,\ 1019,\ 1021,\ 1031,\ 1033,\ 1039,\ \ldots] \tag{4}$$

Stream(Integer)

Here is a stream of primes between 1000 and 1200.

```
[p for p in smallPrimes while p < 1200]
```

$$[1009,\ 1013,\ 1019,\ 1021,\ 1031,\ 1033,\ 1039,\ \ldots] \tag{5}$$

Stream(Integer)

To get these expanded into a finite stream, you call complete on the stream.

```
complete %
```

$$[1009,\ 1013,\ 1019,\ 1021,\ 1031,\ 1033,\ 1039,\ \ldots] \tag{6}$$

Stream(Integer)

Twin primes are consecutive odd number pairs which are prime. Here is the stream of twin primes.

```
twinPrimes := [[p,p+2] for p in primes | prime?(p + 2)]
```

$$[[3,\ 5],\ [5,\ 7],\ [11,\ 13],\ [17,\ 19],\ [29,\ 31],\ [41,\ 43],\ [59,\ 61],\ \ldots] \tag{7}$$

Stream(List(Integer))

Since we already have the primes computed we can avoid the call to prime? by using a double iteration. This time we'll just generate a stream of the first of the twin primes.

```
firstOfTwins:= [p for p in primes for q in rest primes | q=p+2]
```

$$[3,\ 5,\ 11,\ 17,\ 29,\ 41,\ 59,\ \ldots] \tag{8}$$

Stream(Integer)

Let's try to compute the infinite stream of triplet primes, the set of primes `p` such that `[p,p+2,p+4]` are primes. For example, `[3,5,7]` is a triple prime. We could do this by a triple `for` iteration. A more economical way is to use **firstOfTwins**. This time however, put a semicolon at the end of the line.

Put a semicolon at the end so that no elements are computed.

```
firstTriplets := [p for p in firstOfTwins for q in rest firstOfTwins | q = p+2];
```

Stream(Integer)

What happened? As you know, by default FriCAS displays the first ten elements of a stream when you first display it. And, therefore, it needs to compute them! If you want *no* elements computed, just terminate the expression by a semicolon ("`;`").[3]

Compute the first triplet prime.

```
firstTriplets.1
```

$$3 \tag{10}$$

PositiveInteger

If you want to compute another, just ask for it. But wait a second! Given three consecutive odd integers, one of them must be divisible by 3. Thus there is only one triplet prime. But suppose that you did not know this and wanted to know what was the tenth triplet prime.

[3]Why does this happen? The semi-colon prevents the display of the result of evaluating the expression. Since no stream elements are needed for display (or anything else, so far), none are computed.

```
firstTriplets.10
```

To compute the tenth triplet prime, FriCAS first must compute the second, the third, and so on. But since there isn't even a second triplet prime, FriCAS will compute forever. Nonetheless, this effort can produce a useful result. After waiting a bit, hit **Ctrl**–**c**. The system responds as follows.

```
>> System error:
Console interrupt.
You are being returned to the top level of
the interpreter.
```

Let's say that you want to know how many primes have been computed. Issue

```
numberOfComputedEntries primes
```

and, for this discussion, let's say that the result is `2045.` How big is the 2045^{th} prime?

```
primes.2045
```

$$17837 \tag{11}$$

PositiveInteger

What you have learned is that there are no triplet primes between 5 and 17837. Although this result is well known (some might even say trivial), there are many experiments you could make where the result is not known. What you see here is a paradigm for testing of hypotheses. Here our hypothesis could have been: "there is more than one triplet prime." We have tested this hypothesis for 17837 cases. With streams, you can let your machine run, interrupt it to see how far it has progressed, then start it up and let it continue from where it left off.

Chapter 6

User-Defined Functions, Macros and Rules

In this chapter we show you how to write functions and macros, and we explain how FriCAS looks for and applies them. We show some simple one-line examples of functions, together with larger ones that are defined piece-by-piece or through the use of piles.

6.1 Functions vs. Macros

A function is a program to perform some computation. Most functions have names so that it is easy to refer to them. A simple example of a function is one named abs which computes the absolute value of an integer. This is a use of the "absolute value" library function for integers.

```
abs(-8)
```

$$8 \tag{1}$$

PositiveInteger

This is an unnamed function that does the same thing, using the "maps-to" syntax "+->" that we discuss in Section 6.17 on page 179.

```
(x +-> if x < 0 then -x else x)(-8)
```

$$8 \tag{2}$$

PositiveInteger

Functions can be used alone or serve as the building blocks for larger programs. Usually they return a value that you might want to use in the next stage of a computation, but not always (for example, see 'Exit' on page 399 and 'Void' on page 605). They may also read data from your keyboard, move information from one place to another, or format and display results on your screen.

In FriCAS, as in mathematics, functions are usually *parameterized*. Each time you *call* (some people say *apply* or *invoke*) a function, you give values to the parameters (variables). Such a value is called

an *argument* of the function. FriCAS uses the arguments for the computation. In this way you get different results depending on what you "feed" the function.

Functions can have local variables or refer to global variables in the workspace. FriCAS can often *compile* functions so that they execute very efficiently. Functions can be passed as arguments to other functions.

Macros are textual substitutions. They are used to clarify the meaning of constants or expressions and to be templates for frequently used expressions. Macros can be parameterized but they are not objects that can be passed as arguments to functions. In effect, macros are extensions to the FriCAS expression parser.

6.2 Macros

A *macro* provides general textual substitution of an FriCAS expression for a name. You can think of a macro as being a generalized abbreviation. You can only have one macro in your workspace with a given name, no matter how many arguments it has.

The two general forms for macros are

`macro` *name* `==` *body*
`macro` *name(arg1,...)* `==` *body*

where the body of the macro can be any FriCAS expression.

For example, suppose you decided that you like to use `df` for D. You define the macro `df` like this.

```
macro df == D
```

Whenever you type `df`, the system expands it to D.

```
df(x^2 + x + 1,x)
```

$$2\,x + 1 \tag{2}$$

Polynomial(Integer)

Macros can be parameterized and so can be used for many different kinds of objects.

```
macro ff(x) == x^2 + 1
```

Apply it to a number, a symbol, or an expression.

```
ff z
```

$$z^2 + 1 \tag{4}$$

Polynomial(Integer)

Macros can also be nested, but you get an error message if you run out of space because of an infinite nesting loop.

```
macro gg(x) == ff(2*x - 2/3)
```

This new macro is fine as it does not produce a loop.

```
gg(1/w)
```

$$\frac{13\,w^2 - 24\,w + 36}{9\,w^2} \tag{6}$$

Fraction(Polynomial(Integer))

This, however, loops since `gg` is defined in terms of `ff`.

```
macro ff(x) == gg(-x)
```

The body of a macro can be a block.

```
macro next == (past := present; present := future; future := past + present)
```

Before entering `next`, we need values for `present` and `future`.

```
present : Integer := 0
```

$$0 \tag{9}$$

Integer

```
future : Integer := 1
```

$$1 \tag{10}$$

Integer

Repeatedly evaluating `next` produces the next Fibonacci number.

```
next
```

$$1 \tag{11}$$

Integer

And the next one.

```
next
```

$$2 \tag{12}$$

Integer

Here is the infinite stream of the rest of the Fibonacci numbers.

```
[next for i in 1..]
```

$$[3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ \ldots] \tag{13}$$

Stream(Integer)

Bundle all the above lines into a single macro.

```
macro fibStream ==
  present : Integer := 1
  future : Integer := 1
  [next for i in 1..] where
    macro next ==
      past := present
      present := future
      future := past + present
```

Use concat to start with the first two Fibonacci numbers.

```
concat([1,1],fibStream)
```

$$[1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ \ldots] \tag{15}$$

Stream(Integer)

An easier way to compute these numbers is to use the library operation fibonacci.

```
[fibonacci i for i in 1..]
```

$$[1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ \ldots] \tag{16}$$

Stream(Integer)

6.3 Introduction to Functions

Each name in your workspace can refer to a single object. This may be any kind of object including a function. You can use interactively any function from the library or any that you define in the workspace. In the library, the same name can have many functions, but you can have only one function with a given name, although it can have any number of arguments that you choose.

If you define a function in the workspace that has the same name and number of arguments as one in the library, then your definition takes precedence. In fact, to get the library function you must *package-call* it (see Section 2.9 on page 89).

To use a function in FriCAS, you apply it to its arguments. Most functions are applied by entering the name of the function followed by its argument or arguments.

```
factor(12)
```

$$2^2\,3 \tag{1}$$

Factored(Integer)

Some functions like + have *infix operators* as names.

```
3 + 4
```

$$7 \tag{2}$$

PositiveInteger

The function + has two arguments. When you give it more than two arguments, FriCAS groups the arguments to the left. This expression is equivalent to `(1 + 2)+ 7`.

```
1 + 2 + 7
```

$$10 \tag{3}$$

PositiveInteger

All operations, including infix operators, can be written in prefix form, that is, with the operation name followed by the arguments in parentheses. For example, `2 + 3` can alternatively be written as `+(2,3)`. But `+(2,3,4)` is an error since + takes only two arguments.

Prefix operations are generally applied before the infix operation. Thus `factorial 3 + 1` means `factorial(3)+ 1` producing 7, and `- 2 + 5` means `(-2) + 5` producing 3. An example of a prefix operator is prefix -. For example, `- 2 + 5` converts to `(- 2) + 5` producing the value 3. Any prefix function taking two arguments can be written in an infix manner by putting an ampersand ("&") before the name. Thus `D(2*x,x)` can be written as `2*x &D x` returning 2.

Every function in FriCAS is identified by a *name* and *type*.[1] The type of a function is always a mapping of the form `Source →Target` where `Source` and `Target` are types. To enter a type from the keyboard, enter the arrow by using a hyphen "-" followed by a greater-than sign ">", e.g. **Integer→Integer**.

Let's go back to +. There are many + functions in the FriCAS library: one for integers, one for floats, another for rational numbers, and so on. These + functions have different types and thus are different functions. You've seen examples of this *overloading* before—using the same name for different functions. Overloading is the rule rather than the exception. You can add two integers, two polynomials, two matrices or two power series. These are all done with the same function name but with different functions.

6.4 Declaring the Type of Functions

In Section 2.3 on page 74 we discussed how to declare a variable to restrict the kind of values that can be assigned to it. In this section we show how to declare a variable that refers to function objects.

A function is an object of type

`Source→Type`

where `Source` and `Target` can be any type. A common type for `Source` is $(T_1, \ldots, T_n)$, to indicate a function of `n` arguments.

If `g` takes an **Integer**, a **Float** and another **Integer**, and returns a **String**, the declaration is written this way.

[1] An exception is an "anonymous function" discussed in Section 6.17 on page 179.

```
g: (Integer,Float,Integer) -> String
```

The types need not be written fully; using abbreviations, the above declaration is:

```
g: (INT,FLOAT,INT) -> STRING
```

It is possible for a function to take no arguments. If `h` takes no arguments but returns a **Polynomial Integer**, any of the following declarations is acceptable.

```
h: () -> POLY INT
```

```
h: () -> Polynomial INT
```

```
h: () -> POLY Integer
```

> Functions can also be declared when they are being defined. The syntax for combined declaration/definition is:
>
> $$functionName(parm_1\texttt{:}\ parmType_1\texttt{,}\ \ldots\texttt{,}\ parm_N\texttt{:}\ parmType_N)\texttt{:}\ functionReturnType$$

The following definition fragments show how this can be done for the functions `g` and `h` above.

```
g(arg1: INT, arg2: FLOAT, arg3: INT): STRING == ...
```

```
h(): POLY INT == ...
```

A current restriction on function declarations is that they must involve fully specified types (that is, cannot include modes involving explicit or implicit "?"). For more information on declaring things in general, see Section 2.3 on page 74.

6.5 One-Line Functions

As you use FriCAS, you will find that you will write many short functions to codify sequences of operations that you often perform. In this section we write some simple one-line functions.

This is a simple recursive factorial function for positive integers.

```
fac n == if n < 3 then n else n * fac(n-1)
```

```
fac 10
```

```
   Compiling function fac with type Integer -> Integer
```

$$3628800 \tag{2}$$

PositiveInteger

This function computes `1 + 1/2 + 1/3 + ... + 1/n`.

```
s n == reduce(+,[1/i for i in 1..n])
```

```
s 50
```

```
   Compiling function s with type PositiveInteger -> Fraction(Integer)
```

$$\frac{13943237577224054960759}{309904450424599670640 0} \tag{4}$$

Fraction (Integer)

This function computes a Mersenne number, several of which are prime.

```
mersenne i == 2^i - 1
```

If you type `mersenne`, FriCAS shows you the function definition.

```
mersenne
```

$$mersenne\,(i) \quad == \; 2^i - 1 \tag{6}$$

FunctionCalled (mersenne)

Generate a stream of Mersenne numbers.

```
[mersenne i for i in 1..]
```

```
   Compiling function mersenne with type PositiveInteger -> Integer
```

$$[1,\ 3,\ 7,\ 15,\ 31,\ 63,\ 127,\ \ldots] \tag{7}$$

Stream(Integer)

Create a stream of those values of `i` such that `mersenne(i)` is prime.

```
mersenneIndex := [n for n in 1.. | prime?(mersenne(n))]
```

$$[2,\ 3,\ 5,\ 7,\ 13,\ 17,\ 19,\ \ldots] \tag{8}$$

Stream(PositiveInteger)

Finally, write a function that returns the n^{th} Mersenne prime.

```
mersennePrime n == mersenne mersenneIndex(n)
```

```
mersennePrime 5
```

```
   Compiling function mersennePrime with type PositiveInteger -> Integer
```

$$8191 \tag{10}$$

PositiveInteger

6.6 Declared vs. Undeclared Functions

If you declare the type of a function, you can apply it to any data that can be converted to the source type of the function.

Define **f** with type **Integer→Integer**.

```
f(x: Integer): Integer == x + 1
```

```
   Function declaration f : Integer -> Integer has been added to workspace.
```

The function **f** can be applied to integers, ...

```
f 9
```

```
   Compiling function f with type Integer -> Integer
```

$$10 \tag{2}$$

PositiveInteger

and to values that convert to integers, ...

```
f(-2.0)
```

$$-1 \tag{3}$$

Integer

but not to values that cannot be converted to integers.

```
f(2/3)
```

```
   Conversion failed in the compiled user function f .
   Cannot convert the value from type Fraction(Integer) to Integer .
```

To make the function over a wide range of types, do not declare its type. Give the same definition with no declaration.

```
g x == x + 1
```

If `x + 1` makes sense, you can apply **g** to `x`.

```
g 9
```

```
   Compiling function g with type PositiveInteger -> PositiveInteger
```

$$10 \tag{5}$$

PositiveInteger

A version of **g** with different argument types gets compiled for each new kind of argument used.

```
g(2/3)
```

```
   Compiling function g with type Fraction(Integer) -> Fraction(Integer)
```

$$\frac{5}{3} \tag{6}$$

Fraction (Integer)

Here `x+1` for `x = "fricas"` makes no sense.

```
g("fricas")
```

```
   There are 13 exposed and 13 unexposed library operations named + having 2
      argument(s) but none was determined to be applicable. Use HyperDoc Browse,
      or issue
                                    )display op +
      to learn more about the available operations. Perhaps package-calling the
      operation or using coercions on the arguments will allow you to apply the
      operation.
```

```
Cannot find a definition or applicable library operation named + with argument
   type(s)
                                    String
                                PositiveInteger

   Perhaps you should use "@" to indicate the required return type, or "$" to
   specify which version of the function you need.
FriCAS will attempt to step through and interpret the code.
There are 13 exposed and 13 unexposed library operations named + having 2
   argument(s) but none was determined to be applicable. Use HyperDoc Browse,
   or issue
                                 )display op +
   to learn more about the available operations. Perhaps package-calling the
   operation or using coercions on the arguments will allow you to apply the
   operation.
Cannot find a definition or applicable library operation named + with argument
   type(s)
                                    String
                                PositiveInteger

   Perhaps you should use "@" to indicate the required return type, or "$" to
   specify which version of the function you need.
```

As you will see in Chapter 12, FriCAS has a formal idea of categories for what "makes sense."

6.7 Functions vs. Operations

A function is an object that you can create, manipulate, pass to, and return from functions (for some interesting examples of library functions that manipulate functions, see '`MappingPackage1`' on page 500). Yet, we often seem to use the term *operation* and function interchangeably in FriCAS. What is the distinction?

First consider values and types associated with some variable `n` in your workspace. You can make the declaration `n : Integer`, then assign `n` an integer value. You then speak of the integer `n`. However, note that the integer is not the name `n` itself, but the value that you assign to `n`.

Similarly, you can declare a variable `f` in your workspace to have type **Integer→Integer**, then assign `f`, through a definition or an assignment of an anonymous function. You then speak of the function `f`. However, the function is not `f`, but the value that you assign to `f`.

A function is a value, in fact, some machine code for doing something. Doing what? Well, performing some *operation*. Formally, an operation consists of the constituent parts of `f` in your workspace, excluding the value; thus an operation has a name and a type. An operation is what domains and packages export. Thus **Ring** exports one operation `+`. Every ring also exports this operation. Also, the author of every ring in the system is obliged under contract (see Section 11.3 on page 649) to provide an implementation for this operation.

This chapter is all about functions—how you create them interactively and how you apply them to meet your needs. In Chapter 11 you will learn how to create them for the FriCAS library. Then in Chapter 12, you will learn about categories and exported operations.

6.8 Delayed Assignments vs. Functions with No Arguments

In Section 5.1 on page 123 we discussed the difference between immediate and delayed assignments. In this section we show the difference between delayed assignments and functions of no arguments.

A function of no arguments is sometimes called a *nullary function.*

```
sin24() == sin(24.0)
```

You must use the parentheses ("`()`") to evaluate it. Like a delayed assignment, the right-hand-side of a function evaluation is not evaluated until the left-hand-side is used.

```
sin24()
```

```
   Compiling function sin24 with type () -> Float
```

$$-0.90557836200662384514 \tag{2}$$

Float

If you omit the parentheses, you just get the function definition.

```
sin24
```

$$sin24() \quad == \quad \sin(24.0) \tag{3}$$

FunctionCalled(sin24)

You do not use the parentheses "`()`" in a delayed assignment...

```
cos24 == cos(24.0)
```

nor in the evaluation.

```
cos24
```

```
   Compiling body of rule cos24 to compute value of type Float
```

$$0.42417900733699697594 \tag{5}$$

Float

The only syntactic difference between delayed assignments and nullary functions is that you use "`()`" in the latter case.

6.9 How FriCAS Determines What Function to Use

What happens if you define a function that has the same name as a library function? Well, if your function has the same name and number of arguments (we sometimes say *arity*) as another function in the library, then your function covers up the library function. If you want then to call the library function, you will have to package-call it. FriCAS can use both the functions you write and those that come from the library. Let's do a simple example to illustrate this. Suppose you (wrongly!) define **sin** in this way.

```
sin x == 1.0
```

The value `1.0` is returned for any argument.

```
sin 4.3
```

```
   Compiling function sin with type Float -> Float
```

$$1.0 \tag{2}$$

Float

If you want the library operation, we have to package-call it (see Section 2.9 on page 89 for more information).

```
sin(4.3)$Float
```

$$-0.91616593674945498404 \tag{3}$$

Float

```
sin(34.6)$Float
```

$$-0.042468034716950101543 \tag{4}$$

Float

Even worse, say we accidentally used the same name as a library function in the function.

```
sin x == sin x
```

```
   Compiled code for sin has been cleared.
   1 old definition(s) deleted for function or rule sin
```

Then FriCAS definitely does not understand us.

```
sin 4.3
```

```
   FriCAS cannot determine the type of sin because it cannot analyze the
      non-recursive part, if that exists. This may be remedied by declaring the
      function.
```

Again, we could package-call the inside function.

```
sin x == sin(x)$Float
```

```
   1 old definition(s) deleted for function or rule sin
```

```
sin 4.3
```

```
   Compiling function sin with type Float -> Float
```

$$-0.91616593674945498404 \tag{7}$$

Float

Of course, you are unlikely to make such obvious errors. It is more probable that you would write a function and in the body use a function that you think is a library function. If you had also written a function by that same name, the library function would be invisible.

How does FriCAS determine what library function to call? It very much depends on the particular example, but the simple case of creating the polynomial `x + 2/3` will give you an idea.

1. The `x` is analyzed and its default type is **Variable(x)**.
2. The `2` is analyzed and its default type is **PositiveInteger**.

3. The `3` is analyzed and its default type is **PositiveInteger**.
4. Because the arguments to `/` are integers, FriCAS gives the expression `2/3` a default target type of **Fraction(Integer)**.
5. FriCAS looks in **PositiveInteger** for `/`. It is not found.
6. FriCAS looks in **Fraction(Integer)** for `/`. It is found for arguments of type **Integer**.
7. The `2` and `3` are converted to objects of type **Integer** (this is trivial) and `/` is applied, creating an object of type **Fraction(Integer)**.
8. No `+` for arguments of types **Variable(x)** and **Fraction(Integer)** are found in either domain.
9. FriCAS resolves (see Section 2.10 on page 92) the types and gets **Polynomial (Fraction (Integer))**.
10. The `x` and the `2/3` are converted to objects of this type and `+` is applied, yielding the answer, an object of type **Polynomial (Fraction (Integer))**.

6.10 Compiling vs. Interpreting

When possible, FriCAS completely determines the type of every object in a function, then translates the function definition to Common LISP or to machine code (see next section). This translation, called *compilation*, happens the first time you call the function and results in a computational delay. Subsequent function calls with the same argument types use the compiled version of the code without delay.

If FriCAS cannot determine the type of everything, the function may still be executed but in *interpret-code mode*: each statement in the function is analyzed and executed as the control flow indicates. This process is slower than executing a compiled function, but it allows the execution of code that may involve objects whose types change.

> If FriCAS decides that it cannot compile the code, it issues a message stating the problem and then the following message:
>
> **We will attempt to step through and interpret the code.**
>
> This is not a time to panic. Rather, it just means that what you gave to FriCAS is somehow ambiguous: either it is not specific enough to be analyzed completely, or it is beyond FriCAS's present interactive compilation abilities.

This function runs in interpret-code mode, but it does not compile.

```
varPolys(vars) ==
  for var in vars repeat
    output(1 :: UnivariatePolynomial(var,Integer))
```

For `vars` equal to `['x, 'y, 'z]`, this function displays `1` three times.

```
varPolys ['x,'y,'z]
```

```
   Cannot compile conversion for types involving local variables. In particular,
      could not compile the expression involving :: UnivariatePolynomial(var,
      Integer)
   FriCAS will attempt to step through and interpret the code.
   1
   1
   1
```

The type of the argument to output changes in each iteration, so FriCAS cannot compile the function. In this case, even the inner loop by itself would have a problem:

```
for var in [’x,’y,’z] repeat
  output(1 :: UnivariatePolynomial(var,Integer))
```

```
   Cannot compile conversion for types involving local variables. In particular,
      could not compile the expression involving :: UnivariatePolynomial(var,
      Integer)
   FriCAS will attempt to step through and interpret the code.
   1
   1
   1
```

Sometimes you can help a function to compile by using an extra conversion or by using pretend. See Section 2.8 on page 86 for details.

When a function is compilable, you have the choice of whether it is compiled to Common LISP and then interpreted by the Common LISP interpreter or then further compiled from Common LISP to machine code. The option is controlled via `)set functions compile`. Issue `)set functions compile on` to compile all the way to machine code. With the default setting `)set functions compile off`, FriCAS has its Common LISP code interpreted because the overhead of further compilation is larger than the run-time of most of the functions our users have defined. You may find that selectively turning this option on and off will give you the best performance in your particular application. For example, if you are writing functions for graphics applications where hundreds of points are being computed, it is almost certainly true that you will get the best performance by issuing `)set functions compile on`.

6.11 Piece-Wise Function Definitions

To move beyond functions defined in one line, we introduce in this section functions that are defined piece-by-piece. That is, we say “use this definition when the argument is such-and-such and use this other definition when the argument is that-and-that.”

6.11.1 A Basic Example

There are many other ways to define a factorial function for nonnegative integers. You might say factorial of 0 is 1; otherwise, factorial of n is n times factorial of n-1. Here is one way to do this in FriCAS. Here is the value for `n = 0`.

```
fact(0) == 1
```

Here is the value for `n > 0`. The vertical bar “|” means “such that”.

```
fact(n | n > 0) == n * fact(n - 1)
```

What is the value for `n = 3`?

```
fact(3)
```

```
   Compiling function fact with type Integer -> Integer
   Compiling function fact as a recurrence relation.
```

$$6 \tag{3}$$

PositiveInteger

What is the value for `n = -3`?

```
fact(-3)
```

```
   You did not define fact for argument -3 .
```

Now for a second definition. Here is the value for `n = 0`.

```
facto(0) == 1
```

Give an error message if `n < 0`.

```
facto(n | n < 0) == error "arguments to facto must be non-negative"
```

Here is the value otherwise.

```
facto(n) == n * facto(n - 1)
```

What is the value for `n = 7`?

```
facto(7)
```

```
   Compiling function facto with type Integer -> Integer
```

$$5040 \tag{7}$$

PositiveInteger

What is the value for `n = -7`?

```
facto(-7)
```

```
   Error signalled from user code in function facto:
      arguments to facto must be non-negative
```

To see the current piece-wise definition of a function, use `)display value`.

```
)display value facto
```

```
   CONCAT
      Definition:
  ,
      facto0 == 1
      facto(n | n < 0) == error(arguments to facto must be non-negative)
      facto(n) == n facto(n - 1)
```

In general a *piece-wise definition* of a function consists of two or more parts. Each part gives a "piece" of the entire definition. FriCAS collects the pieces of a function as you enter them. When you ask for a value of the function, it then "glues" the pieces together to form a function.

The two piece-wise definitions for the factorial function are examples of recursive functions, that is, functions that are defined in terms of themselves. Here is an interesting doubly-recursive function. This function returns the value `11` for all positive integer arguments. Here is the first of two pieces.

```
eleven(n | n < 1) == n + 11
```

And the general case.

```
eleven(m) == eleven(eleven(m - 12))
```

Compute elevens, the infinite stream of values of eleven.

```
elevens := [eleven(i) for i in 0..]
```

```
   Compiling function eleven with type Integer -> Integer
```

$$[11,\ 11,\ 11,\ 11,\ 11,\ 11,\ 11,\ \ldots] \tag{10}$$

Stream(Integer)

What is the value at n = 200?

```
elevens 200
```

$$11 \tag{11}$$

PositiveInteger

What is the FriCAS's definition of eleven?

```
)display value eleven
```

```
   CONCAT
      Definition:
  ,
      eleven(m | m < 1) == m + 11
      eleven(m) == eleven(eleven(m - 12))
```

6.11.2 Picking Up the Pieces

Here are the details about how FriCAS creates a function from its pieces. FriCAS converts the i^{th} piece of a function definition into a conditional expression of the form: `if` $pred_i$ `then` $expression_i$. If any new piece has a $pred_i$ that is identical[2] to an earlier $pred_j$, the earlier piece is removed. Otherwise, the new piece is always added at the end.

> If there are n pieces to a function definition for f, the function defined f is:
>
> `if` $pred_1$ `then` $expression_1$ `else`
>
> . . .
>
> `if` $pred_n$ `then` $expression_n$ `else`
>
> `error "You did not define f for argument <arg>."`

You can give definitions of any number of mutually recursive function definitions, piece-wise or otherwise. No computation is done until you ask for a value. When you do ask for a value, all the relevant definitions are gathered, analyzed, and translated into separate functions and compiled.

Let's recall the definition of **eleven** from the previous section.

```
eleven(n | n < 1) == n + 11
```

[2] after all variables are uniformly named

```
eleven(m) == eleven(eleven(m - 12))
```

A similar doubly-recursive function below produces `-11` for all negative integers. If you haven't worked out why or how **eleven** works, the structure of this definition gives a clue. This definition we write as a block.

```
minusEleven(n) ==
  n >= 0 => n - 11
  minusEleven(5 + minusEleven(n + 7))
```

Define `s(n)` to be the sum of plus and minus "eleven" functions divided by `n`. Since `11 - 11 = 0`, we define `s(0)` to be `1`.

```
s(0) == 1
```

And the general term.

```
s(n) == (eleven(n) + minusEleven(n))/n
```

What are the first ten values of `s`?

```
[s(n) for n in 0..]
```

```
   Compiling function eleven with type Integer -> Integer
   Compiling function minusEleven with type Integer -> Integer
   Compiling function s with type NonNegativeInteger -> Fraction(Integer)
```

$$[1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ \ldots] \tag{6}$$

Stream(Fraction(Integer))

FriCAS can create infinite streams in the positive direction (for example, for index values $0, 1, \ldots$) or negative direction (for example, for index values $0, -1, -2, \ldots$). Here we would like a stream of values of `s(n)` that is infinite in both directions. The function `t(n)` below returns the n^{th} term of the infinite stream $[s(0), s(1), s(-1), s(2), s(-2), \ldots]$. Its definition has three pieces. Define the initial term.

```
t(1) == s(0)
```

The even numbered terms are the `s(i)` for positive `i`. We use `quo` rather than `/` since we want the result to be an integer.

```
t(n | even?(n)) == s(n quo 2)
```

Finally, the odd numbered terms are the `s(i)` for negative `i`. In piece-wise definitions, you can use different variables to define different pieces. FriCAS will not get confused.

```
t(p) == s(- p quo 2)
```

Look at the definition of `t`. In the first piece, the variable `n` was used; in the second piece, `p`. FriCAS always uses your last variable to display your definitions back to you.

```
)display value t
```

```
   CONCAT
      Definition:
  ,
      t1 == s(0)
      t(p | even?(p)) == s(p quo 2)
      t(p) == s(- p quo 2)
```

Create a series of values of `s` applied to alternating positive and negative arguments.

```
[t(i) for i in 1..]
```

```
   Compiling function s with type Integer -> Fraction(Integer)
   Compiling function t with type PositiveInteger -> Fraction(Integer)
```

$$[1, 1, 1, 1, 1, 1, 1, \ldots] \tag{10}$$

Stream(Fraction(Integer))

Evidently `t(n) = 1` for all `i`. Check it at `n= 100`.

```
t(100)
```

$$1 \tag{11}$$

Fraction (Integer)

6.11.3 Predicates

We have already seen some examples of predicates (Section 6.11.1 on page 161). Predicates are **Boolean**-valued expressions and FriCAS uses them for filtering collections (see Section 5.5 on page 143) and for placing constraints on function arguments. In this section we discuss their latter usage.

The simplest use of a predicate is one you don't see at all.

```
opposite 'right == 'left
```

Here is a longer way to give the "opposite definition."

```
opposite (x | x = 'left) == 'right
```

Try it out.

```
for x in ['right, 'left] repeat output opposite x
```

```
   Compiling function opposite with type OrderedVariableList([right,left]) ->
      Symbol
   left
   right
```

We get an error if there is no definition for a given argument.

```
opposite('inbetween)
```

```
   Compiling function opposite with type Variable(inbetween) -> Symbol
   The function opposite is not defined for the given argument(s).
```

Explicit predicates tell FriCAS that the given function definition piece is to be applied if the predicate evaluates to `true` for the arguments to the function. You can use such "constant" arguments for integers, strings, and quoted symbols. The **Boolean** values `true` and `false` can also be used if qualified with "`@`" or "`$`" and **Boolean**. The following are all valid function definition fragments using constant arguments.

```
a(1) == ...
b("unramified") == ...
c('untested) == ...
d(true@Boolean) == ...
```

If a function has more than one argument, each argument can have its own predicate. However, if a predicate involves two or more arguments, it must be given *after* all the arguments mentioned in the predicate have been given. You are always safe to give a single predicate at the end of the argument list. A function involving predicates on two arguments.

```
inFirstHalfQuadrant(x | x > 0,y | y < x) == true
```

This is incorrect as it gives a predicate on `y` before the argument `y` is given.

```
inFirstHalfQuadrant(x | x > 0 and y < x,y) == true
```

```
   1 old definition(s) deleted for function or rule inFirstHalfQuadrant
```

It is always correct to write the predicate at the end.

```
inFirstHalfQuadrant(x,y | x > 0 and y < x) == true
```

```
   1 old definition(s) deleted for function or rule inFirstHalfQuadrant
```

Here is the rest of the definition.

```
inFirstHalfQuadrant(x,y) == false
```

Try it out.

```
[inFirstHalfQuadrant(i,3) for i in 1..5]
```

```
   Compiling function inFirstHalfQuadrant with type (PositiveInteger,
      PositiveInteger) -> Boolean
```

$$[\texttt{false}, \texttt{false}, \texttt{false}, \texttt{true}, \texttt{true}] \tag{8}$$

List (Boolean)

Remark: Very old versions of FriCAS allowed predicates to be given after a `when` keyword as in `inFirstHalfQuadrant(x ,y) == true when x >0 and y < x`. This is no longer supported, is WRONG, and will cause a syntax error or strange behavior.

6.12 Caching Previously Computed Results

By default, FriCAS does not save the values of any function. You can cause it to save values and not to recompute unnecessarily by using `)set functions cache`. This should be used before the functions are defined or, at least, before they are executed. The word following "cache" should be `0` to turn off caching, a positive integer `n` to save the last `n` computed values or "all" to save all computed values. If you then give a list of names of functions, the caching only affects those functions. Use no list of names when you want to define the default behavior for functions not specifically mentioned in other `)set functions cache` statements. If you give no list of names, all functions will have the caching behavior. If you explicitly turn on caching for one or more names, you must explicitly turn off caching for those names when you want to stop saving their values.

This causes the functions **f** and **g** to have the last three computed values saved.

```
)set functions cache 3 f g
```

```
   function f will cache the last 3 values.
   function g will cache the last 3 values.
```

This is a sample definition for **f**.

```
f x == factorial(2^x)
```

A message is displayed stating what **f** will cache.

```
f(4)
```

```
   Compiling function f with type PositiveInteger -> Integer
   f will cache 3 most recently computed value(s).
```

$$20922789888000 \tag{2}$$

PositiveInteger

This causes all other functions to have all computed values saved by default.

```
)set functions cache all
```

```
   In general, interpreter functions will cache all values.
```

This causes all functions that have not been specifically cached in some way to have no computed values saved.

```
)set functions cache 0
```

```
 In general, functions will cache no returned values.
```

We also make **f** and **g** uncached.

```
)set functions cache 0 f g
```

```
   Caching for function f is turned off
   Caching for function g is turned off
```

> Be careful about caching functions that have *side effects*. Such a function might destructively modify the elements of an array or issue a draw command, for example. A function that you expect to execute every time it is called should not be cached. Also, it is highly unlikely that a function with no arguments should be cached.

You should also be careful about caching functions that depend on free variables. See Section 6.16 on page 175 for an example.

6.13 Recurrence Relations

One of the most useful classes of function are those defined via a "recurrence relation." A *recurrence relation* makes each successive value depend on some or all of the previous values. A simple example is the ordinary "factorial" function:

```
fact(0) == 1
fact(n | n > 0) == n * fact(n-1)
```

The value of `fact(10)` depends on the value of `fact(9)`, `fact(9)` on `fact(8)`, and so on. Because it depends on only one previous value, it is usually called a *first order recurrence relation.* You can easily imagine a function based on two, three or more previous values. The Fibonacci numbers are probably the most famous function defined by a second order recurrence relation. The library function fibonacci computes Fibonacci numbers. It is obviously optimized for speed.

```
[fibonacci(i) for i in 0..]
```

$$[0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ \ldots] \tag{1}$$

Stream(Integer)

Define the Fibonacci numbers ourselves using a piece-wise definition.

```
fib(1) == 1

fib(2) == 1

fib(n) == fib(n-1) + fib(n-2)
```

As defined, this recurrence relation is obviously doubly-recursive. To compute `fib(10)`, we need to compute `fib(9)` and `fib(8)`. And to `fib(9)`, we need to compute `fib(8)` and `fib(7)`. And so on. It seems that to compute `fib(10)` we need to compute `fib(9)` once, `fib(8)` twice, `fib(7)` three times. Look familiar? The number of function calls needed to compute *any* second order recurrence relation in the obvious way is exactly `fib(n)`. These numbers grow! For example, if FriCAS actually did this, then `fib(500)` requires more than 10^{104} function calls. And, given all this, our definition of **fib** obviously could not be used to calculate the five-hundredth Fibonacci number. Let's try it anyway.

```
fib(500)
```

```
   Compiling function fib with type Integer -> PositiveInteger
   Compiling function fib as a recurrence relation.
```

$$139423224561697880139724382870407283950070256587697307264108962948325571622863290691557658876222521294125 \tag{2}$$

PositiveInteger

Since this takes a short time to compute, it obviously didn't do as many as 10^{104} operations! By default, FriCAS transforms any recurrence relation it recognizes into an iteration. Iterations are efficient. To compute the value of the n^{th} term of a recurrence relation using an iteration requires only `n` function calls.[3]

To turn off this special recurrence relation compilation, issue

```
)set functions recurrence off
```

To turn it back on, substitute "`on`" for "`off`".

The transformations that FriCAS uses for **fib** cache the last two values.[4] If, after computing a value for **fib**, you ask for some larger value, FriCAS picks up the cached values and continues computing from there. See Section 6.16 on page 175 for an example of a function definition that has this same behavior. Also see Section 6.12 on page 166 for a more general discussion of how you can cache function values.

Recurrence relations can be used for defining recurrence relations involving polynomials, rational functions, or anything you like. Here we compute the infinite stream of Legendre polynomials. The Legendre polynomial of degree `0`.

```
p(0) == 1
```

The Legendre polynomial of degree `1`.

[3] If you compare the speed of our **fib** function to the library function, our version is still slower. This is because the library fibonacci uses a "powering algorithm" with a computing time proportional to $\log^3(n)$ to compute `fibonacci(n)`.

[4] For a more general k^{th} order recurrence relation, FriCAS caches the last `k` values.

```
p(1) == x
```

The Legendre polynomial of degree `n`.

```
p(n) == ((2*n-1)*x*p(n-1) - (n-1)*p(n-2))/n
```

Compute the Legendre polynomial of degree `6`.

```
p(6)
```

```
   Compiling function p with type Integer -> Polynomial(Fraction(Integer))
   Compiling function p as a recurrence relation.
```

$$\frac{231}{16}x^6 - \frac{315}{16}x^4 + \frac{105}{16}x^2 - \frac{5}{16} \tag{9}$$

Polynomial(Fraction (Integer))

6.14 Making Functions from Objects

There are many times when you compute a complicated expression and then wish to use that expression as the body of a function. FriCAS provides an operation called function to do this. It creates a function object and places it into the workspace. There are several versions, depending on how many arguments the function has. The first argument to function is always the expression to be converted into the function body, and the second is always the name to be used for the function. For more information, see '`MakeFunction`' on page 498.

Start with a simple example of a polynomial in three variables.

```
p := -x + y^2 - z^3
```

$$-z^3 + y^2 - x \tag{1}$$

Polynomial(Integer)

To make this into a function of no arguments that simply returns the polynomial, use the two argument form of function.

```
function(p,'f0)
```

$$f0 \tag{2}$$

Symbol

To avoid possible conflicts (see below), it is a good idea to always quote this second argument.

```
f0
```

$$f0\,() \quad == \quad -z^3 + y^2 - x \tag{3}$$

FunctionCalled (f0)

This is what you get when you evaluate the function.

```
f0()
```

```
Compiling function f0 with type () -> Polynomial(Integer)
```

$$-z^3+y^2-x \tag{4}$$

Polynomial(Integer)

To make a function in x, use a version of function that takes three arguments. The last argument is the name of the variable to use as the parameter. Typically, this variable occurs in the expression and, like the function name, you should quote it to avoid possible confusion.

```
function(p,'f1,'x)
```

$$f1 \tag{5}$$

Symbol

This is what the new function looks like.

```
f1
```

$$f1\,(x) \quad == \quad -z^3+y^2-x \tag{6}$$

FunctionCalled (f1)

This is the value of **f1** at x = 3. Notice that the return type of the function is **Polynomial (Integer)**, the same as p.

```
f1(3)
```

```
Compiling function f1 with type PositiveInteger -> Polynomial(Integer)
```

$$-z^3+y^2-3 \tag{7}$$

Polynomial(Integer)

To use x and y as parameters, use the four argument form of function.

```
function(p,'f2,'x,'y)
```

$$f2 \tag{8}$$

Symbol

```
f2
```

$$f2\,(x,\,y) \quad == \quad -z^3+y^2-x \tag{9}$$

FunctionCalled (f2)

Evaluate f2 at x = 3 and y = 0. The return type of **f2** is still **Polynomial(Integer)** because the variable z is still present and not one of the parameters.

```
f2(3,0)
```

```
Compiling function f2 with type (PositiveInteger, NonNegativeInteger) ->
   Polynomial(Integer)
```

$$-z^3-3 \tag{10}$$

Polynomial(Integer)

Finally, use all three variables as parameters. There is no five argument form of function, so use the one with three arguments, the third argument being a list of the parameters.

```
function(p,'f3,['x,'y,'z])
```

$$f3 \tag{11}$$

Symbol

Evaluate this using the same values for x and y as above, but let z be -6. The result type of **f3** is **Integer**.

```
f3
```

$$f3\,(x,\ y,\ z) \quad == \quad -z^3+y^2-x \tag{12}$$

FunctionCalled(f3)

```
f3(3,0,-6)
```

```
   Compiling function f3 with type (PositiveInteger, NonNegativeInteger, Integer)
       -> Integer
```

$$213 \tag{13}$$

PositiveInteger

The four functions we have defined via p have been undeclared. To declare a function whose body is to be generated by function, issue the declaration *before* the function is created.

```
g: (Integer, Integer) -> Float
```

```
D(sin(x-y)/cos(x+y),x)
```

$$\frac{-\sin(y-x)\,\sin(y+x)+\cos(y-x)\,\cos(y+x)}{\cos(y+x)^2} \tag{15}$$

Expression(Integer)

```
function(%,'g,'x,'y)
```

$$g \tag{16}$$

Symbol

```
g
```

$$g\,(x,\ y) \quad == \quad \frac{-\sin(y-x)\,\sin(y+x)+\cos(y-x)\,\cos(y+x)}{\cos(y+x)^2} \tag{17}$$

FunctionCalled(g)

It is an error to use g without the quote in the penultimate expression since g had been declared but did not have a value. Similarly, since it is common to overuse variable names like x, y, and so on, you avoid problems if you always quote the variable names for function. In general, if x has a value and you use x without a quote in a call to function, then FriCAS does not know what you are trying to do.

What kind of object is allowable as the first argument to function? Let's use the Browse facility of HyperDoc to find out. At the main Browse menu, enter the string `function` and then click on **Operations.** The exposed operations called function all take an object whose type belongs to category **ConvertibleTo InputForm**. What domains are those? Go back to the main Browse menu, erase `function`, enter `ConvertibleTo` in the input area, and click on **categories** on the **Constructors** line. At the bottom of the page, enter `InputForm` in the input area following **S** =. Click on **Cross Reference** and then on **Domains**. The list you see contains over forty domains that belong to the category **ConvertibleTo InputForm**. Thus you can use function for **Integer**, **Float**, **String**, **Complex**, **Expression**, and so on.

6.15 Functions Defined with Blocks

You need not restrict yourself to functions that only fit on one line or are written in a piece-wise manner. The body of the function can be a block, as discussed in Section 5.2 on page 126.

Here is a short function that swaps two elements of a list, array or vector.

```
swap(m,i,j) ==
  temp := m.i
  m.i := m.j
  m.j := temp
```

The significance of **swap** is that it has a destructive effect on its first argument.

```
k := [1,2,3,4,5]
```

$$[1,\ 2,\ 3,\ 4,\ 5] \tag{2}$$

List (PositiveInteger)

```
swap(k,2,4)
```

```
   Compiling function swap with type (List(PositiveInteger), PositiveInteger,
      PositiveInteger) -> PositiveInteger
```

$$2 \tag{3}$$

PositiveInteger

You see that the second and fourth elements are interchanged.

```
k
```

$$[1,\ 4,\ 3,\ 2,\ 5] \tag{4}$$

List (PositiveInteger)

Using this, we write a couple of different sort functions. First, a simple bubble sort. The operation # returns the number of elements in an aggregate.

```
bubbleSort(m) ==
  n := #m
  for i in 1..(n-1) repeat
    for j in n..(i+1) by -1 repeat
      if m.j < m.(j-1) then swap(m,j,j-1)
  m
```

Let this be the list we want to sort.

```
m := [8,4,-3,9]
```

$$[8,\ 4,\ -3,\ 9] \tag{6}$$

List (Integer)

This is the result of sorting.

```
bubbleSort(m)
```

```
   Compiling function swap with type (List(Integer), Integer, Integer) -> Integer
   Compiling function bubbleSort with type List(Integer) -> List(Integer)
```

$$[-3,\ 4,\ 8,\ 9] \tag{7}$$

List (Integer)

Moreover, `m` is destructively changed to be the sorted version.

```
m
```

$$[-3,\ 4,\ 8,\ 9] \tag{8}$$

List (Integer)

This function implements an insertion sort. The basic idea is to traverse the list and insert the i^{th} element in its correct position among the `i-1` previous elements. Since we start at the beginning of the list, the list elements before the i^{th} element have already been placed in ascending order.

```
insertionSort(m) ==
  for i in 2..#m repeat
    j := i
    while j > 1 and m.j < m.(j-1) repeat
      swap(m,j,j-1)
      j := j - 1
  m
```

As with our bubble sort, this is a destructive function.

```
m := [8,4,-3,9]
```

$$[8,\ 4,\ -3,\ 9] \tag{10}$$

List (Integer)

```
insertionSort(m)
```

```
   Compiling function swap with type (List(Integer), NonNegativeInteger, Integer)
       -> Integer
   Compiling function insertionSort with type List(Integer) -> List(Integer)
```

$$[-3,\ 4,\ 8,\ 9] \tag{11}$$

List (Integer)

```
m
```

$$[-3,\ 4,\ 8,\ 9] \tag{12}$$

List (Integer)

Neither of the above functions is efficient for sorting large lists since they reference elements by asking for the j^{th} element of the structure `m`.

Here is a more efficient bubble sort for lists.

```
bubbleSort2(m: List Integer): List Integer ==
  empty?(m) => m
  l := m
  while not empty?(r := l.rest) repeat
     r := bubbleSort2 r
     x := l.first
     if x < r.first then
       l.first := r.first
       r.first := x
     l.rest := r
     l := l.rest
  m
```

```
   Function declaration bubbleSort2 : List(Integer) -> List(Integer) has been
      added to workspace.
```

Try it out.

```
bubbleSort2 [3,7,2]
```

```
   Compiling function bubbleSort2 with type List(Integer) -> List(Integer)
```

$$[7,\ 3,\ 2] \tag{14}$$

List (Integer)

This definition is both recursive and iterative, and is tricky! Unless you are *really* curious about this definition, we suggest you skip immediately to the next section.

Here are the key points in the definition. First notice that if you are sorting a list with less than two elements, there is nothing to do: just return the list. This definition returns immediately if there are zero elements, and skips the entire `while` loop if there is just one element.

The second point to realize is that on each outer iteration, the bubble sort ensures that the minimum element is propagated leftmost. Each iteration of the `while` loop calls **bubbleSort2** recursively to sort all but the first element. When finished, the minimum element is either in the first or second position. The conditional expression ensures that it comes first. If it is in the second, then a swap occurs. In any case, the rest of the original list must be updated to hold the result of the recursive call.

6.16 Free and Local Variables

When you want to refer to a variable that is not local to your function, use a "free" declaration. Variables declared to be free are assumed to be defined globally in the workspace.

This is a global workspace variable.

```
counter := 0
```

$$0 \tag{1}$$

NonNegativeInteger

This function refers to the global counter.

```
f() ==
  free counter
  counter := counter + 1
```

The global counter is incremented by 1.

```
f()
```

```
   Compiling function f with type () -> NonNegativeInteger
```

$$1 \tag{3}$$

PositiveInteger

```
counter
```

$$1 \tag{4}$$

NonNegativeInteger

Usually FriCAS can tell that you mean to refer to a global variable and so free isn't always necessary. However, for clarity and the sake of self-documentation, we encourage you to use it.

Declare a variable to be "local" when you do not want to refer to a global variable by the same name.

This function uses counter as a local variable.

```
g() ==
  local counter
  counter := 7
```

Apply the function.

```
g()
```

```
   Compiling function g with type () -> PositiveInteger
```

$$7 \tag{6}$$

PositiveInteger

Check that the global value of counter is unchanged.

```
counter
```

$$1 \tag{7}$$

NonNegativeInteger

Parameters to a function are local variables in the function. Even if you issue a `free` declaration for a parameter, it is still local.

What happens if you do not declare that a variable `x` in the body of your function is `local` or `free`? Well, FriCAS decides on this basis:

1. FriCAS scans your function line-by-line, from top-to-bottom. The right-hand side of an assignment is looked at before the left-hand side.
2. If `x` is referenced before it is assigned a value, it is a `free` (global) variable.
3. If `x` is assigned a value before it is referenced, it is a `local` variable.

Set two global variables to 1.

```
a := b := 1
```

$$1 \tag{8}$$

PositiveInteger

Refer to `a` before it is assigned a value, but assign a value to `b` before it is referenced.

```
h() ==
  b := a + 1
  a := b + a
```

Can you predict this result?

```
h()
```

```
   Compiling function h with type () -> PositiveInteger
```

$$3 \tag{10}$$

PositiveInteger

How about this one?

```
[a, b]
```

$$[3,\ 1] \tag{11}$$

List (PositiveInteger)

What happened? In the first line of the function body for `h`, `a` is referenced on the right-hand side of the assignment. Thus `a` is a free variable. The variable `b` is not referenced in that line, but it is assigned a value. Thus `b` is a local variable and is given the value `a + 1 = 2`. In the second line, the free variable `a` is assigned the value `b + a` which equals `2 + 1 = 3`. This is the value returned by the function. Since `a` was free in **h**, the global variable `a` has value `3`. Since `b` was local in **h**, the global variable `b` is unchanged—it still has the value `1`.

It is good programming practice always to declare global variables. However, by far the most common situation is to have local variables in your functions. No declaration is needed for this situation, but be sure to initialize their values.

Be careful if you use free variables and you cache the value of your function (see Section 6.12 on page 166). Caching *only* checks if the values of the function arguments are the same as in a function call previously seen. It does not check if any of the free variables on which the function depends have changed between function calls. Turn on caching for **p**.

```
)set fun cache all p
```

```
   function p will cache all values.
```

Define **p** to depend on the free variable N.

```
p(i,x) == ( free N; reduce( + , [ (x-i)^n for n in 1..N ] ) )
```

Set the value of N.

```
N := 1
```

$$1 \tag{13}$$

PositiveInteger

Evaluate **p** the first time.

```
p(0, x)
```

```
   Compiling function p with type (NonNegativeInteger, Variable(x)) -> Polynomial(
      Integer)
   p will cache all previously computed values.
```

$$x \tag{14}$$

Polynomial(Integer)

Change the value of N.

```
N := 2
```

$$2 \tag{15}$$

PositiveInteger

Evaluate **p** the second time.

```
p(0, x)
```

$$x \tag{16}$$

Polynomial(Integer)

If caching had been turned off, the second evaluation would have reflected the changed value of N. Turn off caching for **p**.

```
)set fun cache 0 p
```

```
   Caching for function p is turned off
```

FriCAS does not allow *fluid variables*, that is, variables *bound* by a function `f` that can be referenced by functions called by `f`.

Values are passed to functions by *reference*: a pointer to the value is passed rather than a copy of the value or a pointer to a copy.

This is a global variable that is bound to a record object.

```
r : Record(i : Integer) := [1]
```

$$[i = 1] \tag{17}$$

Record(i : Integer)

This function first modifies the one component of its record argument and then rebinds the parameter to another record.

```
resetRecord rr ==
  rr.i := 2
  rr := [10]
```

Pass `r` as an argument to **resetRecord**.

```
resetRecord r
```

```
   Compiling function resetRecord with type Record(i: Integer) -> Record(i:
      Integer)
```

$$[i = 10] \tag{19}$$

Record(i : Integer)

The value of `r` was changed by the expression `rr.i := 2` but not by `rr := [10]`.

```
r
```

$$[i = 2] \tag{20}$$

Record(i : Integer)

To conclude this section, we give an iterative definition of a function that computes Fibonacci numbers. This definition approximates the definition into which FriCAS transforms the recurrence relation definition of **fib** in Section 6.13 on page 167.

Global variables `past` and `present` are used to hold the last computed Fibonacci numbers.

```
past := present := 1
```

$$1 \tag{21}$$

PositiveInteger

Global variable `index` gives the current index of `present`.

```
index := 2
```

$$2 \tag{22}$$

PositiveInteger

Here is a recurrence relation defined in terms of these three global variables.

```
fib(n) ==
  free past, present, index
  n < 3 => 1
  n = index - 1 => past
  if n < index-1 then
    (past,present) := (1,1)
    index := 2
  while (index < n) repeat
    (past,present) := (present, past+present)
    index := index + 1
  present
```

Compute the infinite stream of Fibonacci numbers.

```
fibs := [fib(n) for n in 1..]
```

```
   Compiling function fib with type PositiveInteger -> PositiveInteger
```

$$[1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ \ldots] \tag{24}$$

Stream(PositiveInteger)

What is the 1000th Fibonacci number?

```
fibs 1000
```

434665576869374564356885276750406258025646605173717804024817290895365554179490518904038798402792551692959225930

PositiveInteger

As an exercise, we suggest you write a function in an iterative style that computes the value of the recurrence relation $p(n) = p(n-1) - 2\,p(n-2) + 4\,p(n-3)$ having the initial values $p(1) = 1$, $p(2) = 3$, and $p(3) = 9$, How would you write the function using an element **OneDimensionalArray** or **Vector** to hold the previously computed values?

6.17 Anonymous Functions

An *anonymous function* is a function that is defined by giving a list of parameters, the "maps-to" compound symbol "+->"(from the mathematical symbol $\mapsto$), and by an expression involving the parameters, the evaluation of which determines the return value of the function.

($parm_1$, $parm_2$, ..., $parm_N$) +-> *expression*

You can apply an anonymous function in several ways.

1. Place the anonymous function definition in parentheses directly followed by a list of arguments.
2. Assign the anonymous function to a variable and then use the variable name when you would normally use a function name.

3. Use "==" to use the anonymous function definition as the arguments and body of a regular function definition.

4. Have a named function contain a declared anonymous function and use the result returned by the named function.

6.17.1 Some Examples

Anonymous functions are particularly useful for defining functions "on the fly." That is, they are handy for simple functions that are used only in one place. In the following examples, we show how to write some simple anonymous functions.

This is a simple absolute value function.

```
x +-> if x < 0 then -x else x
```

$$x \mapsto if\ \ x < 0 \quad then\ \ -x \quad else\ x \tag{1}$$

AnonymousFunction

```
abs1 := %
```

$$x \mapsto if\ \ x < 0 \quad then\ \ -x \quad else\ x \tag{2}$$

AnonymousFunction

This function returns `true` if the absolute value of the first argument is greater than the absolute value of the second, `false` otherwise.

```
(x,y) +-> abs1(x) > abs1(y)
```

$$(x,\ y) \mapsto \text{abs1}(x) > \text{abs1}(y) \tag{3}$$

AnonymousFunction

We use the above function to "sort" a list of integers.

```
sort(%,[3,9,-4,10,-3,-1,-9,5])
```

$$[10,\ -9,\ 9,\ 5,\ -4,\ -3,\ 3,\ -1] \tag{4}$$

List (Integer)

This function returns `1` if `i + j` is even, `-1` otherwise.

```
ev := ( (i,j) +-> if even?(i+j) then 1 else -1)
```

$$(i,\ j) \mapsto if\ \ \text{even?}(i+j) \quad then\ 1 \quad else\ -1 \tag{5}$$

AnonymousFunction

We create a four-by-four matrix containing 1 or -1 depending on whether the row plus the column index is even or not.

```
matrix([[ev(row,col) for row in 1..4] for col in 1..4])
```

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix} \tag{6}$$

Matrix(Integer)

This function returns `true` if a polynomial in x has multiple roots, `false` otherwise. It is defined and applied in the same expression.

```
( p +-> not one?(gcd(p,D(p,x))) )(x^2+4*x+4)
```

$$\texttt{true} \tag{7}$$

Boolean

This and the next expression are equivalent.

```
g(x,y,z) == cos(x + sin(y + tan(z)))
```

The one you use is a matter of taste.

```
g == (x,y,z) +-> cos(x + sin(y + tan(z)))
```

```
   1 old definition(s) deleted for function or rule g
```

6.17.2 Declaring Anonymous Functions

If you declare any of the arguments you must declare all of them. Thus,

```
(x: INT,y): FRAC INT +-> (x + 2*y)/(y - 1)
```

is not legal.

This is an example of a fully declared anonymous function. The output shown just indicates that the object you created is a particular kind of map, that is, function.

```
(x: INT,y: INT): FRAC INT +-> (x + 2*y)/(y - 1)
```

$$\text{theMap(anonymousFunction)} \tag{1}$$

((Integer , Integer) → Fraction(Integer))

FriCAS allows you to declare the arguments and not declare the return type.

```
(x: INT,y: INT) +-> (x + 2*y)/(y - 1)
```

$$\text{theMap(anonymousFunction)} \tag{2}$$

$$((\text{Integer}, \text{Integer}) \to \text{Fraction}(\text{Integer}))$$

The return type is computed from the types of the arguments and the body of the function. You cannot declare the return type if you do not declare the arguments. Therefore,

```
(x,y): FRAC INT +-> (x + 2*y)/(y - 1)
```

is not legal.

This and the next expression are equivalent.

```
h(x: INT,y: INT): FRAC INT == (x + 2*y)/(y - 1)
```

```
   Function declaration h : (Integer, Integer) -> Fraction(Integer) has been added
      to workspace.
```

The one you use is a matter of taste.

```
h == (x: INT,y: INT): FRAC INT +-> (x + 2*y)/(y - 1)
```

```
   Function declaration h : (Integer, Integer) -> Fraction(Integer) has been added
      to workspace.
   1 old definition(s) deleted for function or rule h
```

When should you declare an anonymous function?

1. If you use an anonymous function and FriCAS can't figure out what you are trying to do, declare the function.
2. If the function has nontrivial argument types or a nontrivial return type that FriCAS may be able to determine eventually, but you are not willing to wait that long, declare the function.
3. If the function will only be used for arguments of specific types and it is not too much trouble to declare the function, do so.
4. If you are using the anonymous function as an argument to another function (such as map or sort), consider declaring the function.
5. If you define an anonymous function inside a named function, you *must* declare the anonymous function.

This is an example of a named function for integers that returns a function.

```
addx x == ((y: Integer): Integer +-> x + y)
```

We define **g** to be a function that adds 10 to its argument.

```
g := addx 10
```

```
   Compiling function addx with type PositiveInteger -> (Integer -> Integer)
```

$$\text{theMap(?)} \tag{6}$$

$$(\text{Integer} \to \text{Integer})$$

Try it out.

```
g 3
```

$$13 \tag{7}$$

PositiveInteger

```
g(-4)
```

$$6 \tag{8}$$

PositiveInteger

An anonymous function cannot be recursive: since it does not have a name, you cannot even call it within itself! If you place an anonymous function inside a named function, the anonymous function must be declared.

6.18 Example: A Database

This example shows how you can use FriCAS to organize a database of lineage data and then query the database for relationships.

The database is entered as "assertions" that are really pieces of a function definition.

```
children("albert") == ["albertJr","richard","diane"]
```

Each piece `children(x)== y` means "the children of `x` are `y`".

```
children("richard") == ["douglas","daniel","susan"]
```

This family tree thus spans four generations.

```
children("douglas") == ["dougie","valerie"]
```

Say "no one else has children."

```
children(x) == []
```

We need some functions for computing lineage. Start with `childOf`.

```
childOf(x,y) == member?(x,children(y))
```

To find the `parentOf` someone, you have to scan the database of people applying `children`.

```
parentOf(x) ==
  for y in people repeat
    (if childOf(x,y) then return y)
  "unknown"
```

And a grandparent of `x` is just a parent of a parent of `x`.

```
grandParentOf(x) == parentOf parentOf x
```

The grandchildren of `x` are the people `y` such that `x` is a grandparent of `y`.

```
grandchildren(x) == [y for y in people | grandParentOf(y) = x]
```

Suppose you want to make a list of all great-grandparents. Well, a great-grandparent is a grandparent of a person who has children.

```
greatGrandParents == [x for x in people |
  reduce(_or,[not empty? children(y) for y in grandchildren(x)],false)]
```

Define `descendants` to include the parent as well.

```
descendants(x) ==
  kids := children(x)
  empty?(kids) => [x]
  concat(x,reduce(concat,[descendants(y)
    for y in kids],[]))
```

Finally, we need a list of people. Since all people are descendants of "albert", let's say so.

```
people == descendants "albert"
```

We have used "`==`" to define the database and some functions to query the database. But no computation is done until we ask for some information. Then, once and for all, the functions are analyzed and compiled to machine code for run-time efficiency. Notice that no types are given anywhere in this example. They are not needed.

Who are the grandchildren of "richard"?

```
grandchildren "richard"
```

```
   Compiling function children with type String -> List(String)
   Compiling function descendants with type String -> List(String)
   Compiling body of rule people to compute value of type List(String)
   Compiling function childOf with type (String, String) -> Boolean
   Compiling function parentOf with type String -> String
   Compiling function grandParentOf with type String -> String
   Compiling function grandchildren with type String -> List(String)
```

["dougie", "valerie"] (12)

List (String)

Who are the great-grandparents?

```
greatGrandParents
```

```
   Compiling body of rule greatGrandParents to compute value of type List(String)
```

["albert"] (13)

List (String)

6.19 Example: A Famous Triangle

In this example we write some functions that display Pascal's triangle. It demonstrates the use of piece-wise definitions and some output operations you probably haven't seen before.

To make these output operations available, we have to *expose* the domain **OutputForm**. See Section 2.11 on page 94 for more information about exposing domains and packages.

```
)set expose add constructor OutputForm
```

```
   OutputForm is now explicitly exposed in frame initial
```

Define the values along the first row and any column `i`.

```
pascal(1,i) == 1
```

Define the values for when the row and column index `i` are equal. Repeating the argument name indicates that the two index values are equal.

```
pascal(n,n) == 1
```

```
pascal(i,j | 1 < i and i < j) ==
   pascal(i-1,j-1)+pascal(i,j-1)
```

Now that we have defined the coefficients in Pascal's triangle, let's write a couple of one-liners to display it. First, define a function that gives the n^{th} row.

```
pascalRow(n) == [pascal(i,n) for i in 1..n]
```

Next, we write the function **displayRow** to display the row, separating entries by blanks and centering.

```
displayRow(n) == output center blankSeparate pascalRow(n)
```

Here we have used three output operations. Operation output displays the printable form of objects on the screen, center centers a printable form in the width of the screen, and blankSeparate takes a list of printable forms and inserts a blank between successive elements. Look at the result.

```
for i in 1..7 repeat displayRow i
```

```
   Compiling function pascal with type (Integer, Integer) -> PositiveInteger
   Compiling function pascalRow with type PositiveInteger -> List(PositiveInteger)
   Compiling function displayRow with type PositiveInteger -> Void
                                          1
                                        1 1
                                      1 2 1
                                    1 3 3 1
                                  1 4 6 4 1
                                1 5 10 10 5 1
                              1 6 15 20 15 6 1
```

Being purists, we find this less than satisfactory. Traditionally, elements of Pascal's triangle are centered between the left and right elements on the line above. To fix this misalignment, we go back and redefine **pascalRow** to right adjust the entries within the triangle within a width of four characters.

```
pascalRow(n) == [right(pascal(i,n),4) for i in 1..n]
```

```
   Compiled code for pascalRow has been cleared.
   Compiled code for displayRow has been cleared.
   1 old definition(s) deleted for function or rule pascalRow
```

Finally let's look at our purely reformatted triangle.

```
for i in 1..7 repeat displayRow i
```

```
   Compiling function pascalRow with type PositiveInteger -> List(OutputForm)
   Compiling function displayRow with type PositiveInteger -> Void
                                        1
                                      1   1
                                    1   2   1
                                  1   3   3   1
                                1   4   6   4   1
                              1   5   10  10  5   1
                            1   6   15  20  15  6   1
```

Unexpose **OutputForm** so we don't get unexpected results later.

```
)set expose drop constructor OutputForm
```

```
   OutputForm is now explicitly hidden in frame initial
```

6.20 Example: Testing for Palindromes

In this section we define a function **pal?** that tests whether its argument is a *palindrome*, that is, something that reads the same backwards and forwards. For example, the string "Madam I'm Adam" is a palindrome (excluding blanks and punctuation) and so is the number `123454321`. The definition works for any datatype that has `n` components that are accessed by the indices $1 \ldots n$.

Here is the definition for **pal?**. It is simply a call to an auxiliary function called **palAux?**. We are following the convention of ending a function's name with "`?`" if the function returns a **Boolean** value.

```
pal? s ==  palAux?(s,1,#s)
```

Here is **palAux?**. It works by comparing elements that are equidistant from the start and end of the object.

```
palAux?(s,i,j) ==
  j > i =>
    (s.i = s.j) and palAux?(s,i+1,j-1)
  true
```

Try **pal?** on some examples. First, a string.

```
pal? "Oxford"
```

```
   Compiling function palAux? with type (String, Integer, Integer) -> Boolean
   Compiling function pal? with type String -> Boolean
```

$$\texttt{false} \tag{3}$$

Boolean

A list of polynomials.

```
pal? [4,a,x-1,0,x-1,a,4]
```

```
   Compiling function palAux? with type (List(Polynomial(Integer)), Integer,
      Integer) -> Boolean
   Compiling function pal? with type List(Polynomial(Integer)) -> Boolean
```

$$\texttt{true} \tag{4}$$

Boolean

A list of integers from the example in the last section.

```
pal? [1,6,15,20,15,6,1]
```

```
   Compiling function palAux? with type (List(PositiveInteger), Integer, Integer)
       -> Boolean
   Compiling function pal? with type List(PositiveInteger) -> Boolean
```

$$\texttt{true} \tag{5}$$

Boolean

To use **pal?** on an integer, first convert it to a string.

```
pal?(1441::String)
```

$$\texttt{true} \tag{6}$$

Boolean

Compute an infinite stream of decimal numbers, each of which is an obvious palindrome.

```
ones := [reduce(+,[10^j for j in 0..i]) for i in 1..]
```

$$[11,\ 111,\ 1111,\ 11111,\ 111111,\ 1111111,\ 11111111,\ \ldots] \tag{7}$$

Stream(PositiveInteger)

How about their squares?

```
squares := [x^2 for x in ones]
```

$$[121,\ 12321,\ 1234321,\ 123454321,\ 12345654321,\ 1234567654321,\ 123456787654321,\ 12345678987654321,\ 1234567900987654321,\ \ldots] \tag{8}$$

Stream(PositiveInteger)

Well, let's test them all!

```
[pal?(x::String) for x in squares]
```

$$[\texttt{true},\ \texttt{true},\ \texttt{true},\ \texttt{true},\ \texttt{true},\ \texttt{true},\ \texttt{true},\ \texttt{true},\ \texttt{false},\ \ldots] \tag{9}$$

Stream(Boolean)

6.21 Rules and Pattern Matching

A common mathematical formula is

$$\log(x) + \log(y) = \log(xy) \quad \forall\, x \text{ and } y.$$

The presence of "∀" indicates that `x` and `y` can stand for arbitrary mathematical expressions in the above formula. You can use such mathematical formulas in FriCAS to specify "rewrite rules". Rewrite rules are objects in FriCAS that can be assigned to variables for later use, often for the purpose of simplification. Rewrite rules look like ordinary function definitions except that they are preceded by the reserved word `rule`. For example, a rewrite rule for the above formula is:

```
rule log(x) + log(y) == log(x * y)
```

Like function definitions, no action is taken when a rewrite rule is issued. Think of rewrite rules as functions that take one argument. When a rewrite rule `A = B` is applied to an argument `f`, its meaning is: "rewrite every subexpression of `f` that *matches* `A` by `B`." The left-hand side of a rewrite rule is called a *pattern*; its right-hand side is called its *substitution*.

Create a rewrite rule named **logrule**. The generated symbol beginning with a "%" is a place-holder for any other terms that might occur in the sum.

```
logrule := rule log(x) + log(y) == log(x * y)
```

$$\log(y) + \log(x) + \%B == \log(x\,y) + \%B \tag{1}$$

RewriteRule(Integer , Integer , Expression(Integer))

Create an expression with logarithms.

```
f := log sin x + log x
```

$$\log(\sin(x)) + \log(x) \tag{2}$$

Expression(Integer)

Apply **logrule** to `f`.

```
logrule f
```

$$\log(x\ \sin(x)) \tag{3}$$

Expression(Integer)

The meaning of our example rewrite rule is: "for all expressions `x` and `y`, rewrite `log(x) + log(y)` by `log(x * y)`." Patterns generally have both operation names (here, log and +) and variables (here, `x` and `y`). By default, every operation name stands for itself. Thus log matches only "`log`" and not any other operation such as sin. On the other hand, variables do not stand for themselves. Rather, a variable denotes a *pattern variable* that is free to match any expression whatsoever.

When a rewrite rule is applied, a process called *pattern matching* goes to work by systematically scanning the subexpressions of the argument. When a subexpression is found that "matches" the pattern, the subexpression is replaced by the right-hand side of the rule. The details of what happens will be covered later.

The customary FriCAS notation for patterns is actually a shorthand for a longer, more general notation. Pattern variables can be made explicit by using a percent ("%") as the first character of the variable name. To say that a name stands for itself, you can prefix that name with a quote operator ("'"). Although the current FriCAS parser does not let you quote an operation name, this more general notation gives you an alternate way of giving the same rewrite rule:

```
rule log(%x) + log(%y) == log(x * y)
```

This longer notation gives you patterns that the standard notation won't handle. For example, the rule

```
rule %f(c * 'x) ==  c*%f(x)
```

means "for all `f` and `c`, replace `f(y)` by `c * f(x)` when `y` is the product of `c` and the explicit variable `x`."

Thus the pattern can have several adornments on the names that appear there. Normally, all these adornments are dropped in the substitution on the right-hand side.

To summarize:

To enter a single rule in FriCAS, use the following syntax:

`rule` *leftHandSide* `==` *rightHandSide*

The *leftHandSide* is a pattern to be matched and the *rightHandSide* is its substitution. The rule is an object of type **RewriteRule** that can be assigned to a variable and applied to expressions to transform them.

Rewrite rules can be collected into rulesets so that a set of rules can be applied at once. Here is another simplification rule for logarithms.

$$y\log(x) = \log(x^y) \quad \forall\, x \text{ and } y.$$

If instead of giving a single rule following the reserved word `rule` you give a "pile" of rules, you create what is called a *ruleset*. Like rules, rulesets are objects in FriCAS and can be assigned to variables. You will find it useful to group commonly used rules into input files, and read them in as needed. Create a ruleset named `logrules`.

```
logrules := rule
  log(x) + log(y) == log(x * y)
  y * log x       == log(x ^ y)
```

$$\{\log(y) + \log(x) + \%C == \log(x\,y) + \%C,\ y\ \log(x) == \log(x^y)\} \tag{4}$$

Ruleset (Integer , Integer , Expression (Integer))

Again, create an expression `f` containing logarithms.

```
f := a * log(sin x) - 2 * log x
```

$$a\ \log(\sin(x)) - 2\ \log(x) \tag{5}$$

Expression (Integer)

Apply the ruleset **logrules** to `f`.

```
logrules f
```

$$\log\left(\frac{\sin(x)^a}{x^2}\right) \tag{6}$$

Expression (Integer)

We have allowed pattern variables to match arbitrary expressions in the above examples. Often you want a variable only to match expressions satisfying some predicate. For example, we may want to apply the transformation

$$y\log(x) = \log(x^y)$$

only when `y` is an integer. The way to restrict a pattern variable `y` by a predicate `f(y)` is by using a vertical bar "`|`", which means "such that," in much the same way it is used in function definitions. You do this only once, but at the earliest (meaning deepest and leftmost) part of the pattern. This restricts the logarithmic rule to create integer exponents only.

```
logrules2 := rule
  log(x) + log(y)          == log(x * y)
  (y | integer? y) * log x == log(x ^ y)
```

$$\{\log(y) + \log(x) + \%E == \log(x\,y) + \%E,\ y\,\log(x) == \log(x^y)\} \tag{7}$$

Ruleset(Integer , Integer , Expression(Integer))

Compare this with the result of applying the previous set of rules.

```
f
```

$$a\,\log(\sin(x)) - 2\,\log(x) \tag{8}$$

Expression(Integer)

```
logrules2 f
```

$$a\,\log(\sin(x)) + \log\left(\frac{1}{x^2}\right) \tag{9}$$

Expression(Integer)

You should be aware that you might need to apply a function like integer within your predicate expression to actually apply the test function. Here we use integer because n has type **Expression Integer** but even? is an operation defined on integers.

```
evenRule := rule cos(x)^(n | integer? n and even? integer n)==(1-sin(x)^2)^(n/2)
```

$$\cos(x)^n == \left(-\sin(x)^2 + 1\right)^{\frac{n}{2}} \tag{10}$$

RewriteRule(Integer , Integer , Expression(Integer))

Here is the application of the rule.

```
evenRule( cos(x)^2 )
```

$$-\sin(x)^2 + 1 \tag{11}$$

Expression(Integer)

This is an example of some of the usual identities involving products of sines and cosines.

```
sinCosProducts == rule
  sin(x) * sin(y) == (cos(x-y) - cos(x + y))/2
  cos(x) * cos(y) == (cos(x-y) + cos(x+y))/2
  sin(x) * cos(y) == (sin(x-y) + sin(x + y))/2
```

```
g := sin(a)*sin(b) + cos(b)*cos(a) + sin(2*a)*cos(2*a)
```

$$\sin(a)\,\sin(b) + \cos(2\,a)\,\sin(2\,a) + \cos(a)\,\cos(b) \tag{13}$$

Expression(Integer)

```
sinCosProducts g
```

```
Compiling body of rule sinCosProducts to compute value of type Ruleset(Integer,
   Integer,Expression(Integer))
```

$$\frac{\sin(4\,a) + 2\,\cos(b - a)}{2} \tag{14}$$

Expression(Integer)

Another qualification you will often want to use is to allow a pattern to match an identity element. Using the pattern `x + y`, for example, neither `x` nor `y` matches the expression `0`. Similarly, if a pattern contains a product `x*y` or an exponentiation `x^y`, then neither `x` nor `y` matches `1`. If identical elements were matched, pattern matching would generally loop. Here is an expansion rule for exponentials.

```
exprule := rule exp(a + b) == exp(a) * exp(b)
```

$$e^{b+a} == e^{a}\,e^{b} \tag{15}$$

RewriteRule(Integer , Integer , Expression(Integer))

This rule would cause infinite rewriting on this if either `a` or `b` were allowed to match `0`.

```
exprule exp x
```

$$e^{x} \tag{16}$$

Expression(Integer)

There are occasions when you do want a pattern variable in a sum or product to match `0` or `1`. If so, prefix its name with a "`?`" whenever it appears in the left-hand side of a rule. For example, consider the following rule for the exponential integral:

$$\int \left(\frac{y + e^{x}}{x}\right) dx = \int \frac{y}{x}\,dx + \mathrm{Ei}(x) \quad \forall\, x \text{ and } y.$$

This rule is valid for `y = 0`. One solution is to create a **Ruleset** with two rules, one with and one without `y`. A better solution is to use an "optional" pattern variable. Define rule `eirule` with a pattern variable `?y` to indicate that an expression may or may not occur.

```
eirule := rule integral((?y + exp x)/x,x) == integral(y/x,x) + Ei x
```

$$\int^{x} \frac{e^{\%D} + y}{\%D}\,d\%D == \int^{x} \frac{y}{\%D}\,d\%D + \mathrm{Ei}(x) \tag{17}$$

RewriteRule(Integer , Integer , Expression(Integer))

Apply rule `eirule` to an integral without this term.

```
eirule integral(exp u/u, u)
```

$$\mathrm{Ei}(u) \tag{18}$$

Expression(Integer)

Apply rule `eirule` to an integral with this term.

```
eirule integral((sin u + exp u)/u, u)
```

$$\int^{u} \frac{\sin(\%D)}{\%D}\, d\%D + \mathrm{Ei}(u) \tag{19}$$

Expression(Integer)

Here is one final adornment you will find useful. When matching a pattern of the form `x + y` to an expression containing a long sum of the form $a + \ldots + b$, there is no way to predict in advance which subset of the sum matches `x` and which matches `y`. Aside from efficiency, this is generally unimportant since the rule holds for any possible combination of matches for `x` and `y`. In some situations, however, you may want to say which pattern variable is a sum (or product) of several terms, and which should match only a single term. To do this, put a prefix colon "`:`" before the pattern variable that you want to match multiple terms. The remaining rules involve operators `u` and `v`.

```
u := operator 'u
```

$$u \tag{20}$$

BasicOperator

These definitions tell FriCAS that `u` and `v` are formal operators to be used in expressions.

```
v := operator 'v
```

$$v \tag{21}$$

BasicOperator

First define `myRule` with no restrictions on the pattern variables `x` and `y`.

```
myRule := rule u(x + y) == u x + v y
```

$$u(y + x) == v(y) + u(x) \tag{22}$$

RewriteRule(Integer , Integer , Expression(Integer))

Apply `myRule` to an expression.

```
myRule u(a + b + c + d)
```

$$v(c + b + a) + u(d) \tag{23}$$

Expression(Integer)

Define `myOtherRule` to match several terms so that the rule gets applied recursively.

```
myOtherRule := rule u(:x + y) == u x + v y
```

$$u(y + x) == v(y) + u(x) \tag{24}$$

RewriteRule(Integer , Integer , Expression(Integer))

Apply `myOtherRule` to the same expression.

```
myOtherRule u(a + b + c + d)
```

$$v(d) + v(c) + v(b) + u(a) \tag{25}$$

Expression (Integer)

Summary of pattern variable adornments:

`(x | predicate?(x))`	means that the substitution s for x must satisfy `predicate?(s) = true`.
`?x`	means that x can match an identity element (0 or 1).
`:x`	means that x can match several terms in a sum.

Here are some final remarks on pattern matching. Pattern matching provides a very useful paradigm for solving certain classes of problems, namely, those that involve transformations of one form to another and back. However, it is important to recognize its limitations.

First, pattern matching slows down as the number of rules you have to apply increases. Thus it is good practice to organize the sets of rules you use optimally so that irrelevant rules are never included.

Second, careless use of pattern matching can lead to wrong answers. You should avoid using pattern matching to handle hidden algebraic relationships that can go undetected by other programs. As a simple example, a symbol such as "J" can easily be used to represent the square root of `-1` or some other important algebraic quantity. Many algorithms branch on whether an expression is zero or not, then divide by that expression if it is not. If you fail to simplify an expression involving powers of `J` to `-1`, algorithms may incorrectly assume an expression is non-zero, take a wrong branch, and produce a meaningless result.

Pattern matching should also not be used as a substitute for a domain. In FriCAS, objects of one domain are transformed to objects of other domains using well-defined coerce operations. Pattern matching should be used on objects that are all the same type. Thus if your application can be handled by type **Expression** in FriCAS and you think you need pattern matching, consider this choice carefully. You may well be better served by extending an existing domain or by building a new domain of objects for your application.

Chapter 7

Graphics

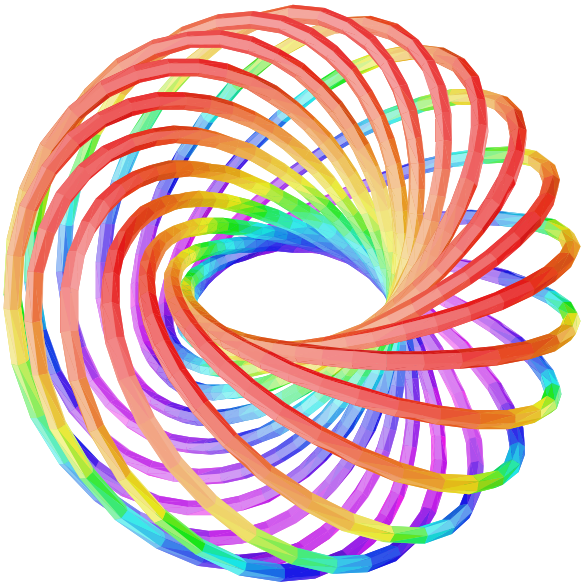


Figure 7.1: Torus knot of type (15,17).

This chapter shows how to use the FriCAS graphics facilities under the X Window System. FriCAS has two-dimensional and three-dimensional drawing and rendering packages that allow the drawing, coloring, transforming, mapping, clipping, and combining of graphic output from FriCAS computations. This facility is particularly useful for investigating problems in areas such as topology. The graphics package is capable of plotting functions of one or more variables or plotting parametric surfaces and curves. Various coordinate systems are also available, such as polar and spherical.

A graph is displayed in a viewport window and it has a control-panel that uses interactive mouse commands. PostScript and other output forms are available so that FriCAS images can be printed or used by other programs.[1]

[1] PostScript is a trademark of Adobe Systems Incorporated, registered in the United States.

7.1 Two-Dimensional Graphics

The FriCAS two-dimensional graphics package provides the ability to display

- curves defined by functions of a single real variable
- curves defined by parametric equations
- implicit non-singular curves defined by polynomial equations
- planar graphs generated from lists of point components.

These graphs can be modified by specifying various options, such as calculating points in the polar coordinate system or changing the size of the graph viewport window.

7.1.1 Plotting Two-Dimensional Functions of One Variable

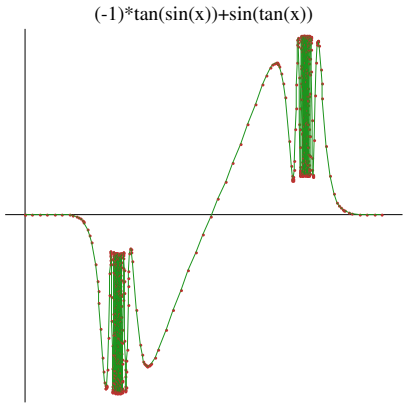
The first kind of two-dimensional graph is that of a curve defined by a function `y = f(x)` over a finite interval of the `x` axis.

> The general format for drawing a function defined by a formula `f(x)` is:
>
> `draw(f(x), x = a..b,` *options*`)`
>
> where `a..b` defines the range of `x`, and where *options* prescribe zero or more options as described in Section 7.1.4 on page 200. An example of an option is `curveColor == bright red()`. An alternative format involving functions `f` and `g` is also available.

A simple way to plot a function is to use a formula. The first argument is the formula. For the second argument, write the name of the independent variable (here, `x`), followed by an "`=`", and the range of values.

Display this formula over the range $0 \leq x \leq 6$. FriCAS converts your formula to a compiled function so that the results can be computed quickly and efficiently.

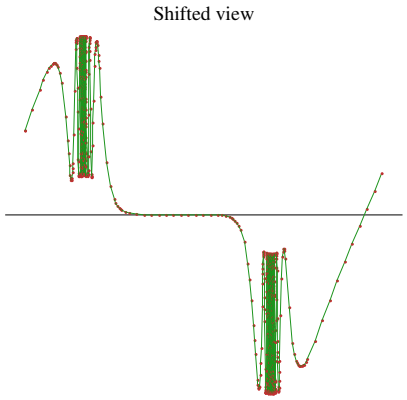
```
draw(sin(tan(x)) - tan(sin(x)),x = 0..6)
```



Notice that FriCAS compiled the function before the graph was put on the screen.

Here is the same graph on a different interval. This time we give the graph a title.

```
draw(sin(tan(x)) - tan(sin(x)), x = 10..16, title=="Shifted view")
```

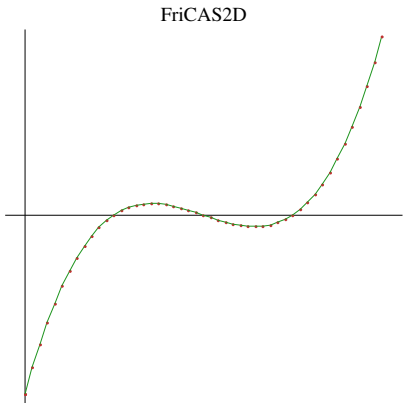


Once again the formula is converted to a compiled function before any points are computed. If you want to graph the same function on several intervals, it is a good idea to define the function first so that the function has to be compiled only once. This time we first define the function.

```
f(x) == (x-1)*(x-2)*(x-3)
```

To draw the function, the first argument is its name and the second is just the range with no independent variable.

```
draw(f, 0..4)
```



7.1.2 Plotting Two-Dimensional Parametric Plane Curves

The second kind of two-dimensional graph is that of curves produced by parametric equations. Let `x = f(t)` and `y = g(t)` be formulas or two functions `f` and `g` as the parameter `t` ranges over an interval `[a,b]`. The function curve takes the two functions `f` and `g` as its parameters.

The general format for drawing a two-dimensional plane curve defined by parametric formulas `x = f(t)` and `y = g(t)` is:

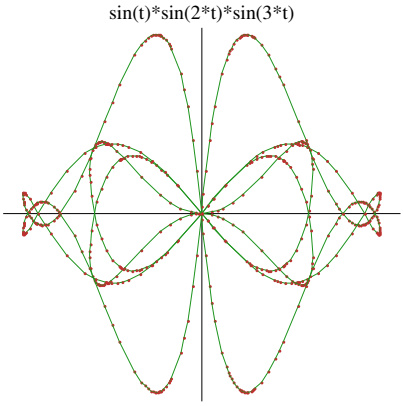
`draw(curve(f(t), g(t)), t = a..b,` *options*`)`

where `a..b` defines the range of the independent variable `t`, and where *options* prescribe zero or more options as described in Section 7.2.4 on page 225. An example of an option is `curveColor == bright red()`.

Here's an example:

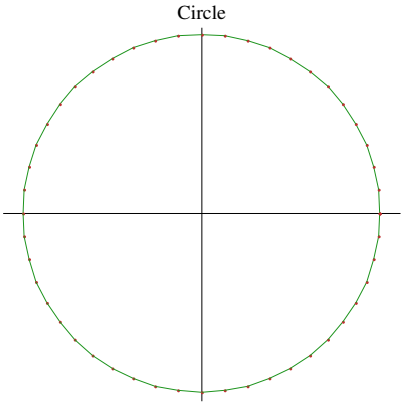
Define a parametric curve using a range involving `%pi`, FriCAS's way of saying π. For parametric curves, FriCAS compiles two functions, one for each of the functions `f` and `g`.

```
draw(curve(sin(t)*sin(2*t)*sin(3*t), sin(4*t)*sin(5*t)*sin(6*t)), t = 0..2*%pi)
```



The title may be an arbitrary string and is an optional argument to the draw command.

```
draw(curve(cos(t), sin(t)), t = 0..2*%pi, title=="Circle")
```



If you plan on plotting `x = f(t)`, `y = g(t)` as `t` ranges over several intervals, you may want to define functions `f` and `g` first, so that they need not be recompiled every time you create a new graph. Here's an example: As before, you can first define the functions you wish to draw.

```
f(t:DFLOAT):DFLOAT == sin(3*t/4)
```

```
   Function declaration f : DoubleFloat -> DoubleFloat has been added to
      workspace.
```

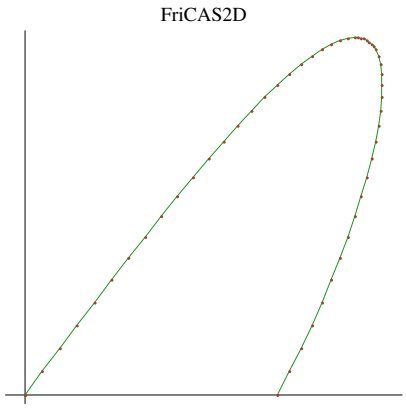
FriCAS compiles them to map **DoubleFloat** values to **DoubleFloat** values.

```
g(t:DFLOAT):DFLOAT == sin(t)
```

```
   Function declaration g : DoubleFloat -> DoubleFloat has been added to
      workspace.
```

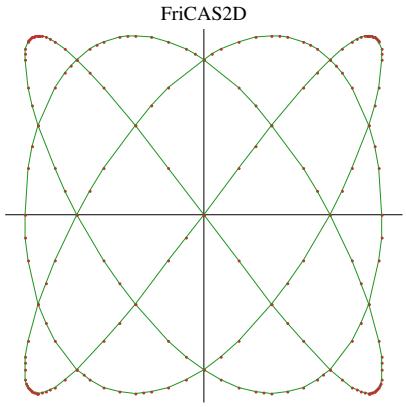
Give to `curve` the names of the functions, then write the range without the name of the independent variable.

```
draw(curve(f,g),0..%pi)
```



Here is another look at the same curve but over a different range. Notice that `f` and `g` are not recompiled. Also note that FriCAS provides a default title based on the first function specified in curve.

```
draw(curve(f,g),-4*%pi..4*%pi)
```



7.1.3 Plotting Plane Algebraic Curves

A third kind of two-dimensional graph is a non-singular "solution curve" in a rectangular region of the plane. A solution curve is a curve defined by a polynomial equation `p(x,y) = 0`. Non-singular means

that the curve is "smooth" in that it does not cross itself or come to a point (cusp). Algebraically, this means that for any point `(x,y)` on the curve, that is, a point such that `p(x,y) = 0`, the partial derivatives $\frac{\partial p}{\partial x}(x,y)$ and $\frac{\partial p}{\partial y}(x,y)$ are not both zero.

The general format for drawing a non-singular solution curve given by a polynomial of the form `p(x,y) = 0` is:

`draw(p(x,y) = 0, x, y, range == [a..b, c..d],` *options*`)`

where the second and third arguments name the first and second independent variables of `p`. A `range` option is always given to designate a bounding rectangular region of the plane $a \le x \le b, c \le y \le d$. Zero or more additional options as described in Section 7.1.4 on page 200 may be given.

We require that the polynomial has rational or integral coefficients. Here is an algebraic curve example ("Cartesian ovals"):

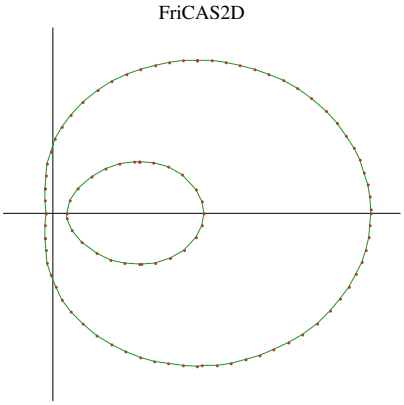
```
p := ((x^2 + y^2 + 1) - 8*x)^2 - (8*(x^2 + y^2 + 1)-4*x-1)
```

$$y^4 + \left(2\,x^2 - 16\,x - 6\right)\,y^2 + x^4 - 16\,x^3 + 58\,x^2 - 12\,x - 6 \tag{1}$$

Polynomial(Integer)

The first argument is always expressed as an equation of the form `p = 0` where `p` is a polynomial.

```
draw(p = 0, x, y, range == [-1..11, -7..7])
```



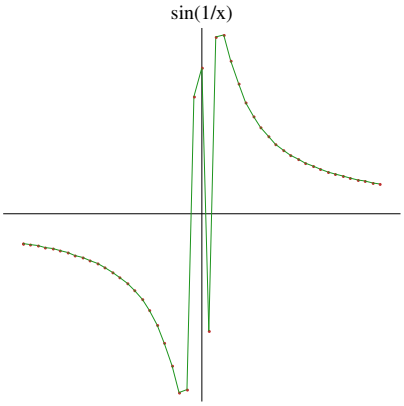
7.1.4 Two-Dimensional Options

The draw commands take an optional list of options, such as `title` shown above. Each option is given by the syntax: *name* `==` *value*. Here is a list of the available options in the order that they are described below.

adaptive	clip	unit
clip	curveColor	range
toScale	pointColor	coordinates

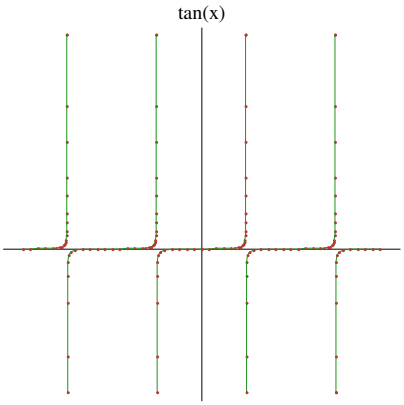
The `adaptive` option turns adaptive plotting on or off. Adaptive plotting uses an algorithm that traverses a graph and computes more points for those parts of the graph with high curvature. The higher the curvature of a region is, the more points the algorithm computes. The `adaptive` option is normally on. Here we turn it off.

```
draw(sin(1/x),x=-2*%pi..2*%pi, adaptive == false)
```



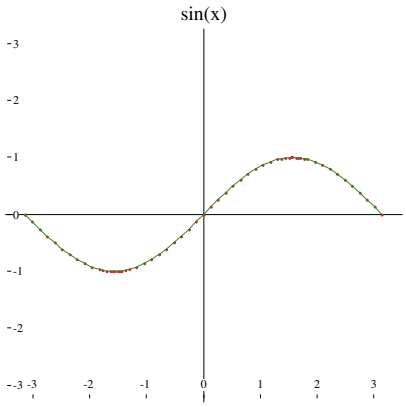
The `clip` option turns clipping on or off. If on, large values are cut off according to clipPointsDefault.

```
draw(tan(x),x=-2*%pi..2*%pi, clip == true)
```



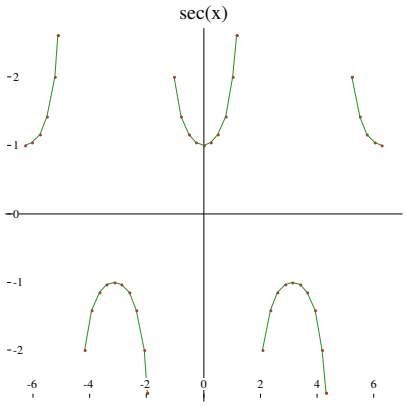
Option `toScale` does plotting to scale if `true` or uses the entire viewport if `false`. The default can be determined using drawToScale.

```
draw(sin(x),x=-%pi..%pi, toScale == true, unit == [1.0,1.0])
```



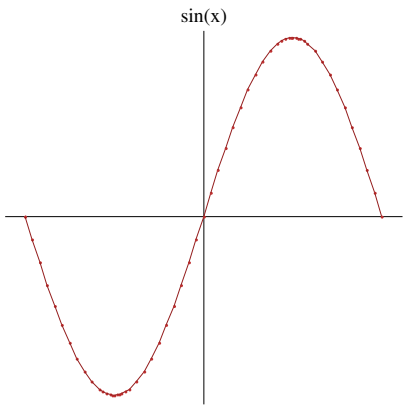
Option `clip` with a range sets point clipping of a graph within the ranges specified in the list `[x range , y range]`. If only one range is specified, clipping applies to the y-axis.

```
draw(sec(x),x=-2*%pi..2*%pi, clip == [-2*%pi..2*%pi,-%pi..%pi], unit == [1.0,1.0])
```



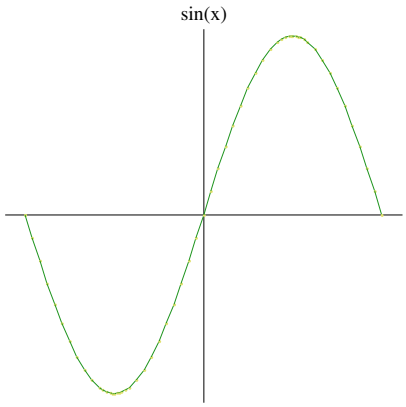
Option `curveColor` sets the color of the graph curves or lines to be the indicated palette color (see Section 7.1.5 on page 205 and Section 7.1.6 on page 206).

```
draw(sin(x),x=-%pi..%pi, curveColor == bright red())
```



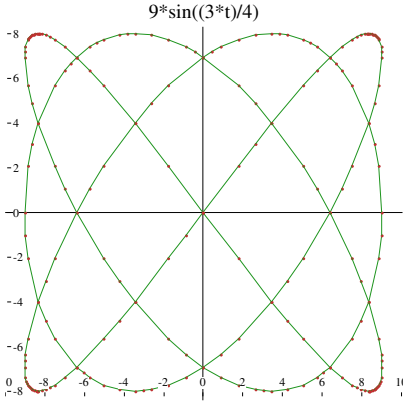
Option `pointColor` sets the color of the graph points to the indicated palette color (see Section 7.1.5 on page 205 and Section 7.1.6 on page 206).

```
draw(sin(x),x=-%pi..%pi, pointColor == pastel yellow())
```



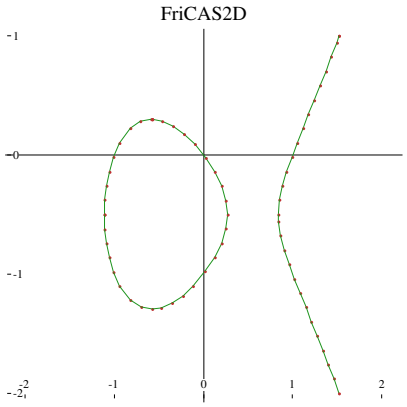
Option `unit` sets the intervals at which the axis units are plotted according to the indicated steps [`x` interval, `y` interval].

```
draw(curve(9*sin(3*t/4),8*sin(t)), t = -4*%pi..4*%pi, unit == [2.0,1.0])
```



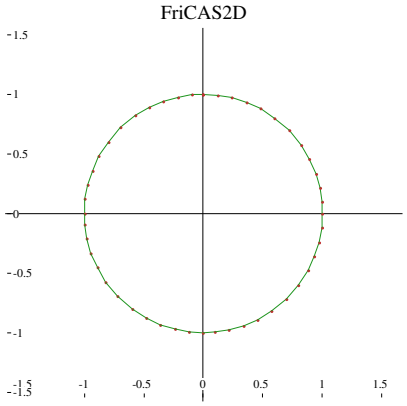
Option `range` sets the range of variables in a graph to be within the ranges for solving plane algebraic curve plots.

```
draw(y^2 + y - (x^3 - x) = 0, x, y, range == [-2..2,-2..1], unit==[1.0,1.0])
```



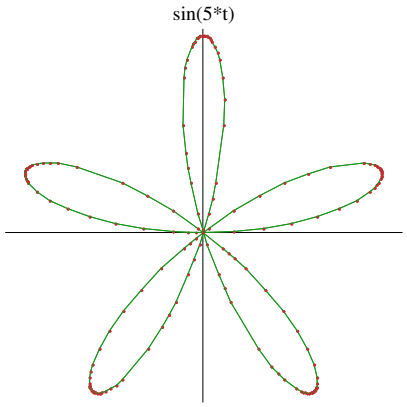
A second example of a solution plot.

```
draw(x^2 + y^2 = 1, x, y, range == [-3/2..3/2,-3/2..3/2], unit==[0.5,0.5])
```



Option `coordinates` indicates the coordinate system in which the graph is plotted. The default is to use the Cartesian coordinate system. For more details, see Section 7.2.7 on page 235.

```
draw(curve(sin(5*t),t),t=0..2*%pi, coordinates == polar)
```



7.1.5 Color

The domain **Color** provides operations for manipulating colors in two-dimensional graphs. Colors are objects of **Color**. Each color has a *hue* and a *weight*. Hues are represented by integers that range from `1` to the numberOfHues(), normally `27`. Weights are floats and have the value `1.0` by default.

color (*integer*)
: creates a color of hue *integer* and weight `1.0`.

hue (*color*)
: returns the hue of *color* as an integer.

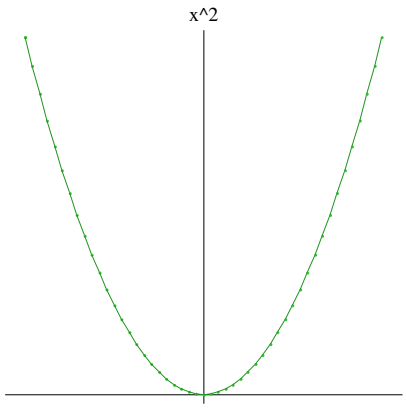
red (), blue() green(), and yellow()
: create colors of that hue with weight `1.0`.

$color_1$ `+` $color_2$ returns the color that results from additively combining the indicated $color_1$ and $color_2$. Color addition is not commutative: changing the order of the arguments produces different results.

integer `*` *color* changes the weight of *color* by *integer* without affecting its hue. For example, `red() + 3*yellow()` produces a color closer to yellow than to red. Color multiplication is not associative: changing the order of grouping produces different results.

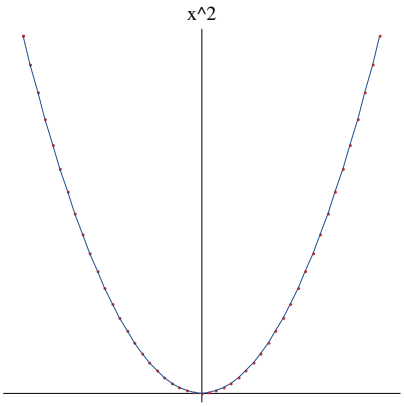
These functions can be used to change the point and curve colors for two- and three-dimensional graphs. Use the `pointColor` option for points.

```
draw(x^2,x=-1..1,pointColor == green())
```



Use the `curveColor` option for curves.

```
draw(x^2,x=-1..1,curveColor == color(13) + 2*blue())
```



7.1.6 Palette

Domain **Palette** is the domain of shades of colors: dark, dim, bright, pastel, and light, designated by the integers 1 through 5, respectively. Colors are normally "bright."

```
shade red()
```

3 (1)

PositiveInteger

To change the shade of a color, apply the name of a shade to it.

```
myFavoriteColor := dark blue()
```

[Hue: 22 Weight: 1.0] from the Dark palette (2)

Palette

The expression `shade(color)` returns the value of a shade of `color`.

```
shade myFavoriteColor
```

1 (3)

PositiveInteger

The expression `hue(color)` returns its hue.

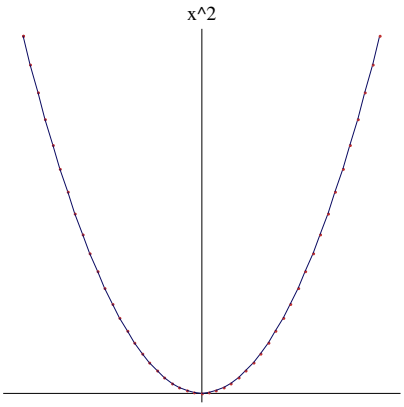
```
hue myFavoriteColor
```

Hue: 22 Weight: 1.0 (4)

Color

Palettes can be used in specifying colors in two-dimensional graphs.

```
draw(x^2,x=-1..1,curveColor == dark blue())
```



7.1.7 Two-Dimensional Control-Panel

Once you have created a viewport, move your mouse to the viewport and click with your left mouse button to display a control-panel. The panel is displayed on the side of the viewport closest to where you clicked. Each of the buttons which toggle on and off show the current state of the graph.

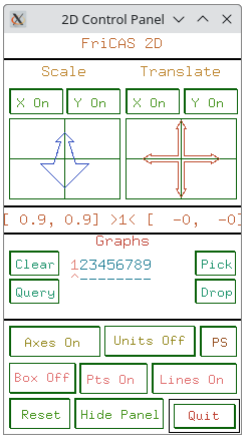


Figure 7.2: Two-dimensional control-panel.

Transformations

Object transformations are executed from the control-panel by mouse-activated potentiometer windows.

Scale: To scale a graph, click on a mouse button within the **Scale** window in the upper left corner of the control-panel. The axes along which the scaling is to occur are indicated by setting the toggles above the arrow. With `X On` and `Y On` appearing, both axes are selected and scaling is uniform. If either is not selected, for example, if `X Off` appears, scaling is non-uniform.

Translate: To translate a graph, click the mouse in the **Translate** window in the direction you wish the graph to move. This window is located in the upper right corner of the control-panel. Along the top of the **Translate** window are two buttons for selecting the direction of translation. Translation along both coordinate axes results when `X On` and `Y On` appear or along one axis when one is on, for example, `X On` and `Y Off` appear.

Messages

The window directly below the transformation potentiometer windows is used to display system messages relating to the viewport and the control-panel. The following format is displayed:

[scaleX, scaleY] >graph< [translateX, translateY]

The two values to the left show the scale factor along the `X` and `Y` coordinate axes. The two values to the right show the distance of translation from the center in the `X` and `Y` directions. The number in the center shows which graph in the viewport this data pertains to. When multiple graphs exist in the same viewport, the graph must be selected (see "Multiple Graphs," below) in order for its transformation data to be shown, otherwise the number is 1.

Multiple Graphs

The **Graphs** window contains buttons that allow the placement of two-dimensional graphs into one of nine available slots in any other two-dimensional viewport. In the center of the window are numeral buttons from one to nine that show whether a graph is displayed in the viewport. Below each number button is a button showing whether a graph that is present is selected for application of some transformation. When the caret symbol is displayed, then the graph in that slot will be manipulated. Initially, the graph for which the viewport is created occupies the first slot, is displayed, and is selected.

Clear: The **Clear** button deselects every viewport graph slot. A graph slot is reselected by selecting the button below its number.

Query: The **Query** button is used to display the scale and translate data for the indicated graph. When this button is selected the message "Click on the graph to query" appears. Select a slot number button from the **Graphs** window. The scaling factor and translation offset of the graph are then displayed in the message window.

Pick: The **Pick** button is used to select a graph to be placed or dropped into the indicated viewport. When this button is selected, the message "Click on the graph to pick" appears. Click on the slot with the graph number of the desired graph. The graph information is held waiting for you to execute a **Drop** in some other graph.

Drop: Once a graph has been picked up using the **Pick** button, the **Drop** button places it into a new viewport slot. The message "Click on the graph to drop" appears in the message window when the **Drop** button is selected. By selecting one of the slot number buttons in the **Graphs** window, the graph currently being held is dropped into this slot and displayed.

Buttons

Axes turns the coordinate axes on or off.

Units turns the units along the `x` and `y` axis on or off.

Box encloses the area of the viewport graph in a bounding box, or removes the box if already enclosed.

Pts turns on or off the display of points.

Lines turns on or off the display of lines connecting points.

PS writes the current viewport contents to a file **fricas2D.ps**. The file is placed in the directory from which FriCAS or the **viewAlone** program was invoked.

Reset resets the object transformation characteristics and attributes back to their initial states.

Hide makes the control-panel disappear.

Quit queries whether the current viewport session should be terminated.

7.1.8 Operations for Two-Dimensional Graphics

Here is a summary of useful FriCAS operations for two-dimensional graphics. Each operation name is followed by a list of arguments. Each argument is written as a variable informally named according to the type of the argument (for example, *integer*). If appropriate, a default value for an argument is given in parentheses immediately following the name.

adaptive `([`*boolean*`(true)])`
sets or indicates whether graphs are plotted according to the adaptive refinement algorithm.

axesColorDefault `([`*color*`(dark blue())])`
sets or indicates the default color of the axes in a two-dimensional graph viewport.

clipPointsDefault `([`*boolean*`(false)])`
sets or indicates whether point clipping is to be applied as the default for graph plots.

drawToScale `([`*boolean*`(false)])`
sets or indicates whether the plot of a graph is "to scale" or uses the entire viewport space as the default.

lineColorDefault `([`*color*`(pastel yellow())])`
sets or indicates the default color of the lines or curves in a two-dimensional graph viewport.

maxPoints `([`*integer*`(500)])`
sets or indicates the default maximum number of possible points to be used when constructing a two-dimensional graph.

minPoints `([`*integer*`(21)])`
sets or indicates the default minimum number of possible points to be used when constructing a two-dimensional graph.

pointColorDefault ([*color*(bright red())])
sets or indicates the default color of the points in a two-dimensional graph viewport.

pointSizeDefault ([*integer*(5)])
sets or indicates the default size of the dot used to plot points in a two-dimensional graph.

screenResolution ([*integer*(600)])
sets or indicates the default screen resolution constant used in setting the computation limit of adaptively generated curve plots.

unitsColorDefault ([*color*(dim green())])
sets or indicates the default color of the unit labels in a two-dimensional graph viewport.

viewDefaults ()
resets the default settings for the following attributes: point color, line color, axes color, units color, point size, viewport upper left-hand corner position, and the viewport size.

viewPosDefault ([*list*([100,100])])
sets or indicates the default position of the upper left-hand corner of a two-dimensional viewport, relative to the display root window. The upper left-hand corner of the display is considered to be at the (0, 0) position.

viewSizeDefault ([*list*([200,200])])
sets or indicates the default size in which two dimensional viewport windows are shown. It is defined by a width and then a height.

viewWriteAvailable ([*list*(["pixmap", "bitmap", "postscript", "image"])])
indicates the possible file types that can be created with the 'write' function.

viewWriteDefault ([*list*([])])
sets or indicates the default types of files, in addition to the **data** file, that are created when a write function is executed on a viewport.

units (*viewport*, *integer*(1), *string*("off"))
turns the units on or off for the graph with index *integer*.

axes (*viewport*, *integer*(1), *string*("on"))
turns the axes on or off for the graph with index *integer*.

close (*viewport*)
closes *viewport*.

connect (*viewport*, *integer*(1), *string*("on"))
declares whether lines connecting the points are displayed or not.

controlPanel (*viewport*, *string*("off"))
declares whether the two-dimensional control-panel is automatically displayed or not.

graphs (*viewport*)
returns a list describing the state of each graph. If the graph state is not being used this is shown by "undefined", otherwise a description of the graph's contents is shown.

graphStates (*viewport*)
displays a list of all the graph states available for *viewport*, giving the values for every property.

key (*viewport*)
returns the process ID number for *viewport*.

move (*viewport*, $integer_{\text{x}}$*(viewPosDefault)*, $integer_{\text{y}}$*(viewPosDefault)*)
moves *viewport* on the screen so that the upper left-hand corner of *viewport* is at the position *(x,y)*.

options (*viewport*)
returns a list of all the **DrawOption**s used by *viewport*.

points (*viewport*, *integer*`(1)`, *string*`("on")`)
specifies whether the graph points for graph *integer* are to be displayed or not.

region (*viewport*, *integer*`(1)`, *string*`("off")`)
declares whether graph *integer* is or is not to be displayed with a bounding rectangle.

reset (*viewport*)
resets all the properties of *viewport*.

resize (*viewport*, $integer_{\text{width}}$, $integer_{\text{height}}$)
resizes *viewport* with a new *width* and *height*.

scale (*viewport*, $integer_{\text{n}}$`(1)`, $integer_{\text{x}}$`(0.9)`, $integer_{\text{y}}$`(0.9)`)
scales values for the x and y coordinates of graph n.

show (*viewport*, $integer_{\text{n}}$`(1)`, *string*`("on")`)
indicates if graph n is shown or not.

title (*viewport*, *string*`("FriCAS 2D")`)
designates the title for *viewport*.

translate (*viewport*, $integer_{\text{n}}$`(1)`, $float_{\text{x}}$`(0.0)`, $float_{\text{y}}$`(0.0)`)
causes graph n to be moved x and y units in the respective directions.

write (*viewport*, $string_{\text{directory}}$, `[`*strings*`]`)
if no third argument is given, writes the **data** file into the directory with extension **data**. The third argument can be a single string or a list of strings with some or all the entries `"pixmap"`, `"bitmap"`, `"postscript"`, and `"image"`.

7.1.9 Addendum: Building Two-Dimensional Graphs

In this section we demonstrate how to create two-dimensional graphs from lists of points and give an example showing how to read the lists of points from a file.

Creating a Two-Dimensional Viewport from a List of Points

FriCAS creates lists of points in a two-dimensional viewport by utilizing the **GraphImage** and **TwoDimensionalViewport** domains. In this example, the makeGraphImage function takes a list of lists of points as a parameter, a list of colors for each point in the graph, a list of colors for each line in the graph, and a list of sizes for each point in the graph. The following expressions create a list of lists of points which will be read by FriCAS and made into a two-dimensional viewport.

```
p1 := point [1,1]$(Point DFLOAT)
```

$$[1.0,\ 1.0] \tag{1}$$

Point(DoubleFloat)

```
p2 := point [0,1]$(Point DFLOAT)
```

$$[0.0,\ 1.0] \tag{2}$$

Point(DoubleFloat)

```
p3 := point [0,0]$(Point DFLOAT)
```

$$[0.0,\ 0.0] \tag{3}$$

Point(DoubleFloat)

```
p4 := point [1,0]$(Point DFLOAT)
```

$$[1.0,\ 0.0] \tag{4}$$

Point(DoubleFloat)

```
p5 := point [1,.5]$(Point DFLOAT)
```

$$[1.0,\ 0.5] \tag{5}$$

Point(DoubleFloat)

```
p6 := point [.5,0]$(Point DFLOAT)
```

$$[0.5,\ 0.0] \tag{6}$$

Point(DoubleFloat)

```
p7 := point [0,0.5]$(Point DFLOAT)
```

$$[0.0,\ 0.5] \tag{7}$$

Point(DoubleFloat)

```
p8 := point [.5,1]$(Point DFLOAT)
```

$$[0.5,\ 1.0] \tag{8}$$

Point(DoubleFloat)

```
p9 := point [.25,.25]$(Point DFLOAT)
```

$$[0.25,\ 0.25] \tag{9}$$

Point(DoubleFloat)

```
p10 := point [.25,.75]$(Point DFLOAT)
```

$$[0.25,\ 0.75] \tag{10}$$

Point(DoubleFloat)

```
p11 := point [.75,.75]$(Point DFLOAT)
```

$$[0.75,\ 0.75] \tag{11}$$

Point(DoubleFloat)

```
p12 := point [.75,.25]$(Point DFLOAT)
```

$$[0.75,\ 0.25] \tag{12}$$

Point(DoubleFloat)

Finally, here is the list.

```
llp := [[p1,p2], [p2,p3], [p3,p4], [p4,p1], [p5,p6], [p6,p7], [p7,p8], [p8,p5], _
    [p9,p10], [p10,p11], [p11,p12], [p12,p9]]
```

$$[[[1.0,\ 1.0],\ [0.0,\ 1.0]],\ [[0.0,\ 1.0],\ [0.0,\ 0.0]],\ [[0.0,\ 0.0],\ [1.0,\ 0.0]],\ [[1.0,\ 0.0],\ [1.0,\ 1.0]],\ [[1.0,\ 0.5],\ [0.5,\ 0.0]],\ [[0.5,\ 0.0],\ [0.0,\ 0.5]],\ [[0.0,\ 0.5],\ [0.5,\ 1.0]],\ [[0.5,\ 1.0],\ [1.0,\ 0.5]],\ [[0.25,\ 0.25],\ [0.25,\ 0.75]],\ [[0.25,\ 0.75],\ [0.75,\ 0.75]],\ [[0.75,\ 0.75],\ [0.75,\ 0.25]],\ [[0.75,\ 0.25],\ [0.25,\ 0.25]]] \tag{13}$$

List (List (Point(DoubleFloat)))

Now we set the point sizes for all components of the graph.

```
size1 := 6::PositiveInteger
```

$$6 \tag{14}$$

PositiveInteger

```
size2 := 8::PositiveInteger
```

$$8 \tag{15}$$

PositiveInteger

```
size3 := 10::PositiveInteger
```

$$10 \tag{16}$$

PositiveInteger

```
lsize := [size1, size1, size1, size1, size2, size2, size2, size2, size3, size3, _
    size3, size3]
```

$$[6,\ 6,\ 6,\ 6,\ 8,\ 8,\ 8,\ 8,\ 10,\ 10,\ 10,\ 10] \tag{17}$$

List (PositiveInteger)

Here are the colors for the points.

```
pc1 := pastel red()
```

```
[Hue: 1  Weight: 1.0] from the Pastel palette
```
(18)

Palette

```
pc2 := dim green()
```

```
[Hue: 14  Weight: 1.0] from the Dim palette
```
(19)

Palette

```
pc3 := pastel yellow()
```

```
[Hue: 11  Weight: 1.0] from the Pastel palette
```
(20)

Palette

```
lpc := [pc1, pc1, pc1, pc1, pc2, pc2, pc2, pc2, pc3, pc3, pc3, pc3]
```

```
[[Hue: 1  Weight: 1.0] from the Pastel palette,
 [Hue: 1  Weight: 1.0] from the Pastel palette,
 [Hue: 1  Weight: 1.0] from the Pastel palette,
 [Hue: 1  Weight: 1.0] from the Pastel palette,
 [Hue: 14  Weight: 1.0] from the Dim palette,
 [Hue: 14  Weight: 1.0] from the Dim palette,
 [Hue: 14  Weight: 1.0] from the Dim palette,
 [Hue: 14  Weight: 1.0] from the Dim palette,
 [Hue: 11  Weight: 1.0] from the Pastel palette,
 [Hue: 11  Weight: 1.0] from the Pastel palette,
 [Hue: 11  Weight: 1.0] from the Pastel palette,
 [Hue: 11  Weight: 1.0] from the Pastel palette]
```
(21)

List (Palette)

Here are the colors for the lines.

```
lc := [pastel blue(), light yellow(), dim green(), bright red(), light green(), dim _
    yellow(), bright blue(), dark red(), pastel red(), light blue(), dim green(), _
    light yellow()]
```

```
[[Hue: 22  Weight: 1.0] from the Pastel palette,
 [Hue: 11  Weight: 1.0] from the Light palette,
 [Hue: 14  Weight: 1.0] from the Dim palette,
 [Hue: 1  Weight: 1.0] from the Bright palette,
 [Hue: 14  Weight: 1.0] from the Light palette,
 [Hue: 11  Weight: 1.0] from the Dim palette,
 [Hue: 22  Weight: 1.0] from the Bright palette,
 [Hue: 1  Weight: 1.0] from the Dark palette,
 [Hue: 1  Weight: 1.0] from the Pastel palette,
 [Hue: 22  Weight: 1.0] from the Light palette,
 [Hue: 14  Weight: 1.0] from the Dim palette,
 [Hue: 11  Weight: 1.0] from the Light palette]
```

(22)

List (Palette)

Now the **GraphImage** is created according to the component specifications indicated above.

```
g := makeGraphImage(llp,lpc,lc,lsize)$GRIMAGE
```

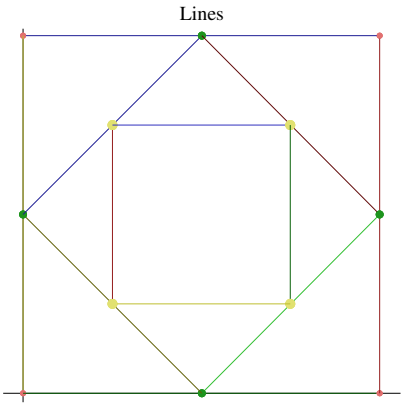
```
Graph with 12 point lists
```

(23)

GraphImage

The makeViewport2D function now creates a **TwoDimensionalViewport** for this graph according to the list of options specified within the brackets.

```
makeViewport2D(g,[title("Lines")])$VIEW2D
```



This example demonstrates the use of the **GraphImage** functions component and appendPoint in adding points to an empty **GraphImage**.

```
)clear all
```

```
   All user variables and function definitions have been cleared.
```

```
g := graphImage()$GRIMAGE
```

```
Graph with 0 point lists                                    (1)
```

GraphImage

```
p1 := point [0,0]$(Point DFLOAT)
```

$$[0.0,\ 0.0] \tag{2}$$

Point(DoubleFloat)

```
p2 := point [.25,.25]$(Point DFLOAT)
```

$$[0.25,\ 0.25] \tag{3}$$

Point(DoubleFloat)

```
p3 := point [.5,.5]$(Point DFLOAT)
```

$$[0.5,\ 0.5] \tag{4}$$

Point(DoubleFloat)

```
p4 := point [.75,.75]$(Point DFLOAT)
```

$$[0.75,\ 0.75] \tag{5}$$

Point(DoubleFloat)

```
p5 := point [1,1]$(Point DFLOAT)
```

$$[1.0,\ 1.0] \tag{6}$$

Point(DoubleFloat)

```
component(g,p1)$GRIMAGE

component(g,p2)$GRIMAGE

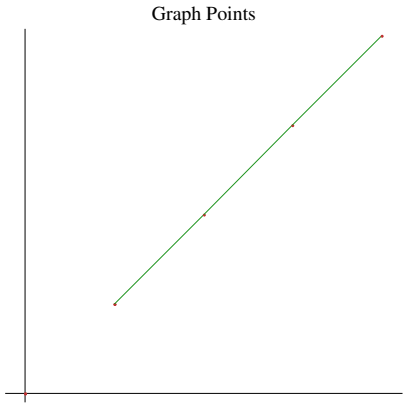
appendPoint(g,p3)$GRIMAGE

appendPoint(g,p4)$GRIMAGE

appendPoint(g,p5)$GRIMAGE
```

Here is the graph.

```
makeViewport2D(g,[title("Graph Points")])$VIEW2D
```



A list of points can also be made into a **GraphImage** by using the operation coerce. It is equivalent to adding each point to g2 using component.

```
g2 := coerce([[p1],[p2],[p3],[p4],[p5]])$GRIMAGE
```

Graph with 5 point lists (12)

GraphImage

Now, create an empty **TwoDimensionalViewport**.

```
v := viewport2D()$VIEW2D
```

Closed or Undefined TwoDimensionalViewport: "FriCAS2D" (13)

TwoDimensionalViewport

```
options(v,[title("Just Points")])$VIEW2D
```

Closed or Undefined TwoDimensionalViewport: "FriCAS2D" (14)

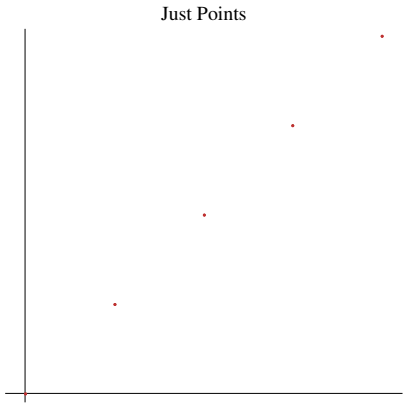
TwoDimensionalViewport

Place the graph into the viewport.

```
putGraph(v,g2,1)$VIEW2D
```

Take a look.

```
makeViewport2D(v)$VIEW2D
```



Creating a Two-Dimensional Viewport of a List of Points from a File

The following three functions read a list of points from a file and then draw the points and the connecting lines. The points are stored in the file in readable form as floating point numbers (specifically, **DoubleFloat** values) as an alternating stream of `x`- and `y`-values. For example,

```
0.0 0.0     1.0 1.0     2.0 4.0
3.0 9.0     4.0 16.0    5.0 25.0
```

```
drawPoints(lp:List Point DoubleFloat):VIEW2D ==
  g := graphImage()$GRIMAGE
  for p in lp repeat
    component(g,p,pointColorDefault(),lineColorDefault(),
      pointSizeDefault())
  makeViewport2D(g,[title("Points")])$VIEW2D

drawLines(lp:List Point DoubleFloat):VIEW2D ==
  g := graphImage()$GRIMAGE
  component(g, lp, pointColorDefault(), lineColorDefault(),
    pointSizeDefault())$GRIMAGE
  makeViewport2D(g,[title("Points")])$VIEW2D

plotData2D(name, title) ==
  f:File(DFLOAT) := open(name,"input")
  lp:LIST(Point DFLOAT) := empty()
  while ((x := readIfCan!(f)) case DFLOAT) repeat
    y : DFLOAT := read!(f)
    lp := cons(point [x,y]$(Point DFLOAT), lp)
    lp
  close!(f)
  drawPoints(lp)
  drawLines(lp)
```

This command will actually create the viewport and the graph if the point data is in the file `"file.data"`.

```
plotData2D("file.data", "2D Data Plot")
```

7.1.10 Addendum: Appending a Graph to a Viewport Window Containing a Graph

This section demonstrates how to append a two-dimensional graph to a viewport already containing other graphs. The default draw command places a graph into the first **GraphImage** slot position of the **TwoDimensionalViewport**.

This graph is in the first slot in its viewport.

```
v1 := draw(sin(x),x=0..2*%pi)
```

```
   Compiling function %C with type DoubleFloat -> DoubleFloat
```

```
                                TwoDimensionalViewport: "sin(x)"                (1)
                                                          TwoDimensionalViewport
```

So is this graph.

```
v2 := draw(cos(x),x=0..2*%pi, curveColor==light red())
```

```
   Compiling function %E with type DoubleFloat -> DoubleFloat
```

```
                                TwoDimensionalViewport: "cos(x)"                (2)
                                                          TwoDimensionalViewport
```

The operation getGraph retrieves the **GraphImage** g1 from the first slot position in the viewport v1.

```
g1 := getGraph(v1,1)
```

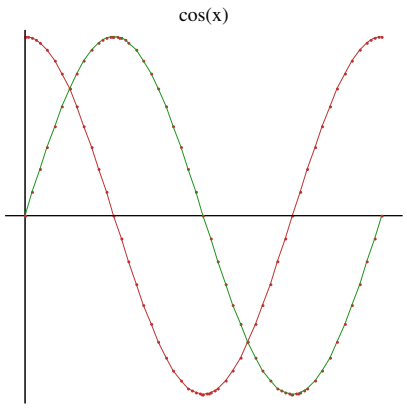
```
                                     Graph with 1 point list                    (3)
                                                                    GraphImage
```

Now putGraph places g1 into the second slot position of v2.

```
putGraph(v2,g1,2)
```

Display the new **TwoDimensionalViewport** containing both graphs.

```
makeViewport2D(v2)
```



Instead of using draw to draw a graph and then extract graph data we can use makeObject. First graph.

```
g3 := makeObject(sin(x),x=-1..%pi,[])
```

```
   Compiling function %G with type DoubleFloat -> DoubleFloat
```

```
                                  Graph with 1 point list                    (5)
                                                                    GraphImage
```

This graph is in the first slot in its viewport.

```
v3 := draw(cos(x),x=-1..%pi, curveColor==light red())
```

```
   Compiling function %I with type DoubleFloat -> DoubleFloat
```

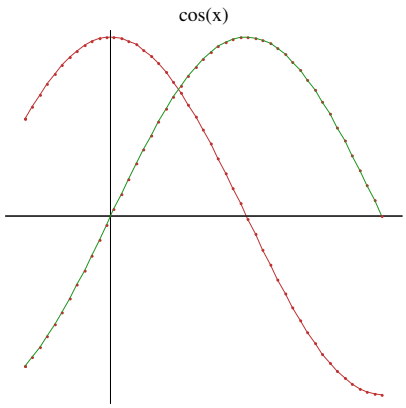
```
                              TwoDimensionalViewport: "cos(x)"               (6)
                                                              TwoDimensionalViewport
```

Now putGraph places g3 into the second slot position of v3.

```
putGraph(v3,g3,2)
```

Display the new **TwoDimensionalViewport** containing both graphs.

```
makeViewport2D(v3)
```



The viewports v1, v2 and v3 are no longer needed so we close them.

```
close(v1); close(v2); close(v3)
```

7.2 Three-Dimensional Graphics

The FriCAS three-dimensional graphics package provides the ability to

- generate surfaces defined by a function of two real variables
- generate space curves and tubes defined by parametric equations

- generate surfaces defined by parametric equations

These graphs can be modified by using various options, such as calculating points in the spherical coordinate system or changing the polygon grid size of a surface.

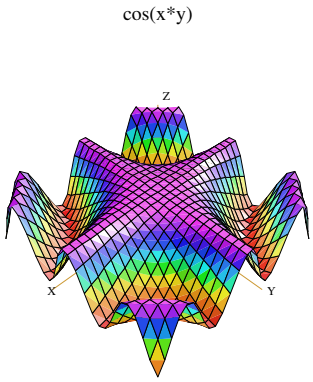
7.2.1 Plotting Three-Dimensional Functions of Two Variables

The simplest three-dimensional graph is that of a surface defined by a function of two variables, `z = f(x,y)`.

> The general format for drawing a surface defined by a formula `f(x,y)` of two variables `x` and `y` is:
>
> `draw(f(x,y), x = a..b, y = c..d,` *options*`)`
>
> where `a..b` and `c..d` define the range of `x` and `y`, and where *options* prescribe zero or more options as described in Section 7.2.4 on page 225. An example of an option is `title == "Title of Graph"`. An alternative format involving a function `f` is also available.

The simplest way to plot a function of two variables is to use a formula. With formulas you always precede the range specifications with the variable name and an "`=`" sign. Notice that FriCAS uses the text of your function as a default title.

```
draw(cos(x*y),x=-3..3,y=-3..3)
```

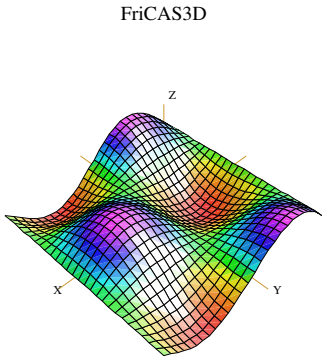


If you intend to use a function more than once, or it is long and complex, then first give its definition to FriCAS.

```
f(x,y) == sin(x)*cos(y)
```

To draw the function, just give its name and drop the variables from the range specifications. FriCAS compiles your function for efficient computation of data for the graph.

```
draw(f,-%pi..%pi,-%pi..%pi)
```



7.2.2 Plotting Three-Dimensional Parametric Space Curves

A second kind of three-dimensional graph is a three-dimensional space curve defined by the parametric equations for `x(t)`, `y(t)`, and `z(t)` as a function of an independent variable `t`.

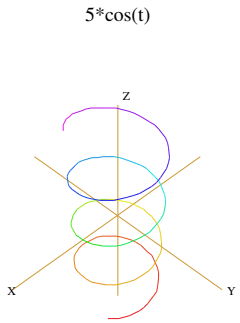
The general format for drawing a three-dimensional space curve defined by parametric formulas `x = f(t)`, `y = g(t)`, and `z = h(t)` is:

`draw(curve(f(t),g(t),h(t)), t = a..b,` *options*`)`

where `a..b` defines the range of the independent variable `t`, and where *options* prescribe zero or more options as described in Section 7.2.4 on page 225. An example of an option is `title == "Title of Graph"`. An alternative format involving functions `f`, `g` and `h` is also available.

If you use explicit formulas to draw a space curve, always precede the range specification with the variable name and an "`=`" sign.

```
draw(curve(5*cos(t), 5*sin(t),t), t=-12..12)
```



Alternatively, you can draw space curves by referring to functions.

```
i1(t:DFLOAT):DFLOAT == sin(t)*cos(3*t/5)
```

```
   Function declaration i1 : DoubleFloat -> DoubleFloat has been added to
      workspace.
```

This is useful if the functions are to be used more than once ...

```
i2(t:DFLOAT):DFLOAT == cos(t)*cos(3*t/5)
```

```
   Function declaration i2 : DoubleFloat -> DoubleFloat has been added to
      workspace.
```

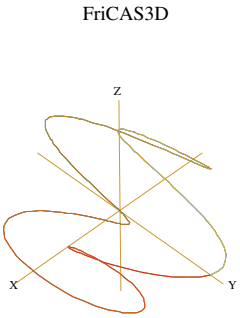
or if the functions are long and complex.

```
i3(t:DFLOAT):DFLOAT == cos(t)*sin(3*t/5)
```

```
   Function declaration i3 : DoubleFloat -> DoubleFloat has been added to
      workspace.
```

Give the names of the functions and drop the variable name specification in the second argument. Again, FriCAS supplies a default title.

```
draw(curve(i1,i2,i3),0..15*%pi)
```



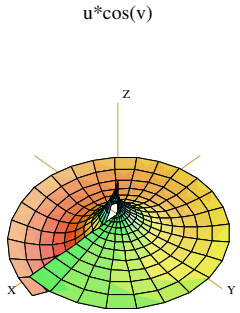
7.2.3 Plotting Three-Dimensional Parametric Surfaces

A third kind of three-dimensional graph is a surface defined by parametric equations for `x(u,v)`, `y(u,v)`, and `z(u,v)` of two independent variables `u` and `v`.

> The general format for drawing a three-dimensional graph defined by parametric formulas `x = f(u,v)`, `y = g(u,v)`, and `z = h(u,v)` is:
>
> `draw(surface(f(u,v),g(u,v),h(u,v)), u = a..b, v = c..d,` *options*`)`
>
> where `a..b` and `c..d` define the range of the independent variables `u` and `v`, and where *options* prescribe zero or more options as described in Section 7.2.4 on page 225. An example of an option is `title == "Title of Graph"`. An alternative format involving functions `f`, `g` and `h` is also available.

This example draws a graph of a surface plotted using the parabolic cylindrical coordinate system option. The values of the functions supplied to surface are interpreted in coordinates as given by a `coordinates` option, here as parabolic cylindrical coordinates (see Section 7.2.7 on page 235).

```
draw(surface(u*cos(v), u*sin(v), v*cos(u)), u=-4..4, v=0..%pi, coordinates== _
    parabolicCylindrical)
```



Again, you can graph these parametric surfaces using functions, if the functions are long and complex. Here we declare the types of arguments and values to be of type **DoubleFloat**.

```
n1(u:DFLOAT,v:DFLOAT):DFLOAT == u*cos(v)
```

```
   Function declaration n1 : (DoubleFloat, DoubleFloat) -> DoubleFloat has been
      added to workspace.
```

As shown by previous examples, these declarations are necessary.

```
n2(u:DFLOAT,v:DFLOAT):DFLOAT == u*sin(v)
```

```
   Function declaration n2 : (DoubleFloat, DoubleFloat) -> DoubleFloat has been
      added to workspace.
```

In either case, FriCAS compiles the functions when needed to graph a result.

```
n3(u:DFLOAT,v:DFLOAT):DFLOAT == u
```

```
   Function declaration n3 : (DoubleFloat, DoubleFloat) -> DoubleFloat has been
      added to workspace.
```

Without these declarations, you have to suffix floats with `@DFLOAT` to get a **DoubleFloat** result. However, a call here with an unadorned float produces a **DoubleFloat**.

```
n3(0.5,1.0)
```

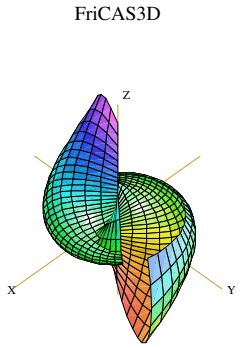
```
   Compiling function n3 with type (DoubleFloat, DoubleFloat) -> DoubleFloat
```

$$0.5 \tag{4}$$

DoubleFloat

Draw the surface by referencing the function names, this time choosing the toroidal coordinate system.

```
draw(surface(n1,n2,n3), 1..4, 1..2*%pi, coordinates == toroidal(1$DFLOAT))
```



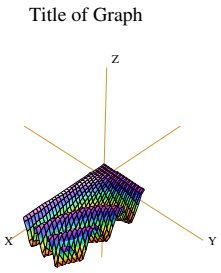
7.2.4 Three-Dimensional Options

The draw commands optionally take an optional list of options such as `coordinates` as shown in the last example. Each option is given by the syntax: `name == value`. Here is a list of the available options in the order that they are described below:

title	coordinates	var1Steps
style	tubeRadius	var2Steps
colorFunction	tubePoints	space

The option `title` gives your graph a title.

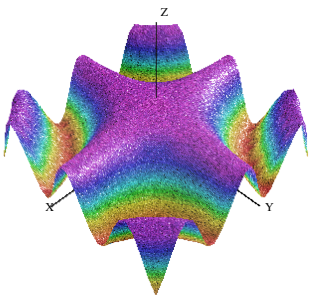
```
draw(cos(x*y),x=0..2*%pi,y=0..%pi,title == "Title of Graph")
```



The `style` determines which of four rendering algorithms is used for the graph. The choices are `"wireMesh"`, `"solid"`, `"shade"`, and `"smooth"`.

```
draw(cos(x*y),x=-3..3,y=-3..3, style=="smooth", title=="Smooth Option")
```

Smooth Option



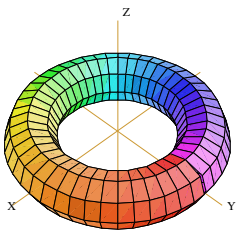
In all but the wire-mesh style, polygons in a surface or tube plot are normally colored in a graph according to their z-coordinate value. Space curves are colored according to their parametric variable value. To change this, you can give a coloring function. The coloring function is sampled across the range of its arguments, then normalized onto the standard FriCAS colormap.

A function of one variable makes the color depend on the value of the parametric variable specified for a tube plot.

```
color1(t) == t
```

```
draw(curve(sin(t), cos(t),0), t=0..2*%pi, tubeRadius == .3, colorFunction == color1)
```

sin(t)

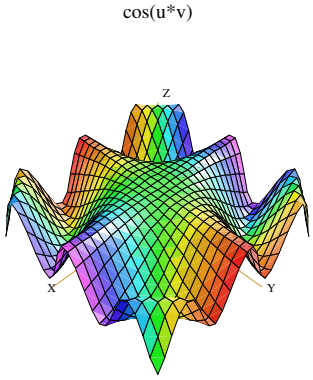


A function of two variables makes the color depend on the values of the independent variables.

```
color2(u,v) == u^2 - v^2
```

Use the option `colorFunction` for special coloring.

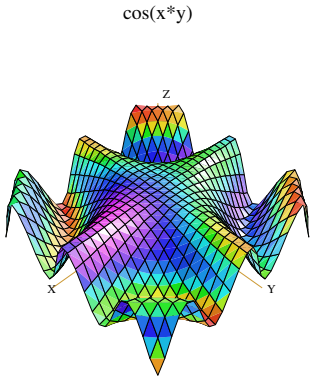
```
draw(cos(u*v), u=-3..3, v=-3..3, colorFunction == color2)
```



With a three variable function, the color also depends on the value of the function.

```
color3(x,y,fxy) == sin(x*fxy) + cos(y*fxy)
```

```
draw(cos(x*y), x=-3..3, y=-3..3, colorFunction == color3)
```



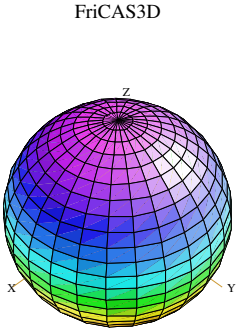
Normally the Cartesian coordinate system is used. To change this, use the `coordinates` option. For details, see Section 7.2.7 on page 235.

```
m(u:DFLOAT,v:DFLOAT):DFLOAT == 1
```

```
   Function declaration m : (DoubleFloat, DoubleFloat) -> DoubleFloat has been
      added to workspace.
```

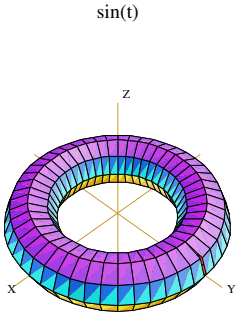
Use the spherical coordinate system.

```
draw(m, 0..2*%pi,0..%pi, coordinates == spherical, style=="shade")
```



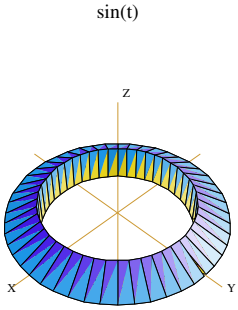
Space curves may be displayed as tubes with polygonal cross sections. Two options, `tubeRadius` and `tubePoints`, control the size and shape of this cross section. The `tubeRadius` option specifies the radius of the tube that encircles the specified space curve.

```
draw(curve(sin(t),cos(t),0),t=0..2*%pi, style=="shade", tubeRadius == .3)
```



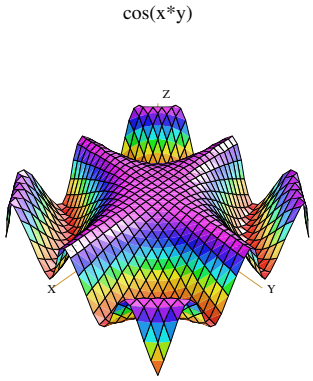
The `tubePoints` option specifies the number of vertices defining the polygon that is used to create a tube around the specified space curve. The larger this number is, the more cylindrical the tube becomes.

```
draw(curve(sin(t), cos(t), 0), t=0..2*%pi, style=="shade", tubeRadius == .25, _
    tubePoints == 3)
```



Options var1Steps and var2Steps specify the number of intervals into which the grid defining a surface plot is subdivided with respect to the first and second parameters of the surface function(s).

```
draw(cos(x*y),x=-3..3,y=-3..3, style=="shade", var1Steps == 30, var2Steps == 30)
```



The space option of a draw command lets you build multiple graphs in three space. To use this option, first create an empty three-space object, then use the space option thereafter. There is no restriction as to the number or kinds of graphs that can be combined this way. Create an empty three-space object.

```
s := create3Space()$(ThreeSpace DFLOAT)
```

$$\text{3-Space with 0 components} \tag{5}$$

ThreeSpace(DoubleFloat)

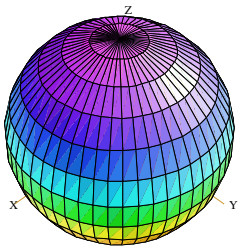
```
m(u:DFLOAT,v:DFLOAT):DFLOAT == 1
```

```
   Function declaration m : (DoubleFloat, DoubleFloat) -> DoubleFloat has been
      added to workspace.
   1 old definition(s) deleted for function or rule m
```

Add a graph to this three-space object. The new graph destructively inserts the graph into s.

```
draw(m,0..%pi,0..2*%pi, coordinates == spherical, space == s)
```

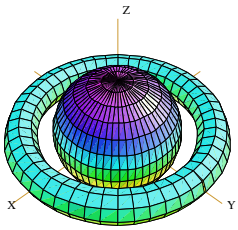
FriCAS3D



Add a second graph to s.

```
v := draw(curve(1.5*sin(t), 1.5*cos(t),0), t=0..2*%pi, tubeRadius == .25, space == s)
```

FriCAS3D



A three-space object can also be obtained from an existing three-dimensional viewport using the subspace command. You can then use makeViewport3D to create a viewport window. Assign to subsp the three-space object in viewport v.

```
subsp := subspace v
```

Reset the space component of v to the value of subsp.

```
subspace(v, subsp)
```

Create a viewport window from a three-space object.

```
makeViewport3D(subsp,"Graphs")
```

7.2.5 The makeObject Command

An alternate way to create multiple graphs is to use makeObject. The makeObject command is similar to the draw command, except that it returns a three-space object rather than a **ThreeDimensionalViewport**. In fact, makeObject is called by the draw command to create the **ThreeSpace** then makeViewport3D to create a viewport window.

```
m(u:DFLOAT,v:DFLOAT):DFLOAT == 1
```

```
   Function declaration m : (DoubleFloat, DoubleFloat) -> DoubleFloat has been
      added to workspace.
```

Do the last example a new way. First use makeObject to create a three-space object sph.

```
sph := makeObject(m, 0..%pi, 0..2*%pi, coordinates==spherical)
```

```
   Compiling function m with type (DoubleFloat, DoubleFloat) -> DoubleFloat
```

$$\text{3-Space with 1 component} \tag{2}$$

ThreeSpace(DoubleFloat)

Add a second object to sph.

```
makeObject(curve(1.5*sin(t), 1.5*cos(t), 0), t=0..2*%pi, space == sph, tubeRadius == _
    .25)
```

```
   Compiling function %M with type DoubleFloat -> DoubleFloat
   Compiling function %O with type DoubleFloat -> DoubleFloat
   Compiling function %Q with type DoubleFloat -> DoubleFloat
```

$$\text{3-Space with 2 components} \tag{3}$$

ThreeSpace(DoubleFloat)

Create and display a viewport containing sph.

```
makeViewport3D(sph,"Multiple Objects")
```

Note that an undefined **ThreeSpace** parameter declared in a makeObject or draw command results in an error. Use the create3Space function to define a **ThreeSpace**, or obtain a **ThreeSpace** that has been previously generated before including it in a command line.

7.2.6 Building Three-Dimensional Objects From Primitives

Rather than using the draw and makeObject commands, you can create three-dimensional graphs from primitives. Operation create3Space creates a three-space object to which points, curves and polygons can be added using the operations from the **ThreeSpace** domain. The resulting object can then be displayed in a viewport using makeViewport3D.

Create the empty three-space object space.

```
space := create3Space()$(ThreeSpace DFLOAT)
```

$$\text{3-Space with 0 components} \tag{1}$$

ThreeSpace(DoubleFloat)

Objects can be sent to this space using the operations exported by the **ThreeSpace** domain. The following examples place curves into space.

Add these eight curves to the space.

```
closedCurve(space,[[0,30,20], [0,30,30], [0,40,30], [0,40,100], _
    [0,30,100],[0,30,110], [0,60,110], [0,60,100], [0,50,100], [0,50,30], [0,60,30], _
    [0,60,20]])
```

3-Space with 1 component (2)

ThreeSpace(DoubleFloat)

```
closedCurve(space,[[80,0,30], [80,0,100], [70,0,110], [40,0,110], [30,0,100], _
    [30,0,90], [40,0,90], [40,0,95], [45,0,100], [65,0,100], [70,0,95], [70,0,35]])
```

3-Space with 2 components (3)

ThreeSpace(DoubleFloat)

```
closedCurve(space,[[70,0,35], [65,0,30], [45,0,30], [40,0,35], [40,0,60], [50,0,60], _
    [50,0,70], [30,0,70], [30,0,30], [40,0,20], [70,0,20], [80,0,30]])
```

3-Space with 3 components (4)

ThreeSpace(DoubleFloat)

```
closedCurve(space,[[0,70,20], [0,70,110], [0,110,110], [0,120,100], [0,120,70], _
    [0,115,65], [0,120,60], [0,120,30], [0,110,20], [0,80,20], [0,80,30], [0,80,20]])
```

3-Space with 4 components (5)

ThreeSpace(DoubleFloat)

```
closedCurve(space,[[0,105,30], [0,110,35], [0,110,55], [0,105,60], [0,80,60], _
    [0,80,70], [0,105,70], [0,110,75], [0,110,95], [0,105,100], [0,80,100], _
    [0,80,20], [0,80,30]])
```

3-Space with 5 components (6)

ThreeSpace(DoubleFloat)

```
closedCurve(space,[[140,0,20], [140,0,110], [130,0,110], [90,0,20], _
    [101,0,20],[114,0,50], [130,0,50], [130,0,60], [119,0,60], [130,0,85], [130,0,20]])
```

3-Space with 6 components (7)

ThreeSpace(DoubleFloat)

```
closedCurve(space,[[0,140,20], [0,140,110], [0,150,110], [0,170,50], [0,190,110], _
    [0,200,110], [0,200,20], [0,190,20], [0,190,75], [0,175,35], _
    [0,165,35],[0,150,75], [0,150,20]])
```

3-Space with 7 components (8)

ThreeSpace(DoubleFloat)

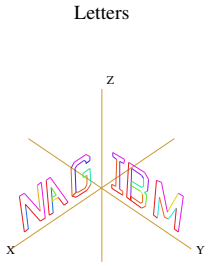
```
closedCurve(space,[[200,0,20], [200,0,110], [189,0,110], [160,0,45], [160,0,110], _
    [150,0,110], [150,0,20], [161,0,20], [190,0,85], [190,0,20]])
```

3-Space with 8 components (9)

ThreeSpace(DoubleFloat)

Create and display the viewport using makeViewport3D. Options may also be given but here are displayed as a list with values enclosed in parentheses.

```
makeViewport3D(space, title == "Letters")
```



Cube Example

As a second example of the use of primitives, we generate a cube using a polygon mesh. It is important to use a consistent orientation of the polygons for correct generation of three-dimensional objects.

Again start with an empty three-space object.

```
spaceC := create3Space()$(ThreeSpace DFLOAT)
```

$$\text{3-Space with 0 components} \tag{10}$$

ThreeSpace(DoubleFloat)

For convenience, give **DoubleFloat** values +1 and -1 names.

```
x: DFLOAT := 1
```

$$1.0 \tag{11}$$

DoubleFloat

```
y: DFLOAT := -1
```

$$-1.0 \tag{12}$$

DoubleFloat

Define the vertices of the cube.

```
a := point [x,x,y,1::DFLOAT]$(Point DFLOAT)
```

$$[1.0,\ 1.0,\ -1.0,\ 1.0] \tag{13}$$

Point(DoubleFloat)

```
b := point [y,x,y,4::DFLOAT]$(Point DFLOAT)
```

$$[-1.0,\ 1.0,\ -1.0,\ 4.0] \tag{14}$$

Point(DoubleFloat)

```
c := point [y,x,x,8::DFLOAT]$(Point DFLOAT)
```

$$[-1.0,\ 1.0,\ 1.0,\ 8.0] \tag{15}$$

Point(DoubleFloat)

```
d := point [x,x,x,12::DFLOAT]$(Point DFLOAT)
```

$$[1.0,\ 1.0,\ 1.0,\ 12.0] \tag{16}$$

Point(DoubleFloat)

```
e := point [x,y,y,16::DFLOAT]$(Point DFLOAT)
```

$$[1.0,\ -1.0,\ -1.0,\ 16.0] \tag{17}$$

Point(DoubleFloat)

```
f := point [y,y,y,20::DFLOAT]$(Point DFLOAT)
```

$$[-1.0,\ -1.0,\ -1.0,\ 20.0] \tag{18}$$

Point(DoubleFloat)

```
g := point [y,y,x,24::DFLOAT]$(Point DFLOAT)
```

$$[-1.0,\ -1.0,\ 1.0,\ 24.0] \tag{19}$$

Point(DoubleFloat)

```
h := point [x,y,x,27::DFLOAT]$(Point DFLOAT)
```

$$[1.0,\ -1.0,\ 1.0,\ 27.0] \tag{20}$$

Point(DoubleFloat)

Add the faces of the cube as polygons to the space using a consistent orientation.

```
polygon(spaceC,[d,c,g,h])
```

$$\texttt{3-Space with 1 component} \tag{21}$$

ThreeSpace(DoubleFloat)

```
polygon(spaceC,[d,h,e,a])
```

3-Space with 2 components (22)

ThreeSpace(DoubleFloat)

```
polygon(spaceC,[c,d,a,b])
```

3-Space with 3 components (23)

ThreeSpace(DoubleFloat)

```
polygon(spaceC,[g,c,b,f])
```

3-Space with 4 components (24)

ThreeSpace(DoubleFloat)

```
polygon(spaceC,[h,g,f,e])
```

3-Space with 5 components (25)

ThreeSpace(DoubleFloat)

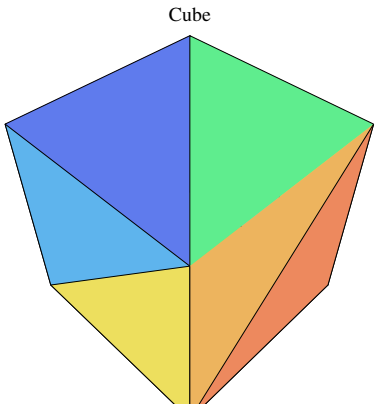
```
polygon(spaceC,[e,f,b,a])
```

3-Space with 6 components (26)

ThreeSpace(DoubleFloat)

Create and display the viewport.

```
makeViewport3D(spaceC, title == "Cube")
```



7.2.7 Coordinate System Transformations

The **CoordinateSystems** package provides coordinate transformation functions that map a given data point from the coordinate system specified into the Cartesian coordinate system. The default

coordinate system, given a triplet `(f(u,v), u, v)`, assumes that `z = f(u, v)`, `x = u` and `y = v`, that is, reads the coordinates in `(z, x, y)` order.

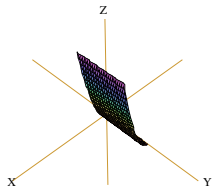
```
m(u:DFLOAT,v:DFLOAT):DFLOAT == u^2
```

```
   Function declaration m : (DoubleFloat, DoubleFloat) -> DoubleFloat has been
      added to workspace.
```

Graph plotted in default coordinate system.

```
draw(m,0..3,0..5)
```

FriCAS3D



The `z` coordinate comes first since the first argument of the draw command gives its values. In general, the coordinate systems FriCAS provides, or any that you make up, must provide a map to an `(x, y, z)` triplet in order to be compatible with the coordinates **DrawOption**. Here is an example.

Define the identity function.

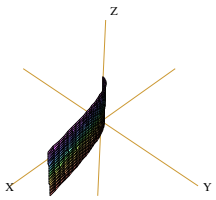
```
cartesian(point:Point DFLOAT):Point DFLOAT == point
```

```
   Function declaration cartesian : Point(DoubleFloat) -> Point(DoubleFloat) has
      been added to workspace.
```

Pass `cartesian` as the coordinates parameter to the draw command.

```
draw(m,0..3,0..5,coordinates==cartesian)
```

FriCAS3D



What happened? The option `coordinates == cartesian` directs FriCAS to treat the dependent variable `m` defined by $m = u^2$ as the `x` coordinate. Thus the triplet of values `(m, u, v)` is transformed to coordinates `(x, y, z)` and so we get the graph of $x = y^2$.

Here is another example. The cylindrical transform takes input of the form `(w,u,v)`, interprets it in the order (r,θ,z) and maps it to the Cartesian coordinates $x = r\cos(\theta)$, $y = r\sin(\theta)$, $z = z$ in which r is the radius, θ is the angle and z is the z-coordinate. An example using the cylindrical coordinates for the constant `r = 3`.

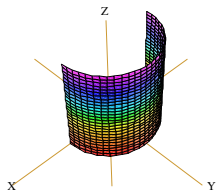
```
f(u:DFLOAT,v:DFLOAT):DFLOAT == 3
```

```
   Function declaration f : (DoubleFloat, DoubleFloat) -> DoubleFloat has been
      added to workspace.
```

Graph plotted in cylindrical coordinates.

```
draw(f,0..%pi,0..6,coordinates==cylindrical)
```

FriCAS3D



Suppose you would like to specify z as a function of r and θ instead of just r? Well, you still can use the cylindrical FriCAS transformation but we have to reorder the triplet before passing it to the transformation.

First, let's create a point to work with and call it `pt` with some color `col`.

```
col := 5
```

$$5 \tag{4}$$

PositiveInteger

```
pt := point[1,2,3,col]$(Point DFLOAT)
```

$$[1.0,\ 2.0,\ 3.0,\ 5.0] \tag{5}$$

Point(DoubleFloat)

The reordering you want is (z, r, θ) to (r, θ, z) so that the first element is moved to the third element, while the second and third elements move forward and the color element does not change. Define a function **reorder** to reorder the point elements.

```
reorder(p:Point DFLOAT):Point DFLOAT == point[p.2, p.3, p.1, p.4]
```

```
   Function declaration reorder : Point(DoubleFloat) -> Point(DoubleFloat) has
      been added to workspace.
```

The function moves the second and third elements forward but the color does not change.

```
reorder pt
```

```
   Compiling function reorder with type Point(DoubleFloat) -> Point(DoubleFloat)
```

$$[2.0,\ 3.0,\ 1.0,\ 5.0] \tag{7}$$

Point(DoubleFloat)

The function **newmap** converts our reordered version of the cylindrical coordinate system to the standard (x, y, z) Cartesian system.

```
newmap(pt:Point DFLOAT):Point DFLOAT == cylindrical(reorder pt)
```

```
   Function declaration newmap : Point(DoubleFloat) -> Point(DoubleFloat) has been
      added to workspace.
```

```
newmap pt
```

```
   Compiling function newmap with type Point(DoubleFloat) -> Point(DoubleFloat)
```

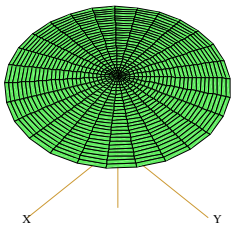
$$[-1.9799849932008908,\ 0.2822400161197344,\ 1.0,\ 5.0] \tag{9}$$

Point(DoubleFloat)

Graph the same function `f` using the coordinate mapping of the function `newmap`, so it is now interpreted as $z = 3$:

```
draw(f,0..3,0..2*%pi,coordinates==newmap)
```

FriCAS3D



The **CoordinateSystems** package exports the following operations: bipolar, bipolarCylindrical, cartesian, conical, cylindrical, elliptic, ellipticCylindrical, oblateSpheroidal, parabolic, parabolicCylindrical, paraboloidal, polar, prolateSpheroidal, spherical, and toroidal. Use Browse or the `)show` system command to get more information.

7.2.8 Three-Dimensional Clipping

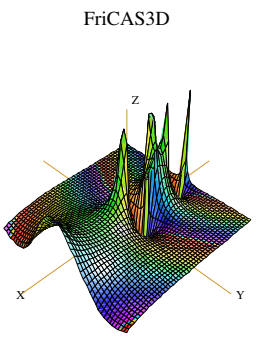
A three-dimensional graph can be explicitly clipped within the draw command by indicating a minimum and maximum threshold for the given function definition. These thresholds can be defined using the

FriCAS min and max functions.

```
gamma(x,y) ==
  g := Gamma complex(x,y)
  point [x, y, max( min(real g, 4), -4), argument g]
```

Here is an example that clips the gamma function in order to eliminate the extreme divergence it creates.

```
draw(gamma,-%pi..%pi,-%pi..%pi,var1Steps==50,var2Steps==50)
```



7.2.9 Three-Dimensional Control-Panel

Once you have created a viewport, move your mouse to the viewport and click with your left mouse button. This displays a control-panel on the side of the viewport that is closest to where you clicked.

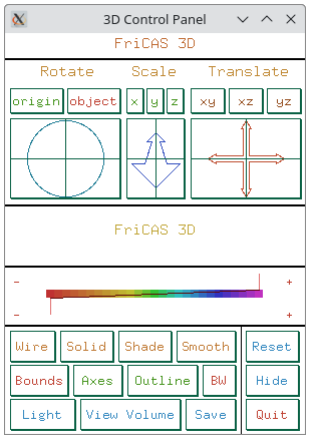


Figure 7.3: Three-dimensional control-panel.

Transformations

We recommend you first select the **Bounds** button while executing transformations since the bounding box displayed indicates the object's position as it changes.

Rotate: A rotation transformation occurs by clicking the mouse within the **Rotate** window in the upper left corner of the control-panel. The rotation is computed in spherical coordinates, using the horizontal mouse position to increment or decrement the value of the longitudinal angle θ within the range of 0 to 2π and the vertical mouse position to increment or decrement the value of the latitudinal angle φ within the range of -π to π. The active mode of rotation is displayed in green on a color monitor or in clear text on a black and white monitor, while the inactive mode is displayed in red for color display or a mottled pattern for black and white.

origin: The **origin** button indicates that the rotation is to occur with respect to the origin of the viewing space, that is indicated by the axes.

object: The **object** button indicates that the rotation is to occur with respect to the center of volume of the object, independent of the axes' origin position.

Scale: A scaling transformation occurs by clicking the mouse within the **Scale** window in the upper center of the control-panel, containing a zoom arrow. The axes along which the scaling is to occur are indicated by selecting the appropriate button above the zoom arrow window. The selected axes are displayed in green on a color monitor or in clear text on a black and white monitor, while the unselected axes are displayed in red for a color display or a mottled pattern for black and white.

uniform: Uniform scaling along the `x`, `y` and `z` axes occurs when all the axes buttons are selected.

non-uniform: If any of the axes buttons are not selected, non-uniform scaling occurs, that is, scaling occurs only in the direction of the axes that are selected.

Translate: Translation occurs by indicating with the mouse in the **Translate** window the direction you want the graph to move. This window is located in the upper right corner of the control-panel and contains a potentiometer with crossed arrows pointing up, down, left and right. Along the top of the **Translate** window are three buttons (**XY**, **XZ**, and **YZ**) indicating the three orthographic projection planes. Each orientates the group as a view into that plane. Any translation of the graph occurs only along this plane.

Messages

The window directly below the potentiometer windows for transformations is used to display system messages relating to the viewport, the control-panel and the current graph displaying status.

Colormap

Directly below the message window is the colormap range indicator window. The FriCAS Colormap shows a sampling of the spectrum from which hues can be drawn to represent the colors of a surface. The Colormap is composed of five shades for each of the hues along this spectrum. By moving the markers above and below the Colormap, the range of hues that are used to color the existing surface are set. The bottom marker shows the hue for the low end of the color range and the top marker shows the hue for the upper end of the range. Setting the bottom and top markers at the same hue results

in monochromatic smooth shading of the graph when **Smooth** mode is selected. At each end of the Colormap are + and - buttons. When clicked on, these increment or decrement the top or bottom marker.

Buttons

Below the Colormap window and to the left are located various buttons that determine the characteristics of a graph. The buttons along the bottom and right hand side all have special meanings; the remaining buttons in the first row indicate the mode or style used to display the graph. The second row are toggles that turn on or off a property of the graph. On a color monitor, the property is on if green (clear text, on a monochrome monitor) and off if red (mottled pattern, on a monochrome monitor). Here is a list of their functions.

Wire displays surface and tube plots as a wireframe image in a single color (blue) with no hidden surfaces removed, or displays space curve plots in colors based upon their parametric variables. This is the fastest mode for displaying a graph. This is very useful when you want to find a good orientation of your graph.

Solid displays the graph with hidden surfaces removed, drawing each polygon beginning with the furthest from the viewer. The edges of the polygons are displayed in the hues specified by the range in the Colormap window.

Shade displays the graph with hidden surfaces removed and with the polygons shaded, drawing each polygon beginning with the furthest from the viewer. Polygons are shaded in the hues specified by the range in the Colormap window using the Phong illumination model.

Smooth displays the graph using a renderer that computes the graph one line at a time. The location and color of the graph at each visible point on the screen are determined and displayed using the Phong illumination model. Smooth shading is done in one of two ways, depending on the range selected in the colormap window and the number of colors available from the hardware and/or window manager. When the top and bottom markers of the colormap range are set to different hues, the graph is rendered by dithering between the transitions in color hue. When the top and bottom markers of the colormap range are set to the same hue, the graph is rendered using the Phong smooth shading model. However, if enough colors cannot be allocated for this purpose, the renderer reverts to the color dithering method until a sufficient color supply is available. For this reason, it may not be possible to render multiple Phong smooth shaded graphs at the same time on some systems.

Bounds encloses the entire volume of the viewgraph within a bounding box, or removes the box if previously selected. The region that encloses the entire volume of the viewport graph is displayed.

Axes displays Cartesian coordinate axes of the space, or turns them off if previously selected.

Outline causes quadrilateral polygons forming the graph surface to be outlined in black when the graph is displayed in **Shade** mode.

BW converts a color viewport to black and white, or vice-versa. When this button is selected the control-panel and viewport switch to an immutable colormap composed of a range of grey scale patterns or tiles that are used wherever shading is necessary.

Light takes you to a control-panel described below.

ViewVolume takes you to another control-panel as described below.

Save creates a menu of the possible file types that can be written using the control-panel. The **Exit** button leaves the save menu. The **Pixmap** button writes an FriCAS pixmap of the current viewport contents. The file is called **fricas3D.pixmap** and is located in the directory from which FriCAS or **viewAlone** was started. The **PS** button writes the current viewport contents to PostScript output rather than to the viewport window. By default the file is called **fricas3D.ps**; however, if a file name is specified in the user's **.Xdefaults** file it is used. The file is placed in the directory from which the FriCAS or **viewAlone** session was begun. See also the 'write' function.

Reset returns the object transformation characteristics back to their initial states.

Hide causes the control-panel for the corresponding viewport to disappear from the screen.

Quit queries whether the current viewport session should be terminated.

Light

The **Light** button changes the control-panel into the **Lighting Control-Panel**. At the top of this panel, the three axes are shown with the same orientation as the object. A light vector from the origin of the axes shows the current position of the light source relative to the object. At the bottom of the panel is an **Abort** button that cancels any changes to the lighting that were made, and a **Return** button that carries out the current set of lighting changes on the graph.

XY: The **XY** lighting axes window is below the **Lighting Control-Panel** title and to the left. This changes the light vector within the **XY** view plane.

Z: The **Z** lighting axis window is below the **Lighting Control-Panel** title and in the center. This changes the **Z** location of the light vector.

Intensity: Below the **Lighting Control-Panel** title and to the right is the light intensity meter. Moving the intensity indicator down decreases the amount of light emitted from the light source. When the indicator is at the top of the meter the light source is emitting at 100% intensity. At the bottom of the meter the light source is emitting at a level slightly above ambient lighting.

View Volume

The **View Volume** button changes the control-panel into the **Viewing Volume Panel**. At the bottom of the viewing panel is an **Abort** button that cancels any changes to the viewing volume that were made and a *Return* button that carries out the current set of viewing changes to the graph.

Eye Reference: At the top of this panel is the **Eye Reference** window. It shows a planar projection of the viewing pyramid from the eye of the viewer relative to the location of the object. This has a bounding region represented by the rectangle on the left. Below the object rectangle is the **Hither** window. By moving the slider in this window the hither clipping plane sets the front of the view volume. As a result of this depth clipping all points of the object closer to the eye than this hither plane are not shown. The **Eye Distance** slider to the right of the **Hither** slider is used to change the degree of perspective in the image.

Clip Volume: The **Clip Volume** window is at the bottom of the **Viewing Volume Panel**. On the right is a **Settings** menu. In this menu are buttons to select viewing attributes. Selecting the **Perspective** button computes the image using perspective projection. The **Show Region** button indicates whether the clipping region of the volume is to be drawn in the viewport and the **Clipping On** button shows whether the view volume clipping is to be in effect when the image is drawn. The left side of the **Clip Volume** window shows the clipping boundary of the graph. Moving the knobs along the **X**, **Y**, and **Z** sliders adjusts the volume of the clipping region accordingly.

7.2.10 Operations for Three-Dimensional Graphics

Here is a summary of useful FriCAS operations for three-dimensional graphics. Each operation name is followed by a list of arguments. Each argument is written as a variable informally named according to the type of the argument (for example, *integer*). If appropriate, a default value for an argument is given in parentheses immediately following the name.

adaptive3D? `()`
tests whether space curves are to be plotted according to the adaptive refinement algorithm.

axes (*viewport, string*`("on")`)
turns the axes on and off.

close (*viewport*)
closes the viewport.

colorDef (*viewport, color*$_1$`(1)`, *color*$_2$`(27)`)
sets the colormap range to be from *color*$_1$ to *color*$_2$.

controlPanel (*viewport, string*`("off")`)
declares whether the control-panel for the viewport is to be displayed or not.

diagonals (*viewport, string*`("off")`)
declares whether the polygon outline includes the diagonals or not.

drawStyle (*viewport, style*)
selects which of four drawing styles are used: `"wireMesh"`, `"solid"`, `"shade"`, or `"smooth"`.

eyeDistance (*viewport,float*`(500)`)
sets the distance of the eye from the origin of the object for use in the 'perspective'.

key (*viewport*)
returns the operating system process ID number for the viewport.

lighting (*viewport, float*$_x$`(-0.5)`, *float*$_y$`(0.5)`, *float*$_z$`(0.5)`)
sets the Cartesian coordinates of the light source.

modifyPointData (*viewport,integer,point*)
replaces the coordinates of the point with the index *integer* with *point*.

move (*viewport, integer*$_x$`(viewPosDefault)`, *integer*$_y$`(viewPosDefault)`)
moves the upper left-hand corner of the viewport to screen position (*integer*$_x$, *integer*$_y$).

options (*viewport*)
returns a list of all current draw options.

outlineRender (*viewport, string*(`"off"`))
turns polygon outlining off or on when drawing in `"shade"` mode.

perspective (*viewport, string*(`"on"`))
turns perspective viewing on and off.

reset (*viewport*)
resets the attributes of a viewport to their initial settings.

resize (*viewport, integer*$_{\text{width}}$ (`viewSizeDefault`), *integer*$_{\text{height}}$ (`viewSizeDefault`))
resets the width and height values for a viewport.

rotate (*viewport, number*$_{\theta}$(`viewThetaDefault`), *number*$_{\varphi}$(`viewPhiDefault`))
rotates the viewport by rotation angles for longitude (θ) and latitude (φ). Angles designate radians if given as floats, or degrees if given as integers.

setAdaptive3D (*boolean*(`true`))
sets whether space curves are to be plotted according to the adaptive refinement algorithm.

setMaxPoints3D (*integer*(`1000`))
sets the default maximum number of possible points to be used when constructing a three-dimensional space curve.

setMinPoints3D (*integer*(`49`))
sets the default minimum number of possible points to be used when constructing a three-dimensional space curve.

setScreenResolution3D (*integer*(`500`))
sets the default screen resolution constant used in setting the computation limit of adaptively generated three-dimensional space curve plots.

showRegion (*viewport, string*(`"off"`))
declares whether the bounding box of a graph is shown or not.

subspace (*viewport*)
returns the space component.

subspace (*viewport, subspace*)
resets the space component to *subspace*.

title (*viewport, string*)
gives the viewport the title *string*.

translate (*viewport, float*$_{\text{x}}$(`viewDeltaXDefault`), *float*$_{\text{y}}$(`viewDeltaYDefault`))
translates the object horizontally and vertically relative to the center of the viewport.

intensity (*viewport,float*(`1.0`))
resets the intensity I of the light source, $0 \leq I \leq 1$.

tubePointsDefault ([*integer*(`6`)])
sets or indicates the default number of vertices defining the polygon that is used to create a tube around a space curve.

tubeRadiusDefault ([*float*(0.5)])
sets or indicates the default radius of the tube that encircles a space curve.

var1StepsDefault ([*integer*(27)])
sets or indicates the default number of increments into which the grid defining a surface plot is subdivided with respect to the first parameter declared in the surface function.

var2StepsDefault ([*integer*(27)])
sets or indicates the default number of increments into which the grid defining a surface plot is subdivided with respect to the second parameter declared in the surface function.

viewDefaults ([$integer_{\text{point}}$, $integer_{\text{line}}$, $integer_{\text{axes}}$, $integer_{\text{units}}$, $float_{\text{point}}$, $list_{\text{position}}$, $list_{\text{size}}$])
resets the default settings for the point color, line color, axes color, units color, point size, viewport upper left-hand corner position, and the viewport size.

viewDeltaXDefault ([*float*(0)])
resets the default horizontal offset from the center of the viewport, or returns the current default offset if no argument is given.

viewDeltaYDefault ([*float*(0)])
resets the default vertical offset from the center of the viewport, or returns the current default offset if no argument is given.

viewPhiDefault ([*float*(-π/4)])
resets the default latitudinal view angle, or returns the current default angle if no argument is given. φ is set to this value.

viewpoint (*viewport*, $float_{\text{x}}$, $float_{\text{y}}$, $float_{\text{z}}$)
sets the viewing position in Cartesian coordinates.

viewpoint (*viewport*, $float_{\theta}$, $Float_{\varphi}$)
sets the viewing position in spherical coordinates.

viewpoint (*viewport*, $Float_{\theta}$, $Float_{\varphi}$, $Float_{\text{scaleFactor}}$, $Float_{\text{xOffset}}$, $Float_{\text{yOffset}}$)
sets the viewing position in spherical coordinates, the scale factor, and offsets. θ (longitude) and φ (latitude) are in radians.

viewPosDefault ([*list*([0,0])])
sets or indicates the position of the upper left-hand corner of a two-dimensional viewport, relative to the display root window (the upper left-hand corner of the display is [0, 0]).

viewSizeDefault ([*list*([400,400])])
sets or indicates the width and height dimensions of a viewport.

viewThetaDefault ([*float*(π/4)])
resets the default longitudinal view angle, or returns the current default angle if no argument is given. When a parameter is specified, the default longitudinal view angle θ is set to this value.

viewWriteAvailable ([*list*(["pixmap", "bitmap", "postscript", "image"])])
indicates the possible file types that can be created with the ‘write‘ function.

viewWriteDefault ([*list*([])])
sets or indicates the default types of files that are created in addition to the **data** file when a ‘write‘ command is executed on a viewport.

write (*viewport, directory,* [*option*])
writes the file **data** for *viewport* in the directory *directory*. An optional third argument specifies a file type (one of `pixmap`, `bitmap`, `postscript`, or `image`), or a list of file types. An additional file is written for each file type listed.

zoom (*viewport, float*`(2.5)`)
specifies the scaling factor.

7.2.11 Customization using .Xdefaults

Both the two-dimensional and three-dimensional drawing facilities consult the **.Xdefaults** file for various defaults. The list of defaults that are recognized by the graphing routines is discussed in this section. These defaults are preceded by `FriCAS.3D.` for three-dimensional viewport defaults, `FriCAS.2D.` for two-dimensional viewport defaults, or `FriCAS*` (no dot) for those defaults that are acceptable to either viewport type.

`FriCAS*buttonFont:` *font*
This indicates which font type is used for the button text on the control-panel. The default value is `"Rom11"`.

`FriCAS.2D.graphFont:` *font* (2D only)
This indicates which font type is used for displaying the graph numbers and slots in the **Graphs** section of the two-dimensional control-panel. The default value is `"Rom22"`.

`FriCAS.3D.headerFont:` *font*
This indicates which font type is used for the axes labels and potentiometer header names on three-dimensional viewport windows. This is also used for two-dimensional control-panels for indicating which font type is used for potentiometer header names and multiple graph title headers. The default value is `"Itl14"`.

`FriCAS*inverse:` *switch*
This indicates whether the background color is to be inverted from white to black. If `on`, the graph viewports use black as the background color. If `off` or no declaration is made, the graph viewports use a white background. The default value is `"off"`.

`FriCAS.3D.lightingFont:` *font* (3D only)
This indicates which font type is used for the **x**, **y**, and **z** labels of the two lighting axes potentiometers, and for the **Intensity** title on the lighting control-panel. The default value is `"Rom10"`.

`FriCAS.2D.messageFont, FriCAS.3D.messageFont:` *font*
These indicate the font type to be used for the text in the control-panel message window. The default value is `"Rom14"`.

`FriCAS*monochrome:` *switch*
This indicates whether the graph viewports are to be displayed as if the monitor is black and white, that is, a 1 bit plane. If `on` is specified, the viewport display is black and white. If `off` is specified, or no declaration for this default is given, the viewports are displayed in the normal fashion for the monitor in use. The default value is `"off"`.

`FriCAS*titleFont` *font*
: This indicates which font type is used for the title text and, for three-dimensional graphs, in the lighting and viewing-volume control-panel windows. The default value is `"Rom14"`.

`FriCAS.2D.unitFont:` *font* (2D only)
: This indicates which font type is used for displaying the unit labels on two-dimensional viewport graphs. The default value is `"6x10"`.

`FriCAS.3D.volumeFont:` *font* (3D only)
: This indicates which font type is used for the **x**, **y**, and **z** labels of the clipping region sliders; for the **Perspective**, **Show Region**, and **Clipping On** buttons under **Settings**, and above the windows for the **Hither** and **Eye Distance** sliders in the **Viewing Volume Panel** of the three-dimensional control-panel. The default value is `"Rom8"`.

Part II

Advanced Problem Solving and Examples

Chapter 8

Advanced Problem Solving

In this chapter we describe techniques useful in solving advanced problems with FriCAS.

8.1 Numeric Functions

FriCAS provides two basic floating-point types: **Float** and **DoubleFloat**. This section describes how to use numerical operations defined on these types and the related complex types. As we mentioned in Chapter 1, the **Float** type is a software implementation of floating-point numbers in which the exponent and the significand may have any number of digits. See '`Float`' on page 419 for detailed information about this domain. The **DoubleFloat** (see '`DoubleFloat`' on page 395) is usually a hardware implementation of floating point numbers, corresponding to machine double precision. The types **Complex Float** and **Complex DoubleFloat** are the corresponding software implementations of complex floating-point numbers. In this section the term *floating-point type* means any of these four types. The floating-point types implement the basic elementary functions. These include (where "%" means **DoubleFloat**, **Float**, **Complex DoubleFloat**, or **Complex Float**):

```
exp, log: % → %
sin, cos, tan, cot, sec, csc: % → %
asin, acos, atan, acot, asec, acsc: % → %
sinh, cosh, tanh, coth, sech, csch: % → %
asinh, acosh, atanh, acoth, asech, acsch: % → %
pi: () → %
sqrt: % → %
nthRoot: (%, Integer)→ %
^: (%, Fraction Integer)→%
^: (%, %) → %
```

The handling of roots depends on whether the floating-point type is real or complex: for the real floating-point types, **DoubleFloat** and **Float**, if a real root exists the one with the same sign as the radicand is returned; for the complex floating-point types, the principal value is returned. Also, for real floating-point types the inverse functions produce errors if the results are not real. This includes cases such as `asin(1.2)`, `log(-3.2)`, `sqrt(-1.1)`. The default floating-point type is **Float** so to evaluate functions using **Float** or **Complex Float**, just use normal decimal notation.

```
exp(3.1)
```

$$22.197951281441633405 \tag{1}$$

Float

```
exp(3.1 + 4.5 * %i)
```

$$-4.6792348860969899118 - 21.699165928071731864\,i \tag{2}$$

Complex(Float)

To evaluate functions using **DoubleFloat** or **Complex DoubleFloat**, a declaration or conversion is required.

```
r: DFLOAT := 3.1; t: DFLOAT := 4.5; exp(r + t*%i)
```

$$-4.679234886096991 - 21.699165928071732\,i \tag{3}$$

Complex(DoubleFloat)

```
exp(3.1::DFLOAT + 4.5::DFLOAT * %i)
```

$$-4.679234886096991 - 21.699165928071732\,i \tag{4}$$

Complex(DoubleFloat)

A number of special functions are provided by the package **DoubleFloatSpecialFunctions** for the machine-precision floating-point types. The special functions provided are listed below, where `F` stands for the types **DoubleFloat** and **Complex DoubleFloat**. The real versions of the functions yield an error if the result is not real.

Gamma: `F` → `F`
`Gamma(z)` is the Euler gamma function, $\Gamma(z)$, defined by

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt.$$

Beta: `F` → `F`
`Beta(u, v)` is the Euler Beta function, $B(u, v)$, defined by

$$B(u, v) = \int_0^1 t^{u-1}(1-t)^{v-1}dt.$$

This is related to $\Gamma(z)$ by

$$B(u, v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}.$$

logGamma: `F` → `F`
`logGamma(z)` is the natural logarithm of $\Gamma(z)$. This can often be computed even if $\Gamma(z)$ cannot.

digamma: `F` → `F`
`digamma(z)`, also called `psi(z)`, is the function $\psi(z)$, defined by

$$\psi(z) = \Gamma'(z)/\Gamma(z).$$

polygamma: (NonNegativeInteger, F)→F
polygamma(n, z) is the n^{th} derivative of $\psi(z)$, written $\psi^{(n)}(z)$.

besselJ: (F,F) → F
besselJ(v,z) is the Bessel function of the first kind, $J_\nu(z)$. This function satisfies the differential equation

$$z^2w''(z) + zw'(z) + (z^2 - \nu^2)w(z) = 0.$$

besselY: (F,F) → F
besselY(v,z) is the Bessel function of the second kind, $Y_\nu(z)$. This function satisfies the same differential equation as besselJ. The implementation simply uses the relation

$$Y_\nu(z) = \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)}.$$

besselI: (F,F) → F
besselI(v,z) is the modified Bessel function of the first kind, $I_\nu(z)$. This function satisfies the differential equation

$$z^2w''(z) + zw'(z) - (z^2 + \nu^2)w(z) = 0.$$

besselK: (F,F) → F
besselK(v,z) is the modified Bessel function of the second kind, $K_\nu(z)$. This function satisfies the same differential equation as besselI. The implementation simply uses the relation

$$K_\nu(z) = \pi\frac{I_{-\nu}(z) - I_\nu(z)}{2\sin(\nu\pi)}.$$

airyAi: F → F
airyAi(z) is the Airy function $Ai(z)$. This function satisfies the differential equation $w''(z) - zw(z) = 0$. The implementation simply uses the relation

$$Ai(-z) = \frac{1}{3}\sqrt{z}(J_{-1/3}(\frac{2}{3}z^{3/2}) + J_{1/3}(\frac{2}{3}z^{3/2})).$$

airyBi: F → F
airyBi(z) is the Airy function $Bi(z)$. This function satisfies the same differential equation as airyAi. The implementation simply uses the relation

$$Bi(-z) = \frac{1}{3}\sqrt{3z}(J_{-1/3}(\frac{2}{3}z^{3/2}) - J_{1/3}(\frac{2}{3}z^{3/2})).$$

hypergeometric0F1: (F,F) → F
hypergeometric0F1(c,z) is the hypergeometric function ${}_0F_1(;c;z)$.

The package **FloatSpecialFunctions** provides the implementation of some special functions for **Float** or **Complex Float** arguments, including Gamma, Beta, logGamma, digamma. If you give **Float** or **Complex Float** arguments to a special function that has not yet been defined to accept **Float** arguments, these are respectively converted to **DoubleFloat** or **Complex DoubleFloat** arguments.

```
Gamma(0.5)^2
```

$$3.1415926535897932385 \tag{5}$$

Float

```
a := 2.1; b := 1.1; besselI(a + %i*b, b*a + 1)
```

$$2.4894824175473707 - 2.365846038146837\,i \tag{6}$$

Complex(DoubleFloat)

A number of additional operations may be used to compute numerical values. These are special polynomial functions that can be evaluated for values in any commutative ring `R`, and in particular for values in any floating-point type. The following operations are provided by the package **OrthogonalPolynomialFunctions**:

chebyshevT: `(NonNegativeInteger, R)→R`
`chebyshevT(n,z)` is the n^{th} Chebyshev polynomial of the first kind, $T_n(z)$. These are defined by

$$\frac{1-tz}{1-2tz+t^2} = \sum_{n=0}^{\infty} T_n(z)t^n.$$

chebyshevU: `(NonNegativeInteger, R)→R`
`chebyshevU(n,z)` is the n^{th} Chebyshev polynomial of the second kind, $U_n(z)$. These are defined by

$$\frac{1}{1-2tz+t^2} = \sum_{n=0}^{\infty} U_n(z)t^n.$$

hermiteH: `(NonNegativeInteger, R)→R`
`hermiteH(n,z)` is the n^{th} Hermite polynomial, $H_n(z)$. These are defined by

$$e^{2tz-t^2} = \sum_{n=0}^{\infty} H_n(z)\frac{t^n}{n!}.$$

laguerreL: `(NonNegativeInteger, R)→R`
`laguerreL(n,z)` is the n^{th} Laguerre polynomial, $L_n(z)$. These are defined by

$$\frac{e^{-\frac{tz}{1-t}}}{1-t} = \sum_{n=0}^{\infty} L_n(z)\frac{t^n}{n!}.$$

laguerreL: `(NonNegativeInteger, NonNegativeInteger, R)→R`
`laguerreL(m,n,z)` is the associated Laguerre polynomial, $L_n^m(z)$. This is the m^{th} derivative of $L_n(z)$.

legendreP: `(NonNegativeInteger, R)→R`
`legendreP(n,z)` is the n^{th} Legendre polynomial, $P_n(z)$. These are defined by

$$\frac{1}{\sqrt{1-2tz+t^2}} = \sum_{n=0}^{\infty} P_n(z)t^n.$$

These operations require non-negative integers for the indices, but otherwise the argument can be given as desired.

```
[chebyshevT(i, z) for i in 0..5]
```

$$\left[1,\ z,\ 2\,z^2-1,\ 4\,z^3-3\,z,\ 8\,z^4-8\,z^2+1,\ 16\,z^5-20\,z^3+5\,z\right] \tag{7}$$

List (Polynomial(Integer))

The expression `chebyshevT(n,z)` evaluates to the n^{th} Chebyshev polynomial of the first kind.

```
chebyshevT(3, 5.0 + 6.0*%i)
```

$$-1675.0+918.0\,i \tag{8}$$

Complex(Float)

```
chebyshevT(3, 5.0::DoubleFloat)
```

$$485.0 \tag{9}$$

DoubleFloat

The expression `chebyshevU(n,z)` evaluates to the n^{th} Chebyshev polynomial of the second kind.

```
[chebyshevU(i, z) for i in 0..5]
```

$$\left[1,\ 2\,z,\ 4\,z^2-1,\ 8\,z^3-4\,z,\ 16\,z^4-12\,z^2+1,\ 32\,z^5-32\,z^3+6\,z\right] \tag{10}$$

List (Polynomial(Integer))

```
chebyshevU(3, 0.2)
```

$$-0.736 \tag{11}$$

Float

The expression `hermiteH(n,z)` evaluates to the n^{th} Hermite polynomial.

```
[hermiteH(i, z) for i in 0..5]
```

$$\left[1,\ 2\,z,\ 4\,z^2-2,\ 8\,z^3-12\,z,\ 16\,z^4-48\,z^2+12,\ 32\,z^5-160\,z^3+120\,z\right] \tag{12}$$

List (Polynomial(Integer))

```
hermiteH(100, 1.0)
```

$$-0.144870672933793408\,E93 \tag{13}$$

Float

The expression `laguerreL(n,z)` evaluates to the n^{th} Laguerre polynomial.

```
[laguerreL(i, z) for i in 0..4]
```

$$\left[1,\ -z+1,\ z^2-4\,z+2,\ -z^3+9\,z^2-18\,z+6,\ z^4-16\,z^3+72\,z^2-96\,z+24\right] \tag{14}$$

List (Polynomial(Integer))

```
laguerreL(4, 1.2)
```

$$-13.0944 \tag{15}$$

Float

```
[laguerreL(j, 3, z) for j in 0..4]
```

$$\left[-z^3+9\,z^2-18\,z+6,\ -3\,z^2+18\,z-18,\ -6\,z+18,\ -6,\ 0\right] \tag{16}$$

List (Polynomial(Integer))

```
laguerreL(1, 3, 2.1)
```

$$6.57 \tag{17}$$

Float

The expression `legendreP(n,z)` evaluates to the n^{th} Legendre polynomial,

```
[legendreP(i,z) for i in 0..5]
```

$$\left[1,\ z,\ \frac{3}{2}\,z^2-\frac{1}{2},\ \frac{5}{2}\,z^3-\frac{3}{2}\,z,\ \frac{35}{8}\,z^4-\frac{15}{4}\,z^2+\frac{3}{8},\ \frac{63}{8}\,z^5-\frac{35}{4}\,z^3+\frac{15}{8}\,z\right] \tag{18}$$

List (Polynomial(Fraction (Integer)))

```
legendreP(3, 3.0*%i)
```

$$-72.0\,i \tag{19}$$

Complex(Float)

Finally, three number-theoretic polynomial operations may be evaluated. The following operations are provided by the package **NumberTheoreticPolynomialFunctions**. .

bernoulliB: `(NonNegativeInteger, R)→R`
`bernoulliB(n,z)` is the n^{th} Bernoulli polynomial, $B_n(z)$. These are defined by

$$\frac{te^{zt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(z)\frac{t^n}{n!}$$

eulerE: `(NonNegativeInteger, R)→R`
`eulerE(n,z)` is the n^{th} Euler polynomial, $E_n(z)$. These are defined by

$$\frac{2e^{zt}}{e^t+1}=\sum_{n=0}^{\infty}E_n(z)\frac{t^n}{n!}.$$

cyclotomic: `(NonNegativeInteger, R)→R`
`cyclotomic(n,z)` is the n^{th} cyclotomic polynomial $\Phi_n(z)$. This is the polynomial whose roots are

precisely the primitive n^{th} roots of unity. This polynomial has degree given by the Euler totient function $\varphi(n)$.

The expression `bernoulliB(n,z)` evaluates to the n^{th} Bernoulli polynomial.

```
bernoulliB(3, z)
```

$$z^3 - \frac{3}{2}z^2 + \frac{1}{2}z \tag{20}$$

Polynomial(Fraction (Integer))

```
bernoulliB(3, 0.7 + 0.4 * %i)
```

$$-0.138 - 0.116\,i \tag{21}$$

Complex(Float)

The expression `eulerE(n,z)` evaluates to the n^{th} Euler polynomial.

```
eulerE(3, z)
```

$$z^3 - \frac{3}{2}z^2 + \frac{1}{4} \tag{22}$$

Polynomial(Fraction (Integer))

```
eulerE(3, 0.7 + 0.4 * %i)
```

$$-0.238 - 0.316\,i \tag{23}$$

Complex(Float)

The expression `cyclotomic(n,z)` evaluates to the n^{th} cyclotomic polynomial.

```
cyclotomic(3, z)
```

$$z^2 + z + 1 \tag{24}$$

Polynomial(Integer)

```
cyclotomic(3, (-1.0 + 0.0 * %i)^(2/3))
```

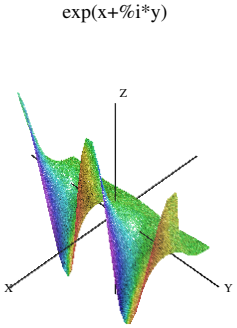
$$0.0 \tag{25}$$

Complex(Float)

Drawing complex functions in FriCAS is presently somewhat awkward compared to drawing real functions. It is necessary to use the draw operations that operate on functions rather than expressions.

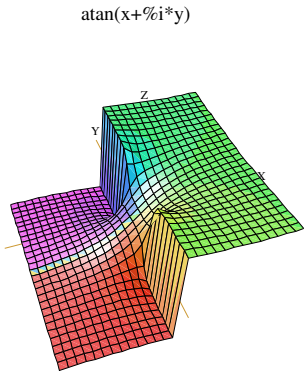
This is the complex exponential function (rotated interactively). When this is displayed in color, the height is the value of the real part of the function and the color is the imaginary part. Red indicates large negative imaginary values, green indicates imaginary values near zero and blue/violet indicates large positive imaginary values.

```
draw((x,y)+-> real exp complex(x,y), -2..2, -2*%pi..2*%pi, colorFunction == (x, y) _
    +->  imag exp complex(x,y), title=="exp(x+%i*y)", style=="smooth")
```



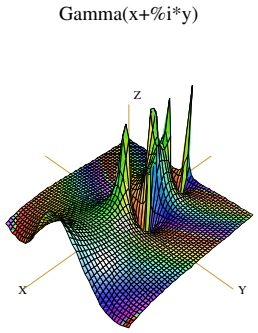
This is the complex arctangent function. Again, the height is the real part of the function value but here the color indicates the function value's phase. The position of the branch cuts are clearly visible and one can see that the function is real only for a real argument.

```
vp := draw((x,y) +-> real atan complex(x,y), -%pi..%pi, -%pi..%pi, _
    colorFunction==(x,y) +->argument atan complex(x,y), title=="atan(x+%i*y)", _
    style=="shade"); rotate(vp,-160,-45); vp
```



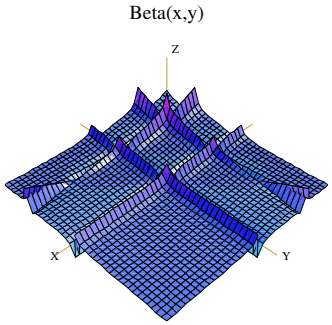
This is the complex Gamma function.

```
draw((x,y) +-> max(min(real Gamma complex(x,y),4),-4), -%pi..%pi, -%pi..%pi, _
    style=="shade", colorFunction == (x,y) +-> argument Gamma complex(x,y), title == _
    "Gamma(x+%i*y)", var1Steps == 50, var2Steps== 50)
```



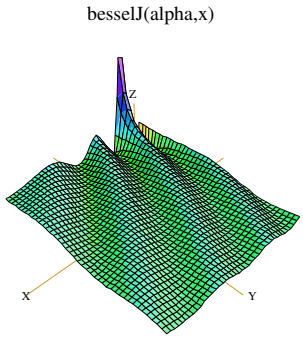
This shows the real Beta function near the origin.

```
draw(Beta(x,y)/100, x=-1.6..1.7, y = -1.6..1.7, style=="shade", title=="Beta(x,y)", _
    var1Steps==40, var2Steps==40)
```



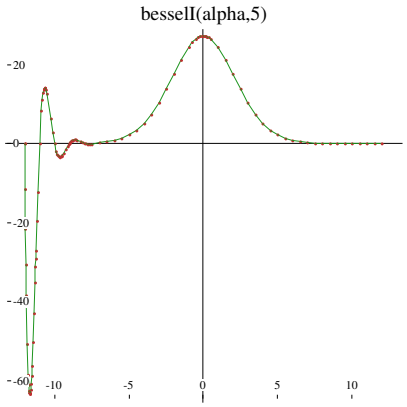
This is the Bessel function $J_\alpha(x)$ for index α in the range `-6..4` and argument x in the range `2..14`.

```
draw((alpha,x) +-> min(max(besselJ(alpha, x+8), -6), 6), -6..4, -6..6, _
    title=="besselJ(alpha,x)", style=="shade", var1Steps==40, var2Steps==40)
```



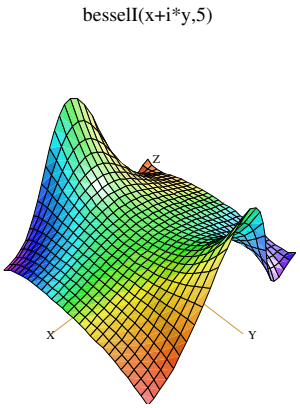
This is the modified Bessel function $I_\alpha(x)$ evaluated for various real values of the index α and fixed argument $x = 5$.

```
draw(besselI(alpha, 5), alpha = -12..12, unit==[5,20])
```



This is similar to the last example except the index α takes on complex values in a `6 x 6` rectangle centered on the origin.

```
draw((x,y) +-> real besselI(complex(x/20, y/20),5), -60..60, -60..60, colorFunction _
    == (x,y)+-> argument besselI(complex(x/20,y/20),5), title=="besselI(x+i*y,5)", _
    style=="shade")
```



8.2 Polynomial Factorization

The FriCAS polynomial factorization facilities are available for all polynomial types and a wide variety of coefficient domains. Here are some examples.

8.2.1 Integer and Rational Number Coefficients

Polynomials with integer coefficients can be factored.

```
v := (4*x^3+2*y^2+1)*(12*x^5-x^3*y+12)
```

$$-2\,x^3\,y^3+\left(24\,x^5+24\right)\,y^2+\left(-4\,x^6-x^3\right)\,y+48\,x^8+12\,x^5+48\,x^3+12 \tag{1}$$

Polynomial(Integer)

```
factor v
```

$$-\left(x^3\,y-12\,x^5-12\right)\,\left(2\,y^2+4\,x^3+1\right) \tag{2}$$

Factored(Polynomial(Integer))

Also, FriCAS can factor polynomials with rational number coefficients.

```
w := (4*x^3+(2/3)*x^2+1)*(12*x^5-(1/2)*x^3+12)
```

$$48\,x^8+8\,x^7-2\,x^6+\frac{35}{3}\,x^5+\frac{95}{2}\,x^3+8\,x^2+12 \tag{3}$$

Polynomial(Fraction (Integer))

```
factor w
```

$$48\,\left(x^3+\frac{1}{6}\,x^2+\frac{1}{4}\right)\,\left(x^5-\frac{1}{24}\,x^3+1\right) \tag{4}$$

Factored(Polynomial(Fraction (Integer)))

8.2.2 Finite Field Coefficients

Polynomials with coefficients in a finite field can be also be factored.

```
u : POLY(PF(19)) :=3*x^4+2*x^2+15*x+18
```

$$3\,x^4+2\,x^2+15\,x+18 \tag{1}$$

Polynomial(PrimeField(19))

These include the integers mod `p`, where `p` is prime, and extensions of these fields.

```
factor u
```

$$3\,\left(x+18\right)\,\left(x^3+x^2+8\,x+13\right) \tag{2}$$

Factored(Polynomial(PrimeField(19)))

Convert this to have coefficients in the finite field with 19^3 elements. See Section 8.11 on page 303 for more information about finite fields.

```
factor(u :: POLY FFX(PF 19,3))
```

$$3\,(x+18)\,\left(x+5\,\%A^2+3\,\%A+13\right)\,\left(x+16\,\%A^2+14\,\%A+13\right)\,\left(x+17\,\%A^2+2\,\%A+13\right) \tag{3}$$

Factored(Polynomial(FiniteFieldExtension (PrimeField(19), 3)))

8.2.3 Simple Algebraic Extension Field Coefficients

Polynomials with coefficients in simple algebraic extensions of the rational numbers can be factored. Here, `aa` and `bb` are symbolic roots of polynomials.

```
aa := rootOf(aa^2+aa+1)
```

$$aa \tag{1}$$

AlgebraicNumber

```
p:=(x^3+aa^2*x+y)*(aa*x^2+aa*x+aa*y^2)^2
```

$$\begin{aligned}&(-aa-1)\,y^5+\left((-aa-1)\,x^3+aa\,x\right)\,y^4+\left((-2\,aa-2)\,x^2+(-2\,aa-2)\,x\right)\,y^3\\&+\left((-2\,aa-2)\,x^5+(-2\,aa-2)\,x^4+2\,aa\,x^3+2\,aa\,x^2\right)\,y^2\\&+\left((-aa-1)\,x^4+(-2\,aa-2)\,x^3+(-aa-1)\,x^2\right)\,y\\&+(-aa-1)\,x^7+(-2\,aa-2)\,x^6-x^5+2\,aa\,x^4+aa\,x^3\end{aligned} \tag{2}$$

Polynomial(AlgebraicNumber)

Note that the second argument to factor can be a list of algebraic extensions to factor over.

```
factor(p,[aa])
```

$$(-aa-1)\,\left(y+x^3+(-aa-1)\,x\right)\,\left(y^2+x^2+x\right)^2 \tag{3}$$

Factored(Polynomial(AlgebraicNumber))

This factors `x^2+3` over the integers.

```
factor(x^2+3)
```

$$x^2+3 \tag{4}$$

Factored(Polynomial(Integer))

Factor the same polynomial over the field obtained by adjoining `aa` to the rational numbers.

```
factor(x^2+3,[aa])
```

$$(x+2\,aa+1)\,(x-2\,aa-1) \tag{5}$$

Factored(Polynomial(AlgebraicNumber))

Factor `x^6+108` over the same field.

```
factor(x^6+108,[aa])
```

$$\left(x^3 + 12\,aa + 6\right)\ \left(x^3 - 12\,aa - 6\right) \tag{6}$$

Factored(Polynomial(AlgebraicNumber))

```
bb:=rootOf(bb^3-2)
```

$$bb \tag{7}$$

AlgebraicNumber

```
factor(x^6+108,[bb])
```

$$\left(x^2 + 3\,bb\,x + 3\,bb^2\right)\ \left(x^2 + 3\,bb^2\right)\ \left(x^2 - 3\,bb\,x + 3\,bb^2\right) \tag{8}$$

Factored(Polynomial(AlgebraicNumber))

Factor again over the field obtained by adjoining both aa and bb to the rational numbers.

```
factor(x^6+108,[aa,bb])
```

$$(x + (aa + 2)\ bb)\ (x + (-2\,aa - 1)\ bb)\ (x + (aa - 1)\ bb)\ (x + (-aa + 1)\ bb)\ (x + (2\,aa + 1)\ bb)\ (x + (-aa - 2)\ bb) \tag{9}$$

Factored(Polynomial(AlgebraicNumber))

8.2.4 Factoring Rational Functions

Since fractions of polynomials form a field, every element (other than zero) divides any other, so there is no useful notion of irreducible factors. Thus the factor operation is not very useful for fractions of polynomials.

There is, instead, a specific operation factorFraction that separately factors the numerator and denominator and returns a fraction of the factored results.

```
factorFraction((x^2-4)/(y^2-4))
```

$$\frac{(x-2)\ (x+2)}{(y-2)\ (y+2)} \tag{1}$$

Fraction(Factored(Polynomial(Integer)))

You can also use map. This expression applies the factor operation to the numerator and denominator.

```
map(factor,(x^2-4)/(y^2-4))
```

$$\frac{(x-2)\ (x+2)}{(y-2)\ (y+2)} \tag{2}$$

Fraction(Factored(Polynomial(Integer)))

8.3 Manipulating Symbolic Roots of a Polynomial

In this section we show you how to work with one root or all roots of a polynomial. These roots are represented symbolically (as opposed to being numeric approximations). See Section 8.5.2 on page 272 and Section 8.5.3 on page 273 for information about solving for the roots of one or more polynomials.

8.3.1 Using a Single Root of a Polynomial

Use rootOf to get a symbolic root of a polynomial: `rootOf(p, x)` returns a root of `p(x)`.

This creates an algebraic number `a`.

```
a := rootOf(a^4+1,a)
```

$$a \tag{1}$$

Expression (Integer)

To find the algebraic relation that defines `a`, use definingPolynomial.

```
definingPolynomial a
```

$$a^4 + 1 \tag{2}$$

Expression (Integer)

You can use `a` in any further expression, including a nested rootOf.

```
b := rootOf(b^2-a-1,b)
```

$$b \tag{3}$$

Expression (Integer)

Higher powers of the roots are automatically reduced during calculations.

```
a + b
```

$$b + a \tag{4}$$

Expression (Integer)

```
% ^ 5
```

$$\left(10\,a^3 + 11\,a^2 + 2\,a - 4\right)\,b + 15\,a^3 + 10\,a^2 + 4\,a - 10 \tag{5}$$

Expression (Integer)

The operation zeroOf is similar to rootOf, except that it may express the root using radicals in some cases.

```
rootOf(c^2+c+1,c)
```

$$c \tag{6}$$

Expression (Integer)

```
zeroOf(d^2+d+1,d)
```

$$\frac{\sqrt{-3}-1}{2} \tag{7}$$

Expression (Integer)

```
rootOf(e^5-2,e)
```

$$e \tag{8}$$

Expression (Integer)

```
zeroOf(f^5-2,f)
```

$$\sqrt[5]{2} \tag{9}$$

Expression (Integer)

8.3.2 Using All Roots of a Polynomial

Use rootsOf to get all symbolic roots of a polynomial: `rootsOf(p, x)` returns a list of all the roots of `p(x)`. If `p(x)` has a multiple root of order `n`, then that root appears `n` times in the list.

Compute all the roots of `x^4 + x + 1`.

```
l := rootsOf(x^4 + x + 1,x)
```

$$[\%x0,\ \%x1,\ \%x2,\ -\%x2-\%x1-\%x0] \tag{1}$$

List (Expression (Integer))

As a side effect, the variables `%x0, %x1` and `%x2` are bound to the first three roots of `x^4 + x + 1`.

```
%x0^5
```

$$-\%x0^2-\%x0 \tag{2}$$

Expression (Integer)

Although they all satisfy `x^4 + x + 1 = 0, %x0, %x1`, and `%x2` are different algebraic numbers. To find the algebraic relation that defines each of them, use definingPolynomial.

```
definingPolynomial %x0
```

$$\%x0^4+\%x0+1 \tag{3}$$

Expression (Integer)

```
definingPolynomial %x1
```

$$\%x0^3 + \%x1\,\%x0^2 + \%x1^2\,\%x0 + \%x1^3 + 1 \tag{4}$$

Expression (Integer)

```
definingPolynomial %x2
```

$$\%x1^2 + (\%x0 + \%x2)\ \%x1 + \%x0^2 + \%x2\,\%x0 + \%x2^2 \tag{5}$$

Expression (Integer)

We can check that the sum and product of the roots of `x^4 + x + 1` are its trace and norm.

```
x3 := last l
```

$$-\%x2 - \%x1 - \%x0 \tag{6}$$

Expression (Integer)

```
%x0 + %x1 + %x2 + x3
```

$$0 \tag{7}$$

Expression (Integer)

```
%x0 * %x1 * %x2 * x3
```

$$1 \tag{8}$$

Expression (Integer)

Note, that in general roots are expressions in new symbols. For example for `x^4 + 1` the second root is a product.

```
rootsOf(x^4 + 1, x)
```

$$[\%x4,\ \%x4\,\%x5,\ -\%x4,\ -\%x4\,\%x5] \tag{9}$$

List (Expression (Integer))

Corresponding to the pair of operations rootOf/zeroOf in Section 8.5.2 on page 272, there is an operation zerosOf that, like rootsOf, computes all the roots of a given polynomial, but which expresses some of them in terms of radicals.

```
zerosOf(y^4 + y + 1, y)
```

$$\left[\%x0,\ \%x1,\ \frac{\sqrt{-3\,\%x1^2 - 2\,\%x0\,\%x1 - 3\,\%x0^2} - \%x1 - \%x0}{2},\ \frac{-\sqrt{-3\,\%x1^2 - 2\,\%x0\,\%x1 - 3\,\%x0^2} - \%x1 - \%x0}{2}\right] \tag{10}$$

List (Expression (Integer))

As you see, only two implicit algebraic numbers were created (`%y0,%y1`), and its defining equations are below. The other two roots are expressed in radicals.

```
definingPolynomial %y0
```

$$\%x0^4 + \%x0 + 1 \tag{11}$$

Expression (Integer)

```
definingPolynomial %y1
```

$$\%x0^3 + \%x1\,\%x0^2 + \%x1^2\,\%x0 + \%x1^3 + 1 \tag{12}$$

Expression (Integer)

For `x^4 + 1` all roots can be expressed in radicals.

```
zerosOf(x^4 + 1)
```

$$\left[\frac{\sqrt{-1}+1}{\sqrt{2}}, \frac{\sqrt{-1}-1}{\sqrt{2}}, \frac{-\sqrt{-1}-1}{\sqrt{2}}, \frac{-\sqrt{-1}+1}{\sqrt{2}}\right] \tag{13}$$

List (AlgebraicNumber)

8.4 Computation of Eigenvalues and Eigenvectors

In this section we show you some of FriCAS's facilities for computing and manipulating eigenvalues and eigenvectors, also called characteristic values and characteristic vectors, respectively.

Let's first create a matrix with integer entries.

```
m1 := matrix [[1,2,1],[2,1,-2],[1,-2,4]]
```

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -2 \\ 1 & -2 & 4 \end{bmatrix} \tag{1}$$

Matrix(Integer)

To get a list of the *rational* eigenvalues, use the operation eigenvalues.

```
leig := eigenvalues(m1)
```

$$[5,\ \%Q \mid \%Q^2 - \%Q - 5] \tag{2}$$

List (Union(Fraction(Polynomial(Integer)), SuchThat(Symbol, Polynomial(Integer))))

Given an explicit eigenvalue, eigenvector computes the eigenvectors corresponding to it.

```
eigenvector(first(leig),m1)
```

$$\left[\left[\begin{matrix}0\\ -\frac{1}{2}\\ 1\end{matrix}\right]\right] \tag{3}$$

List (Matrix(Fraction (Polynomial(Fraction (Integer)))))

The operation eigenvectors returns a list of pairs of values and vectors. When an eigenvalue is rational, FriCAS gives you the value explicitly; otherwise, its minimal polynomial is given, (the polynomial of lowest degree with the eigenvalues as roots), together with a parametric representation of the eigenvector using the eigenvalue. This means that if you ask FriCAS to solve the minimal polynomial, then you can substitute these roots into the parametric form of the corresponding eigenvectors.

You must be aware that unless an exact eigenvalue has been computed, the eigenvector may be badly in error.

```
eigenvectors(m1)
```

$$\left[\left[eigval = 5,\ eigmult = 1,\ eigvec = \left[\left[\begin{matrix}0\\ -\frac{1}{2}\\ 1\end{matrix}\right]\right]\right],\ \left[eigval = \left(\%S \mid \%S^2 - \%S - 5\right),\ eigmult = 1,\ eigvec = \left[\left[\begin{matrix}\%S\\ 2\\ 1\end{matrix}\right]\right]\right]\right] \tag{4}$$

List (Record(eigval : Union(Fraction(Polynomial(Integer)), SuchThat(Symbol, Polynomial(Integer))), eigmult : NonNegativeInteger, eigvec : List (Matrix(Fraction (Polynomial(Integer))))))

Another possibility is to use the operation radicalEigenvectors, which tries to compute explicitly the eigenvectors in terms of radicals.

```
radicalEigenvectors(m1)
```

$$\left[\left[radval = \frac{\sqrt{21}+1}{2},\ radmult = 1,\ radvect = \left[\left[\begin{matrix}\frac{\sqrt{21}+1}{2}\\ 2\\ 1\end{matrix}\right]\right]\right],\ \left[radval = \frac{-\sqrt{21}+1}{2},\ radmult = 1,\ radvect = \left[\left[\begin{matrix}\frac{-\sqrt{21}+1}{2}\\ 2\\ 1\end{matrix}\right]\right]\right],\ \left[radval = 5,\ radmult = 1,\ radvect = \left[\left[\begin{matrix}0\\ -\frac{1}{2}\\ 1\end{matrix}\right]\right]\right]\right] \tag{5}$$

List (Record(radval : Expression(Integer), radmult : Integer , radvect : List (Matrix(Expression (Integer)))))

Alternatively, FriCAS can compute real or complex approximations to the eigenvectors and eigenvalues using the operations realEigenvectors or complexEigenvectors. They each take an additional argument ϵ to specify the "precision" required. In the real case, this means that each approximation will be within $\pm\epsilon$ of the actual result. In the complex case, this means that each approximation will be within $\pm\epsilon$ of the actual result in each of the real and imaginary parts.

The precision can be specified as a **Float** if the results are desired in floating-point notation, or as **Fraction Integer** if the results are to be expressed using rational (or complex rational) numbers.

```
realEigenvectors(m1,1/1000)
```

$$\left[\left[outval = 5,\ outmult = 1,\ outvect = \left[\left[\begin{matrix}0\\ -\frac{1}{2}\\ 1\end{matrix}\right]\right]\right],\ \left[outval = \frac{5717}{2048},\ outmult = 1,\ outvect = \left[\left[\begin{matrix}\frac{5717}{2048}\\ 2\\ 1\end{matrix}\right]\right]\right],\ \left[outval = -\frac{3669}{2048},\ outmult = 1,\ outvect = \left[\left[\begin{matrix}-\frac{3669}{2048}\\ 2\\ 1\end{matrix}\right]\right]\right]\right] \tag{6}$$

List(Record(outval: Fraction(Integer), outmult: Integer, outvect: List(Matrix(Fraction(Integer))))))

If an n by n matrix has n distinct eigenvalues (and therefore n eigenvectors) the operation eigenMatrix gives you a matrix of the eigenvectors.

```
eigenMatrix(m1)
```

$$\begin{bmatrix}\frac{\sqrt{21}+1}{2} & \frac{-\sqrt{21}+1}{2} & 0\\ 2 & 2 & -\frac{1}{2}\\ 1 & 1 & 1\end{bmatrix} \tag{7}$$

Union(Matrix(Expression(Integer)), ...)

```
m2 := matrix [[-5,-2],[18,7]]
```

$$\begin{bmatrix}-5 & -2\\ 18 & 7\end{bmatrix} \tag{8}$$

Matrix(Integer)

```
eigenMatrix(m2)
```

$$\text{"failed"} \tag{9}$$

Union("failed", ...)

If a symmetric matrix has a basis of orthonormal eigenvectors, then orthonormalBasis computes a list of these vectors.

```
m3 := matrix [[1,2],[2,1]]
```

$$\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix} \tag{10}$$

Matrix(Integer)

```
orthonormalBasis(m3)
```

$$\left[\left[\begin{matrix}-\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{matrix}\right],\ \left[\begin{matrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{matrix}\right]\right] \tag{11}$$

List(Matrix(Expression(Integer)))

8.5 Solution of Linear and Polynomial Equations

In this section we discuss the FriCAS facilities for solving systems of linear equations, finding the roots of polynomials and solving systems of polynomial equations. For a discussion of the solution of differential equations, see Section 8.10 on page 296.

8.5.1 Solution of Systems of Linear Equations

You can use the operation solve to solve systems of linear equations.

The operation solve takes two arguments, the list of equations and the list of the unknowns to be solved for. A system of linear equations need not have a unique solution.

To solve the linear system:

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 8 \\ 3x & - & 2y & + & z & = & 0 \\ x & + & 2y & + & 2z & = & 17 \end{array}$$

evaluate this expression.

```
solve([x+y+z=8,3*x-2*y+z=0,x+2*y+2*z=17],[x,y,z])
```

$$[[x = -1,\ y = 2,\ z = 7]] \tag{1}$$

List (List (Equation(Fraction (Polynomial(Integer)))))

Parameters are given as new variables starting with a percent sign and "%" and the variables are expressed in terms of the parameters. If the system has no solutions then the empty list is returned.

When you solve the linear system

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 2 \\ 2x & + & 3y & + & 4z & = & 2 \\ 3x & + & 4y & + & 5z & = & 2 \end{array}$$

with this expression you get a solution involving a parameter.

```
solve([x+2*y+3*z=2,2*x+3*y+4*z=2,3*x+4*y+5*z=2],[x,y,z])
```

$$[[x = \%X - 2,\ y = -2\,\%X + 2,\ z = \%X]] \tag{2}$$

List (List (Equation(Fraction (Polynomial(Integer)))))

The system can also be presented as a matrix and a vector. The matrix contains the coefficients of the linear equations and the vector contains the numbers appearing on the right-hand sides of the equations. You may input the matrix as a list of rows and the vector as a list of its elements.

To solve the system:

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 8 \\ 3x & - & 2y & + & z & = & 0 \\ x & + & 2y & + & 2z & = & 17 \end{array}$$

in matrix form you would evaluate this expression.

```
solve([[1,1,1],[3,-2,1],[1,2,2]],[8,0,17])
```

$$[particular = [-1,\ 2,\ 7]\,,\ basis = []] \tag{3}$$

Record(particular : Union(Vector(Fraction(Integer)), " failed "), basis : List (Vector(Fraction (Integer))))

The solutions are presented as a **Record** with two components: the component *particular* contains a particular solution of the given system or the item `"failed"` if there are no solutions, the component *basis* contains a list of vectors that are a basis for the space of solutions of the corresponding homogeneous system. If the system of linear equations does not have a unique solution, then the *basis* component contains non-trivial vectors.

This happens when you solve the linear system

$$\begin{array}{rcrcrcl} x & + & 2y & + & 3z & = & 2 \\ 2x & + & 3y & + & 4z & = & 2 \\ 3x & + & 4y & + & 5z & = & 2 \end{array}$$

with this command.

```
solve([[1,2,3],[2,3,4],[3,4,5]],[2,2,2])
```

$$[particular = [-2,\ 2,\ 0]\,,\ basis = [[1,\ -2,\ 1]]] \tag{4}$$

Record(particular : Union(Vector(Fraction(Integer)), " failed "), basis : List (Vector(Fraction (Integer))))

All solutions of this system are obtained by adding the particular solution with a linear combination of the *basis* vectors.

When no solution exists then `"failed"` is returned as the *particular* component.

For example:

```
solve([[1,2,3],[2,3,4],[3,4,5]],[2,3,2])
```

$$[particular = \texttt{"failed"},\ basis = [[1,\ -2,\ 1]]] \tag{5}$$

Record(particular : Union(Vector(Fraction(Integer)), " failed "), basis : List (Vector(Fraction (Integer))))

When you want to solve a system of homogeneous equations (that is, a system where the numbers on the right-hand sides of the equations are all zero) in the matrix form you can omit the second argument and use the nullSpace operation.

This computes the solutions of the following system of equations:

$$\begin{array}{rcrcrcl} x & + & 2y & + & 3z & = & 0 \\ 2x & + & 3y & + & 4z & = & 0 \\ 3x & + & 4y & + & 5z & = & 0 \end{array}$$

The result is given as a list of vectors and these vectors form a basis for the solution space.

```
nullSpace([[1,2,3],[2,3,4],[3,4,5]])
```

$$[[1,\ -2,\ 1]] \tag{6}$$

List (Vector(Integer))

8.5.2 Solution of a Single Polynomial Equation

FriCAS can solve polynomial equations producing either approximate or exact solutions. Exact solutions are either members of the ground field or can be presented symbolically as roots of irreducible polynomials.

This returns the one rational root along with an irreducible polynomial describing the other solutions.

```
solve(x^3  = 8,x)
```

$$\left[x = 2,\ x^2 + 2\,x + 4 = 0\right] \tag{1}$$

List (Equation(Fraction(Polynomial(Integer))))

If you want solutions expressed in terms of radicals you would use this instead.

```
radicalSolve(x^3  = 8,x)
```

$$\left[x = -\sqrt{-3} - 1,\ x = \sqrt{-3} - 1,\ x = 2\right] \tag{2}$$

List (Equation(Expression(Integer)))

The solve command always returns a value but radicalSolve returns only the solutions that it is able to express in terms of radicals.

If the polynomial equation has rational coefficients you can ask for approximations to its real roots by calling solve with a second argument that specifies the "precision" ϵ. This means that each approximation will be within $\pm\epsilon$ of the actual result.

Notice that the type of second argument controls the type of the result.

```
solve(x^4 - 10*x^3 + 35*x^2 - 50*x + 25,.0001)
```

$$[x = 3.618011474609375,\ x = 1.381988525390625] \tag{3}$$

List (Equation(Polynomial(Float)))

If you give a floating-point precision you get a floating-point result; if you give the precision as a rational number you get a rational result.

```
solve(x^3-2,1/1000)
```

$$\left[x = \frac{2581}{2048}\right] \tag{4}$$

List (Equation(Polynomial(Fraction(Integer))))

If you want approximate complex results you should use the command complexSolve that takes the same precision argument ϵ.

```
complexSolve(x^3-2,.0001)
```

$$\begin{aligned}[x &= 1.25992104981560260057449340820312 5,\\ x &= -0.62996052473711410282 - 1.0911236358806490898\, i,\\ x &= -0.62996052473711410282 + 1.0911236358806490898 13232421875\, i]\end{aligned} \tag{5}$$

List (Equation(Polynomial(Complex(Float))))

Each approximation will be within $\pm\epsilon$ of the actual result in each of the real and imaginary parts.

```
complexSolve(x^2-2*%i+1,1/100)
```

$$\left[x = -\frac{29413428673197503666554944402597072279332010963219 9}{37414441915671114706014331717536845303191873100185 6} - \frac{10670475}{8388608}\, i,\right.$$
$$\left. x = \frac{29413428673197503666554944402597072279332010963219 9}{37414441915671114706014331717536845303191873100185 6} + \frac{10670475}{8388608}\, i\right] \tag{6}$$

List (Equation(Polynomial(Complex(Fraction(Integer)))))

Note that if you omit the = from the first argument FriCAS generates an equation by equating the first argument to zero. Also, when only one variable is present in the equation, you do not need to specify the variable to be solved for, that is, you can omit the second argument.

FriCAS can also solve equations involving rational functions. Solutions where the denominator vanishes are discarded.

```
radicalSolve(1/x^3 + 1/x^2 + 1/x = 0,x)
```

$$\left[x = \frac{-\sqrt{-3}-1}{2},\ x = \frac{\sqrt{-3}-1}{2}\right] \tag{7}$$

List (Equation(Expression(Integer)))

8.5.3 Solution of Systems of Polynomial Equations

Given a system of equations of rational functions with exact coefficients:

$$\begin{aligned} &p_1(x_1, \ldots, x_n)\\ &\vdots\\ &p_m(x_1, \ldots, x_n)\end{aligned}$$

FriCAS can find numeric or symbolic solutions. The system is first split into irreducible components, then for each component, a triangular system of equations is found that reduces the problem to sequential solution of univariate polynomials resulting from substitution of partial solutions from the previous stage.

$$\begin{aligned} &q_1(x_1, \ldots, x_n)\\ &\vdots\\ &q_m(x_n)\end{aligned}$$

Symbolic solutions can be presented using "implicit" algebraic numbers defined as roots of irreducible polynomials or in terms of radicals. FriCAS can also find approximations to the real or complex roots of a system of polynomial equations to any user-specified accuracy.

The operation solve for systems is used in a way similar to solve for single equations. Instead of a polynomial equation, one has to give a list of equations and instead of a single variable to solve for, a list of variables. For solutions of single equations see Section 8.5.2 on page 272.

Use the operation solve if you want implicitly presented solutions.

```
solve([3*x^3 + y + 1,y^2 -4],[x,y])
```

$$\left[[x=-1,\ y=2],\ \left[x^2-x+1=0,\ y=2\right],\ \left[3\,x^3-1=0,\ y=-2\right]\right] \tag{1}$$

List (List (Equation(Fraction (Polynomial(Integer)))))

```
solve([x = y^2-19,y = z^2+x+3,z = 3*x],[x,y,z])
```

$$\left[\left[x=\frac{z}{3},\ y=\frac{3\,z^2+z+9}{3},\ 9\,z^4+6\,z^3+55\,z^2+15\,z-90=0\right]\right] \tag{2}$$

List (List (Equation(Fraction (Polynomial(Integer)))))

Use radicalSolve if you want your solutions expressed in terms of radicals.

```
radicalSolve([3*x^3 + y + 1,y^2 -4],[x,y])
```

$$\left[\left[x=\frac{\sqrt{-3}+1}{2},\ y=2\right],\ \left[x=\frac{-\sqrt{-3}+1}{2},\ y=2\right],\ \left[x=\frac{-\sqrt{-1}\,\sqrt{3}-1}{2\,\sqrt[3]{3}},\ y=-2\right],\ \left[x=\frac{\sqrt{-1}\,\sqrt{3}-1}{2\,\sqrt[3]{3}},\ y=-2\right],\ \left[x=\frac{1}{\sqrt[3]{3}},\ y=-2\right],\ [x=-1,\ y=2]\right] \tag{3}$$

List (List (Equation(Expression(Integer))))

To get numeric solutions you only need to give the list of equations and the precision desired. The list of variables would be redundant information since there can be no parameters for the numerical solver.

If the precision is expressed as a floating-point number you get results expressed as floats.

```
solve([x^2*y - 1,x*y^2 - 2],.01)
```

$$[[y=1.5874011516571044921875,\ x=0.7937005758285522460937 5]] \tag{4}$$

List (List (Equation(Polynomial(Float))))

To get complex numeric solutions, use the operation complexSolve, which takes the same arguments as in the real case.

```
complexSolve([x^2*y - 1,x*y^2 - 2],1/1000)
```

$$\left[\left[y = \frac{6981463658331}{4398046511104}, x = \frac{6981463658331}{8796093022208}\right], \left[y = -\frac{92799569462155921798031506794235389730378143437 17}{11692013098647223345629478661730264157247460343808} - \frac{3023062441857}{2199023255552}i, x = -\frac{92799569462155921798031506794235389730378143437 17}{23384026197294446691258957323460528314494920687616} - \frac{3023062441857}{4398046511104}i\right], \left[y = -\frac{92799569462155921798031506794235389730378143437 17}{11692013098647223345629478661730264157247460343808} + \frac{3023062441857}{2199023255552}i, x = -\frac{92799569462155921798031506794235389730378143437 17}{23384026197294446691258957323460528314494920687616} + \frac{3023062441857}{4398046511104}i\right]\right] \tag{5}$$

List (List (Equation(Polynomial(Complex(Fraction(Integer))))))

It is also possible to solve systems of equations in rational functions over the rational numbers. Note that `[x = 0.0, a = 0.0]` is not returned as a solution since the denominator vanishes there.

```
solve([x^2/a = a,a = a*x],.001)
```

$$[[x = 1.0,\ a = -1.0],\ [x = 1.0,\ a = 1.0]] \tag{6}$$

List (List (Equation(Polynomial(Float))))

When solving equations with denominators, all solutions where the denominator vanishes are discarded.

```
radicalSolve([x^2/a + a + y^3 - 1,a*y + a + 1],[x,y])
```

$$\left[\left[x = -\sqrt{\frac{-a^4 + 2\,a^3 + 3\,a^2 + 3\,a + 1}{a^2}}, y = \frac{-a - 1}{a}\right], \left[x = \sqrt{\frac{-a^4 + 2\,a^3 + 3\,a^2 + 3\,a + 1}{a^2}}, y = \frac{-a - 1}{a}\right]\right] \tag{7}$$

List (List (Equation(Expression(Integer))))

8.6 Limits

To compute a limit, you must specify a functional expression, a variable, and a limiting value for that variable. If you do not specify a direction, FriCAS attempts to compute a two-sided limit.

Issue this to compute the limit

$$\lim_{x \to 1} \frac{x^2 - 3x + 2}{x^2 - 1}.$$

```
limit((x^2 - 3*x + 2)/(x^2 - 1),x = 1)
```

$$-\frac{1}{2} \tag{1}$$

Union(OrderedCompletion(Fraction(Polynomial(Integer))), ...)

Sometimes the limit when approached from the left is different from the limit from the right and, in this case, you may wish to ask for a one-sided limit. Also, if you have a function that is only defined on one side of a particular value, you can compute a one-sided limit.

The function `log(x)` is real only to the right of zero, that is, for `x > 0`. Thus, when computing limits of functions involving `log(x)`, you probably want a "right-hand" limit.

```
limit(x * log(x),x = 0,"right")
```

$$0 \tag{2}$$

Union(OrderedCompletion(Expression(Integer)), ...)

When you do not specify `"right"` or `"left"` as the optional fourth argument, limit tries to compute a two-sided limit. Here the limit from the left does not exist, as FriCAS indicates when you try to take a two-sided limit.

```
limit(sin(1/x)*exp(1/x), x=0)
```

$$[leftHandLimit = 0,\ rightHandLimit = \texttt{"failed"}] \tag{3}$$

Union(Record(leftHandLimit: Union(OrderedCompletion(Expression(Integer)), " failed "), rightHandLimit: Union(OrderedCompletion(Expression(Integer)), " failed ")), ...)

A function can be defined on both sides of a particular value, but tend to different limits as its variable approaches that value from the left and from the right. We can construct an example of this as follows: Since $\sqrt{y^2}$ is simply the absolute value of `y`, the function $\sqrt{y^2}/y$ is simply the sign (`+1` or `-1`) of the nonzero real number `y`. Therefore, $\sqrt{y^2}/y = -1$ for `y < 0` and $\sqrt{y^2}/y = +1$ for `y > 0`. This is what happens when we take the limit at `y = 0`. The answer returned by FriCAS gives both a "left-hand" and a "right-hand" limit.

```
limit(sqrt(y^2)/y,y = 0)
```

$$[leftHandLimit = -1,\ rightHandLimit = 1] \tag{4}$$

Union(Record(leftHandLimit: Union(OrderedCompletion(Expression(Integer)), " failed "), rightHandLimit: Union(OrderedCompletion(Expression(Integer)), " failed ")), ...)

Here is another example, this time using a more complicated function.

```
limit(sqrt(1 - cos(t))/t,t = 0)
```

$$\left[leftHandLimit = -\frac{1}{\sqrt{2}},\ rightHandLimit = \frac{1}{\sqrt{2}}\right] \tag{5}$$

Union(Record(leftHandLimit: Union(OrderedCompletion(Expression(Integer)), " failed "), rightHandLimit: Union(OrderedCompletion(Expression(Integer)), " failed ")), ...)

You can compute limits at infinity by passing either $+\infty$ or $-\infty$ as the third argument of limit. To do this, use the constants `%plusInfinity` and `%minusInfinity`.

```
limit(sqrt(3*x^2 + 1)/(5*x),x = %plusInfinity)
```

$$\frac{\sqrt{3}}{5} \tag{6}$$

Union(OrderedCompletion(Expression(Integer)), ...)

```
limit(sqrt(3*x^2 + 1)/(5*x),x = %minusInfinity)
```

$$-\frac{\sqrt{3}}{5} \tag{7}$$

Union(OrderedCompletion(Expression(Integer)), ...)

You can take limits of functions with parameters. As you can see, the limit is expressed in terms of the parameters.

```
limit(sinh(a*x)/tan(b*x),x = 0)
```

$$\frac{a}{b} \tag{8}$$

Union(OrderedCompletion(Expression(Integer)), ...)

When you use limit, you are taking the limit of a real function of a real variable. When you compute this, FriCAS returns 0 because, as a function of a real variable, sin(1/z) is always between -1 and 1, so z * sin(1/z) tends to 0 as z tends to 0.

```
limit(z * sin(1/z),z = 0)
```

$$0 \tag{9}$$

Union(OrderedCompletion(Expression(Integer)), ...)

However, as a function of a *complex* variable, sin(1/z) is badly behaved near 0 (one says that sin(1/z) has an *essential singularity* at z = 0). When viewed as a function of a complex variable, z * sin(1/z) does not approach any limit as z tends to 0 in the complex plane. FriCAS indicates this when we call complexLimit.

```
complexLimit(z * sin(1/z),z = 0)
```

$$\text{"failed"} \tag{10}$$

Union("failed", ...)

You can also take complex limits at infinity, that is, limits of a function of z as z approaches infinity on the Riemann sphere. Use the symbol %infinity to denote "complex infinity." As above, to compute complex limits rather than real limits, use complexLimit.

```
complexLimit((2 + z)/(1 - z),z = %infinity)
```

$$-1 \tag{11}$$

OnePointCompletion(Fraction(Polynomial(Integer)))

In many cases, a limit of a real function of a real variable exists when the corresponding complex limit does not. This limit exists.

```
limit(sin(x)/x,x = %plusInfinity)
```

$$0 \tag{12}$$

Union(OrderedCompletion(Expression(Integer)), ...)

But this limit does not.

```
complexLimit(sin(x)/x,x = %infinity)
```

$$\text{"failed"} \tag{13}$$

Union(" failed ", ...)

8.7 Laplace Transforms

FriCAS can compute some forward Laplace transforms, mostly of elementary functions not involving logarithms, although some cases of special functions are handled. To compute the forward Laplace transform of `F(t)` with respect to `t` and express the result as `f(s)`, issue the command `laplace(F(t), t, s)`.

```
laplace(sin(a*t)*cosh(a*t)-cos(a*t)*sinh(a*t), t, s)
```

$$\frac{4\,a^3}{s^4+4\,a^4} \tag{1}$$

Expression(Integer)

Here are some other non-trivial examples.

```
laplace((exp(a*t) - exp(b*t))/t, t, s)
```

$$-\log(s-a)+\log(s-b) \tag{2}$$

Expression(Integer)

```
laplace(2/t * (1 - cos(a*t)), t, s)
```

$$\log\left(s^2+a^2\right)-2\,\log(s) \tag{3}$$

Expression(Integer)

```
laplace(exp(-a*t) * sin(b*t) / b^2, t, s)
```

$$\frac{1}{b\,s^2+2\,a\,b\,s+b^3+a^2\,b} \tag{4}$$

Expression(Integer)

```
laplace((cos(a*t) - cos(b*t))/t, t, s)
```

$$\frac{\log\left(s^2+b^2\right)-\log\left(s^2+a^2\right)}{2} \tag{5}$$

Expression(Integer)

FriCAS also knows about a few special functions.

```
laplace(exp(a*t+b)*Ei(c*t), t, s)
```

$$\frac{e^{b}\,\log\left(\frac{s+c-a}{c}\right)}{s-a} \tag{6}$$

Expression(Integer)

```
laplace(a*Ci(b*t) + c*Si(d*t), t, s)
```

$$\frac{a\,\log\left(\frac{s^2+b^2}{b^2}\right)+2\,c\,\arctan\left(\frac{d}{s}\right)}{2\,s} \tag{7}$$

Expression(Integer)

When FriCAS does not know about a particular transform, it keeps it as a formal transform in the answer.

```
laplace(sin(a*t) - a*t*cos(a*t) + exp(t^2), t, s)
```

$$\frac{\left(s^4+2\,a^2\,s^2+a^4\right)\,\mathrm{laplace}\left(e^{t^2},t,s\right)+2\,a^3}{s^4+2\,a^2\,s^2+a^4} \tag{8}$$

Expression(Integer)

8.8 Integration

Integration is the reverse process of differentiation, that is, an *integral* of a function `f` with respect to a variable `x` is any function `g` such that `D(g,x)` is equal to `f`. The package **FunctionSpaceIntegration** provides the top-level integration operation, integrate, for integrating real-valued elementary functions.

```
integrate(cosh(a*x)*sinh(a*x), x)
```

$$\frac{\sinh(a\,x)^2+\cosh(a\,x)^2}{4\,a} \tag{1}$$

Union(Expression(Integer), ...)

Unfortunately, antiderivatives of most functions cannot be expressed in terms of elementary functions.

```
integrate(log(1 + sqrt(a * x + b)) / x, x)
```

$$\int^{x}\frac{\log\left(\sqrt{b+\%O\,a}+1\right)}{\%O}\,d\%O \tag{2}$$

Union(Expression(Integer), ...)

Given an elementary function to integrate, FriCAS returns a formal integral as above only when it can prove that the integral is not elementary and not when it cannot determine the integral. In this rare case it prints a message that it cannot determine if an elementary integral exists. Similar functions may have antiderivatives that look quite different because the form of the antiderivative depends on the sign of a constant that appears in the function.

```
integrate(1/(x^2 - 2),x)
```

$$\frac{\log\left(\frac{\left(x^2+2\right)\sqrt{2}-4\,x}{x^2-2}\right)}{2\sqrt{2}} \tag{3}$$

Union(Expression(Integer), ...)

```
integrate(1/(x^2 + 2),x)
```

$$\frac{\arctan\left(\frac{x\sqrt{2}}{2}\right)}{\sqrt{2}} \tag{4}$$

Union(Expression(Integer), ...)

If the integrand contains parameters, then there may be several possible antiderivatives, depending on the signs of expressions of the parameters. In this case FriCAS returns a list of answers that cover all the possible cases. Here you use the answer involving the square root of `a` when `a > 0` and the answer involving the square root of `-a` when `a < 0`.

```
integrate(x^2 / (x^4 - a^2), x)
```

$$\left[\frac{\log\left(\frac{\left(x^2+a\right)\sqrt{a}-2\,a\,x}{x^2-a}\right)+2\,\arctan\left(\frac{x\sqrt{a}}{a}\right)}{4\sqrt{a}},\ \frac{\log\left(\frac{\left(x^2-a\right)\sqrt{-a}+2\,a\,x}{x^2+a}\right)-2\,\arctan\left(\frac{x\sqrt{-a}}{a}\right)}{4\sqrt{-a}}\right] \tag{5}$$

Union(List(Expression(Integer)), ...)

If the parameters and the variables of integration can be complex numbers rather than real, then the notion of sign is not defined. In this case all the possible answers can be expressed as one complex function. To get that function, rather than a list of real functions, use complexIntegrate, which is provided by the package **FunctionSpaceComplexIntegration**.

This operation is used for integrating complex-valued elementary functions.

```
complexIntegrate(x^2 / (x^4 - a^2), x)
```

$$\frac{-\sqrt{\frac{1}{4\,a}}\,\log\left(2\,a\,\sqrt{\frac{1}{4\,a}}+x\right)+\sqrt{-\frac{1}{4\,a}}\,\log\left(2\,a\,\sqrt{-\frac{1}{4\,a}}+x\right)-\sqrt{-\frac{1}{4\,a}}\,\log\left(-2\,a\,\sqrt{-\frac{1}{4\,a}}+x\right)+\sqrt{\frac{1}{4\,a}}\,\log\left(-2\,a\,\sqrt{\frac{1}{4\,a}}+x\right)}{2} \tag{6}$$

Expression(Integer)

As with the real case, antiderivatives for most complex-valued functions cannot be expressed in terms of elementary functions.

```
complexIntegrate(log(1 + sqrt(a * x + b)) / x, x)
```

$$\int^{x} \frac{\log\left(\sqrt{b + \%O\, a} + 1\right)}{\%O}\, d\%O \tag{7}$$

Expression(Integer)

Sometimes integrate can involve symbolic algebraic numbers such as those returned by rootOf. To see how to work with these strange generated symbols (such as %%a0), see Section 8.3.2 on page 265.

Definite integration is the process of computing the area between the x-axis and the curve of a function f(x). The fundamental theorem of calculus states that if f is continuous on an interval a..b and if there exists a function g that is differentiable on a..b and such that D(g, x) is equal to f, then the definite integral of f for x in the interval a..b is equal to g(b) - g(a).

The package **RationalFunctionDefiniteIntegration** provides the top-level definite integration operation, integrate, for integrating real-valued rational functions.

```
integrate((x^4 - 3*x^2 + 6)/(x^6-5*x^4+5*x^2+4), x = 1..2)
```

$$\frac{2\,\arctan\left(\frac{1}{2}\right) + 2\,\arctan(8) + 2\,\arctan(5) + 2\,\arctan(2) - \pi}{2} \tag{8}$$

Union(f1: OrderedCompletion(Expression(Integer)), ...)

FriCAS checks beforehand that the function you are integrating is defined on the interval a..b, and prints an error message if it finds that this is not the case, as in the following example:

```
integrate(1/(x^2-2), x = 1..2)

   >> Error detected within library code:
      Pole in path of integration
      You are being returned to the top level
      of the interpreter.
```

When parameters are present in the function, the function may or may not be defined on the interval of integration.

If this is the case, FriCAS issues a warning that a pole might lie in the path of integration, and does not compute the integral.

```
integrate(1/(x^2-a), x = 1..2)
```

$$\texttt{"potentialPole"} \tag{9}$$

Union(pole: "potentialPole", ...)

If you know that you are using values of the parameter for which the function has no pole in the interval of integration, use the string "noPole" as a third argument to integrate:

The value here is, of course, incorrect if sqrt(a) is between 1 and 2.

```
integrate(1/(x^2-a), x = 1..2, "noPole")
```

$$\left[\frac{-\log\left(\frac{\left(-4\,a^2-4\,a\right)\sqrt{a}+a^3+6\,a^2+a}{a^2-2\,a+1}\right)+\log\left(\frac{\left(-8\,a^2-32\,a\right)\sqrt{a}+a^3+24\,a^2+16\,a}{a^2-8\,a+16}\right)}{4\sqrt{a}},\ \frac{-\arctan\left(\frac{2\sqrt{-a}}{a}\right)+\arctan\left(\frac{\sqrt{-a}}{a}\right)}{\sqrt{-a}}\right] \tag{10}$$

Union(f2: List (OrderedCompletion(Expression(Integer))), ...)

8.9 Working with Power Series

FriCAS has very sophisticated facilities for working with power series. Infinite series are represented by a list of the coefficients that have already been determined, together with a function for computing the additional coefficients if needed. The system command that determines how many terms of a series are displayed is `)set streams calculate`. For the purposes of this book, we have used this system command to display fewer than ten terms. Series can be created from expressions, from functions for the series coefficients, and from applications of operations on existing series. The most general function for creating a series is called series, although you can also use taylor, laurent and puiseux in situations where you know what kind of exponents are involved.

For information about solving differential equations in terms of power series, see Section 8.10.3 on page 302.

8.9.1 Creation of Power Series

This is the easiest way to create a power series. This tells FriCAS that `x` is to be treated as a power series, so functions of `x` are again power series.

```
x := series 'x
```

$$x \tag{1}$$

UnivariatePuiseuxSeries (Expression (Integer), x, 0)

We didn't say anything about the coefficients of the power series, so the coefficients are general expressions over the integers. This allows us to introduce denominators, symbolic constants, and other variables as needed. Here the coefficients are integers (note that the coefficients are the Fibonacci numbers).

```
1/(1 - x - x^2)
```

$$1+x+2\,x^2+3\,x^3+5\,x^4+8\,x^5+13\,x^6+21\,x^7+O\left(x^8\right) \tag{2}$$

UnivariatePuiseuxSeries (Expression (Integer), x, 0)

This series has coefficients that are rational numbers.

```
sin(x)
```

$$x - \frac{1}{6}\,x^3 + \frac{1}{120}\,x^5 - \frac{1}{5040}\,x^7 + O\left(x^9\right) \tag{3}$$

UnivariatePuiseuxSeries (Expression (Integer), x, 0)

When you enter this expression you introduce the symbolic constants `sin(1)` and `cos(1)`.

```
sin(1 + x)
```

$$\sin(1) + \cos(1)\,x - \frac{\sin(1)}{2}\,x^2 - \frac{\cos(1)}{6}\,x^3 + \frac{\sin(1)}{24}\,x^4 + \frac{\cos(1)}{120}\,x^5 - \frac{\sin(1)}{720}\,x^6 - \frac{\cos(1)}{5040}\,x^7 + O\left(x^8\right) \tag{4}$$

UnivariatePuiseuxSeries (Expression (Integer), x, 0)

When you enter the expression the variable `a` appears in the resulting series expansion.

```
sin(a * x)
```

$$a\,x - \frac{a^3}{6}\,x^3 + \frac{a^5}{120}\,x^5 - \frac{a^7}{5040}\,x^7 + O\left(x^9\right) \tag{5}$$

UnivariatePuiseuxSeries (Expression (Integer), x, 0)

You can also convert an expression into a series expansion. This expression creates the series expansion of `1/log(y)` about `y = 1`. For details and more examples, see Section 8.9.5 on page 288.

```
series(1/log(y),y = 1)
```

$$\begin{aligned}&(y-1)^{-1} + \frac{1}{2} - \frac{1}{12}\,(y-1) + \frac{1}{24}\,(y-1)^2 - \frac{19}{720}\,(y-1)^3 \\ &\quad + \frac{3}{160}\,(y-1)^4 - \frac{863}{60480}\,(y-1)^5 + \frac{275}{24192}\,(y-1)^6 + O\left((y-1)^7\right)\end{aligned} \tag{6}$$

UnivariatePuiseuxSeries (Expression (Integer), y, 1)

You can create power series with more general coefficients. You normally accomplish this via a type declaration (see Section 2.3 on page 74). See Section 8.9.4 on page 286 for some warnings about working with declared series.

We declare that `y` is a one-variable Taylor series (**UTS** is the abbreviation for **UnivariateTaylorSeries**) in the variable `z` with **FLOAT** (that is, floating-point) coefficients, centered about `0`. Then, by assignment, we obtain the Taylor expansion of `exp(z)` with floating-point coefficients.

```
y : UTS(FLOAT,'z,0) := exp(z)
```

$$\begin{aligned}&1.0 + z + 0.5\,z^2 + 0.16666666666666666667\,z^3 \\ &\quad + 0.041666666666666666667\,z^4 + 0.0083333333333333333334\,z^5 \\ &\quad + 0.0013888888888888888889\,z^6 + 0.00019841269841269841269\,z^7 + O\left(z^8\right)\end{aligned} \tag{7}$$

UnivariateTaylorSeries (Float, z, 0.0)

You can also create a power series by giving an explicit formula for its n^{th} coefficient. For details and more examples, see Section 8.9.6 on page 290.

To create a series about `w = 0` whose n^{th} Taylor coefficient is `1/n!`, you can evaluate this expression. This is the Taylor expansion of `exp(w)` at `w = 0`.

```
series(1/factorial(n),n,w = 0)
```

$$1 + w + \frac{1}{2}\,w^2 + \frac{1}{6}\,w^3 + \frac{1}{24}\,w^4 + \frac{1}{120}\,w^5 + \frac{1}{720}\,w^6 + \frac{1}{5040}\,w^7 + O\left(w^8\right) \tag{8}$$

UnivariatePuiseuxSeries (Expression (Integer) , w, 0)

8.9.2 Coefficients of Power Series

You can extract any coefficient from a power series—even one that hasn't been computed yet. This is possible because in FriCAS, infinite series are represented by a list of the coefficients that have already been determined, together with a function for computing the additional coefficients. (This is known as *lazy evaluation*.) When you ask for a coefficient that hasn't yet been computed, FriCAS computes whatever additional coefficients it needs and then stores them in the representation of the power series.

Here's an example of how to extract the coefficients of a power series.

```
x := series(x)
```

$$x \tag{1}$$

UnivariatePuiseuxSeries (Expression (Integer) , x, 0)

```
y := exp(x) * sin(x)
```

$$x + x^2 + \frac{1}{3}\,x^3 - \frac{1}{30}\,x^5 - \frac{1}{90}\,x^6 - \frac{1}{630}\,x^7 + O\left(x^9\right) \tag{2}$$

UnivariatePuiseuxSeries (Expression (Integer) , x, 0)

This coefficient is readily available.

```
coefficient(y,6)
```

$$-\frac{1}{90} \tag{3}$$

Expression (Integer)

But let's get the fifteenth coefficient of `y`.

```
coefficient(y,15)
```

$$-\frac{1}{10216206000} \tag{4}$$

Expression (Integer)

If you look at `y` then you see that the coefficients up to order `15` have all been computed.

```
)set stream showall on
```

```
y
```

$$x + x^2 + \frac{1}{3}\,x^3 - \frac{1}{30}\,x^5 - \frac{1}{90}\,x^6 - \frac{1}{630}\,x^7 + \frac{1}{22680}\,x^9 + \frac{1}{113400}\,x^{10} + \frac{1}{1247400}\,x^{11} - \frac{1}{97297200}\,x^{13} - \frac{1}{681080400}\,x^{14} - \frac{1}{10216206000}\,x^{15} + O\left(x^{16}\right) \tag{5}$$

UnivariatePuiseuxSeries (Expression (Integer) , x, 0)

```
)set stream showall off
```

8.9.3 Power Series Arithmetic

You can manipulate power series using the usual arithmetic operations +, -, *, and /.

The results of these operations are also power series.

```
x := series x
```

$$x \tag{1}$$

UnivariatePuiseuxSeries (Expression (Integer) , x, 0)

```
(3 + x) / (1 + 7*x)
```

$$3 - 20\,x + 140\,x^2 - 980\,x^3 + 6860\,x^4 - 48020\,x^5 + 336140\,x^6 - 2352980\,x^7 + O\left(x^8\right) \tag{2}$$

UnivariatePuiseuxSeries (Expression (Integer) , x, 0)

You can also compute f(x) ^ g(x), where f(x) and g(x) are two power series.

```
base := 1 / (1 - x)
```

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + O\left(x^8\right) \tag{3}$$

UnivariatePuiseuxSeries (Expression (Integer) , x, 0)

```
expon := x * base
```

$$x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + O\left(x^9\right) \tag{4}$$

UnivariatePuiseuxSeries (Expression (Integer) , x, 0)

```
base ^ expon
```

$$1 + x^2 + \frac{3}{2}\,x^3 + \frac{7}{3}\,x^4 + \frac{43}{12}\,x^5 + \frac{649}{120}\,x^6 + \frac{241}{30}\,x^7 + O\left(x^8\right) \tag{5}$$

UnivariatePuiseuxSeries (Expression (Integer) , x, 0)

8.9.4 Functions on Power Series

Once you have created a power series, you can apply transcendental functions (for example, exp, log, sin, tan, cosh, etc.) to it.

To demonstrate this, we first create the power series expansion of the rational function $\frac{x^2}{1-6x+x^2}$ about `x = 0`.

```
x := series 'x
```

$$x \tag{1}$$

UnivariatePuiseuxSeries (Expression (Integer), x, 0)

```
rat := x^2 / (1 - 6*x + x^2)
```

$$x^2 + 6\,x^3 + 35\,x^4 + 204\,x^5 + 1189\,x^6 + 6930\,x^7 + 40391\,x^8 + 235416\,x^9 + O\left(x^{10}\right) \tag{2}$$

UnivariatePuiseuxSeries (Expression (Integer), x, 0)

If you want to compute the series expansion of $\sin\left(\frac{x^2}{1-6x+x^2}\right)$ you simply compute the sine of `rat`.

```
sin(rat)
```

$$x^2 + 6\,x^3 + 35\,x^4 + 204\,x^5 + \frac{7133}{6}\,x^6 + 6927\,x^7 + \frac{80711}{2}\,x^8 + 235068\,x^9 + O\left(x^{10}\right) \tag{3}$$

UnivariatePuiseuxSeries (Expression (Integer), x, 0)

Warning: the type of the coefficients of a power series may affect the kind of computations that you can do with that series. This can only happen when you have made a declaration to specify a series domain with a certain type of coefficient.

If you evaluate then you have declared that `y` is a one variable Taylor series (**UTS** is the abbreviation for **UnivariateTaylorSeries**) in the variable `y` with **FRAC INT** (that is, fractions of integer) coefficients, centered about `0`.

```
y : UTS(FRAC INT,y,0) := y
```

$$y \tag{4}$$

UnivariateTaylorSeries (Fraction (Integer), y, 0)

You can now compute certain power series in `y`, *provided* that these series have rational coefficients.

```
exp(y)
```

$$1 + y + \frac{1}{2}\,y^2 + \frac{1}{6}\,y^3 + \frac{1}{24}\,y^4 + \frac{1}{120}\,y^5 + \frac{1}{720}\,y^6 + \frac{1}{5040}\,y^7 + O\left(y^8\right) \tag{5}$$

UnivariateTaylorSeries (Fraction (Integer), y, 0)

You can get examples of such series by applying transcendental functions to series in `y` that have no constant terms.

```
tan(y^2)
```

$$y^2 + \frac{1}{3}\,y^6 + O\left(y^8\right) \tag{6}$$

UnivariateTaylorSeries (Fraction (Integer), y, 0)

```
cos(y + y^5)
```

$$1 - \frac{1}{2}\,y^2 + \frac{1}{24}\,y^4 - \frac{721}{720}\,y^6 + O\left(y^8\right) \tag{7}$$

UnivariateTaylorSeries (Fraction (Integer), y, 0)

Similarly, you can compute the logarithm of a power series with rational coefficients if the constant coefficient is `1`.

```
log(1 + sin(y))
```

$$y - \frac{1}{2}\,y^2 + \frac{1}{6}\,y^3 - \frac{1}{12}\,y^4 + \frac{1}{24}\,y^5 - \frac{1}{45}\,y^6 + \frac{61}{5040}\,y^7 + O\left(y^8\right) \tag{8}$$

UnivariateTaylorSeries (Fraction (Integer), y, 0)

If you wanted to apply, say, the operation exp to a power series with a nonzero constant coefficient a_0, then the constant coefficient of the result would be e^{a_0}, which is *not* a rational number. Therefore, evaluating `exp(2 + tan(y))` would generate an error message.

If you want to compute the Taylor expansion of `exp(2 + tan(y))`, you must ensure that the coefficient domain has an operation exp defined for it. An example of such a domain is **Expression Integer**, the type of formal functional expressions over the integers. When working with coefficients of this type,

```
z : UTS(EXPR INT,z,0) := z
```

$$z \tag{9}$$

UnivariateTaylorSeries (Expression (Integer), z, 0)

this presents no problems.

```
exp(2 + tan(z))
```

$$e^2 + e^2\,z + \frac{e^2}{2}\,z^2 + \frac{e^2}{2}\,z^3 + \frac{3\,e^2}{8}\,z^4 + \frac{37\,e^2}{120}\,z^5 + \frac{59\,e^2}{240}\,z^6 + \frac{137\,e^2}{720}\,z^7 + O\left(z^8\right) \tag{10}$$

UnivariateTaylorSeries (Expression (Integer), z, 0)

Another way to create Taylor series whose coefficients are expressions over the integers is to use taylor which works similarly to series. This is equivalent to the previous computation, except that now we are using the variable `w` instead of `z`.

```
w := taylor 'w
```

$$w \tag{11}$$

UnivariateTaylorSeries (Expression(Integer), w, 0)

```
exp(2 + tan(w))
```

$$e^2 + e^2\,w + \frac{e^2}{2}\,w^2 + \frac{e^2}{2}\,w^3 + \frac{3\,e^2}{8}\,w^4 + \frac{37\,e^2}{120}\,w^5 + \frac{59\,e^2}{240}\,w^6 + \frac{137\,e^2}{720}\,w^7 + O\left(w^8\right) \tag{12}$$

UnivariateTaylorSeries (Expression(Integer), w, 0)

8.9.5 Converting to Power Series

The **ExpressionToUnivariatePowerSeries** package provides operations for computing series expansions of functions.

Evaluate this to compute the Taylor expansion of `sin x` about `x = 0`. The first argument, `sin(x)`, specifies the function whose series expansion is to be computed and the second argument, `x = 0`, specifies that the series is to be expanded in powers of `(x - 0)`, that is, in powers of `x`.

```
taylor(sin(x),x = 0)
```

$$x - \frac{1}{6}\,x^3 + \frac{1}{120}\,x^5 - \frac{1}{5040}\,x^7 + O\left(x^8\right) \tag{1}$$

UnivariateTaylorSeries (Expression(Integer), x, 0)

Here is the Taylor expansion of `sin x` about $x = \frac{\pi}{6}$:

```
taylor(sin(x),x = %pi/6)
```

$$\begin{aligned}&\frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{6}\right)^3 + \frac{1}{48}\left(x - \frac{\pi}{6}\right)^4 \\ &+ \frac{\sqrt{3}}{240}\left(x - \frac{\pi}{6}\right)^5 - \frac{1}{1440}\left(x - \frac{\pi}{6}\right)^6 - \frac{\sqrt{3}}{10080}\left(x - \frac{\pi}{6}\right)^7 + O\left(\left(x - \frac{\pi}{6}\right)^8\right)\end{aligned} \tag{2}$$

UnivariateTaylorSeries (Expression(Integer), x, %pi/6)

The function to be expanded into a series may have variables other than the series variable. For example, we may expand `tan(x*y)` as a Taylor series in `x`

```
taylor(tan(x*y),x = 0)
```

$$y\,x + \frac{y^3}{3}\,x^3 + \frac{2\,y^5}{15}\,x^5 + \frac{17\,y^7}{315}\,x^7 + O\left(x^8\right) \tag{3}$$

UnivariateTaylorSeries (Expression(Integer), x, 0)

or as a Taylor series in `y`.

```
taylor(tan(x*y),y = 0)
```

$$x\,y + \frac{x^3}{3}\,y^3 + \frac{2\,x^5}{15}\,y^5 + \frac{17\,x^7}{315}\,y^7 + O\left(y^8\right) \tag{4}$$

UnivariateTaylorSeries (Expression (Integer) , y , 0)

A more interesting function is $\frac{te^{xt}}{e^t - 1}$.

When we expand this function as a Taylor series in `t` the n^{th} order coefficient is the n^{th} Bernoulli polynomial divided by `n!`.

```
bern := taylor(t*exp(x*t)/(exp(t) - 1),t = 0)
```

$$\begin{aligned} &1 + \frac{2\,x - 1}{2}\,t + \frac{6\,x^2 - 6\,x + 1}{12}\,t^2 + \frac{2\,x^3 - 3\,x^2 + x}{12}\,t^3 \\ &+ \frac{30\,x^4 - 60\,x^3 + 30\,x^2 - 1}{720}\,t^4 + \frac{6\,x^5 - 15\,x^4 + 10\,x^3 - x}{720}\,t^5 \\ &+ \frac{42\,x^6 - 126\,x^5 + 105\,x^4 - 21\,x^2 + 1}{30240}\,t^6 + \frac{6\,x^7 - 21\,x^6 + 21\,x^5 - 7\,x^3 + x}{30240}\,t^7 + O\left(t^8\right) \end{aligned} \tag{5}$$

UnivariateTaylorSeries (Expression (Integer) , t , 0)

Therefore, this and the next expression produce the same result.

```
factorial(6) * coefficient(bern,6)
```

$$\frac{42\,x^6 - 126\,x^5 + 105\,x^4 - 21\,x^2 + 1}{42} \tag{6}$$

Expression (Integer)

```
bernoulliB(6,x)
```

$$x^6 - 3\,x^5 + \frac{5}{2}\,x^4 - \frac{1}{2}\,x^2 + \frac{1}{42} \tag{7}$$

Polynomial(Fraction (Integer))

Technically, a series with terms of negative degree is not considered to be a Taylor series, but, rather, a *Laurent series*. If you try to compute a Taylor series expansion of $\frac{x}{\log x}$ at `x = 1` via `taylor(x/log(x), x = 1)` you get an error message. The reason is that the function has a *pole* at `x = 1`, meaning that its series expansion about this point has terms of negative degree. A series with finitely many terms of negative degree is called a Laurent series. You get the desired series expansion by issuing this.

```
laurent(x/log(x),x = 1)
```

$$\begin{aligned} &(x-1)^{-1} + \frac{3}{2} + \frac{5}{12}\,(x-1) - \frac{1}{24}\,(x-1)^2 + \frac{11}{720}\,(x-1)^3 \\ &- \frac{11}{1440}\,(x-1)^4 + \frac{271}{60480}\,(x-1)^5 - \frac{13}{4480}\,(x-1)^6 + O\left((x-1)^7\right) \end{aligned} \tag{8}$$

UnivariateLaurentSeries (Expression (Integer) , x , 1)

Similarly, a series with terms of fractional degree is neither a Taylor series nor a Laurent series. Such a series is called a *Puiseux series*. The expression `laurent(sqrt(sec(x)),x = 3 * %pi/2)` results in an

error message because the series expansion about this point has terms of fractional degree. However, this command produces what you want.

```
puiseux(sqrt(sec(x)),x = 3 * %pi/2)
```

$$\left(x-\frac{3\,\pi}{2}\right)^{-\frac{1}{2}}+\frac{1}{12}\left(x-\frac{3\,\pi}{2}\right)^{\frac{3}{2}}+O\left(\left(x-\frac{3\,\pi}{2}\right)^{\frac{7}{2}}\right) \tag{9}$$

UnivariatePuiseuxSeries (Expression (Integer), x, (3*%pi)/2)

Finally, consider the case of functions that do not have Puiseux expansions about certain points. An example of this is x^x about `x = 0`. `puiseux(x^x,x=0)` produces an error message because of the type of singularity of the function at `x = 0`. The general function series can be used in this case. Notice that the series returned is not, strictly speaking, a power series because of the `log(x)` in the expansion.

```
series(x^x,x=0)
```

$$1+\log(x)\,x+\frac{\log(x)^2}{2}\,x^2+\frac{\log(x)^3}{6}\,x^3+\frac{\log(x)^4}{24}\,x^4+\frac{\log(x)^5}{120}\,x^5+\frac{\log(x)^6}{720}\,x^6+\frac{\log(x)^7}{5040}\,x^7+O\left(x^8\right) \tag{10}$$

GeneralUnivariatePowerSeries (Expression (Integer), x, 0)

> The operation series returns the most general type of infinite series. The user who is not interested in distinguishing between various types of infinite series may wish to use this operation exclusively.

8.9.6 Power Series from Formulas

The **GenerateUnivariatePowerSeries** package enables you to create power series from explicit formulas for their n^{th} coefficients. In what follows, we construct series expansions for certain transcendental functions by giving formulas for their coefficients. You can also compute such series expansions directly simply by specifying the function and the point about which the series is to be expanded. See Section 8.9.5 on page 288 for more information.

Consider the Taylor expansion of e^x about `x = 0`:

$$\begin{aligned} e^x &= 1+x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots \\ &= \sum_{n=0}^{\infty}\frac{x^n}{n!} \end{aligned}$$

The n^{th} Taylor coefficient is `1/n!`. This is how you create this series in FriCAS.

```
series(n +-> 1/factorial(n),x = 0)
```

$$1+x+\frac{1}{2}\,x^2+\frac{1}{6}\,x^3+\frac{1}{24}\,x^4+\frac{1}{120}\,x^5+\frac{1}{720}\,x^6+\frac{1}{5040}\,x^7+O\left(x^8\right) \tag{1}$$

UnivariatePuiseuxSeries (Expression (Integer), x, 0)

The first argument specifies a formula for the n^{th} coefficient by giving a function that maps `n` to `1/n!`. The second argument specifies that the series is to be expanded in powers of `(x - 0)`, that is, in powers of `x`. Since we did not specify an initial degree, the first term in the series was the term of degree 0 (the constant term). Note that the formula was given as an anonymous function. These are discussed in Section 6.17 on page 179.

Consider the Taylor expansion of `log x` about `x = 1`:

$$\begin{aligned}\log(x) &= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots \\ &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-1)^n}{n}\end{aligned}$$

If you were to evaluate the expression `series(n ↦ (-1)^(n-1)/ n, x = 1)` you would get an error message because FriCAS would try to calculate a term of degree `0` and therefore divide by `0`.

Instead, evaluate this. The third argument, `1..`, indicates that only terms of degree `n = 1, ...` are to be computed.

```
series(n +-> (-1)^(n-1)/n,x = 1,1..)
```

$$(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 - \frac{1}{6}(x-1)^6 + \frac{1}{7}(x-1)^7 - \frac{1}{8}(x-1)^8 + O\left((x-1)^9\right) \tag{2}$$

UnivariatePuiseuxSeries (Expression (Integer), x, 1)

Next consider the Taylor expansion of an odd function, say, `sin(x)`:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

Here every other coefficient is zero and we would like to give an explicit formula only for the odd Taylor coefficients. This is one way to do it. The third argument, `1..`, specifies that the first term to be computed is the term of degree 1. The fourth argument, `2`, specifies that we increment by `2` to find the degrees of subsequent terms, that is, the next term is of degree `1 + 2`, the next of degree `1 + 2 + 2`, etc.

```
series(n +-> (-1)^((n-1)/2)/factorial(n),x = 0,1..,2)
```

$$x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + O\left(x^9\right) \tag{3}$$

UnivariatePuiseuxSeries (Expression (Integer), x, 0)

The initial degree and the increment do not have to be integers. For example, this expression produces a series expansion of $\sin(x^{\frac{1}{3}})$.

```
series(n +-> (-1)^((3*n-1)/2)/factorial(3*n),x = 0,1/3..,2/3)
```

$$x^{\frac{1}{3}} - \frac{1}{6}x + \frac{1}{120}x^{\frac{5}{3}} - \frac{1}{5040}x^{\frac{7}{3}} + O\left(x^3\right) \tag{4}$$

UnivariatePuiseuxSeries (Expression (Integer), x, 0)

While the increment must be positive, the initial degree may be negative. This yields the Laurent expansion of `csc(x)` at `x = 0`.

```
cscx := series(n +-> (-1)^((n-1)/2) * 2 * (2^n-1) * bernoulli(numer(n+1)) / _
    factorial(n+1), x=0, -1..,2)
```

$$x^{-1} + \frac{1}{6}\,x + \frac{7}{360}\,x^3 + \frac{31}{15120}\,x^5 + O\left(x^7\right) \tag{5}$$

UnivariatePuiseuxSeries (Expression (Integer), x, 0)

Of course, the reciprocal of this power series is the Taylor expansion of `sin(x)`.

```
1/cscx
```

$$x - \frac{1}{6}\,x^3 + \frac{1}{120}\,x^5 - \frac{1}{5040}\,x^7 + O\left(x^9\right) \tag{6}$$

UnivariatePuiseuxSeries (Expression (Integer), x, 0)

As a final example, here is the Taylor expansion of `asin(x)` about `x = 0`.

```
asinx := series(n +-> binomial(n-1,(n-1)/2)/(n*2^(n-1)),x=0,1..,2)
```

$$x + \frac{1}{6}\,x^3 + \frac{3}{40}\,x^5 + \frac{5}{112}\,x^7 + O\left(x^9\right) \tag{7}$$

UnivariatePuiseuxSeries (Expression (Integer), x, 0)

When we compute the `sin` of this series, we get `x` (in the sense that all higher terms computed so far are zero).

```
sin(asinx)
```

$$x + O\left(x^9\right) \tag{8}$$

UnivariatePuiseuxSeries (Expression (Integer), x, 0)

As we discussed in Section 8.9.5 on page 288, you can also use the operations taylor, laurent and puiseux instead of series if you know ahead of time what kind of exponents a series has. You can't go wrong using series, though.

8.9.7 Substituting Numerical Values in Power Series

Use eval to substitute a numerical value for a variable in a power series. For example, here's a way to obtain numerical approximations of `%e` from the Taylor series expansion of `exp(x)`.

First you create the desired Taylor expansion.

```
f := taylor(exp(x))
```

$$1 + x + \frac{1}{2}\,x^2 + \frac{1}{6}\,x^3 + \frac{1}{24}\,x^4 + \frac{1}{120}\,x^5 + \frac{1}{720}\,x^6 + \frac{1}{5040}\,x^7 + O\left(x^8\right) \tag{1}$$

UnivariateTaylorSeries (Expression (Integer), x, 0)

Then you evaluate the series at the value 1.0. The result is a sequence of the partial sums.

```
eval(f,1.0)
```

$$[1.0,\ 2.0,\ 2.5,\ 2.6666666666666666667,\ 2.7083333333333333333,\ 2.7166666666666666667,\ 2.7180555555555555556,\ \ldots] \tag{2}$$

Stream(Expression(Float))

8.9.8 Example: Bernoulli Polynomials and Sums of Powers

FriCAS provides operations for computing definite and indefinite sums.

You can compute the sum of the first ten fourth powers by evaluating this. This creates a list whose entries are m^4 as m ranges from 1 to 10, and then computes the sum of the entries of that list.

```
reduce(+,[m^4 for m in 1..10])
```

$$25333 \tag{1}$$

PositiveInteger

You can also compute a formula for the sum of the first k fourth powers, where k is an unspecified positive integer.

```
sum4 := sum(m^4, m = 1..k)
```

$$\frac{6\,k^5 + 15\,k^4 + 10\,k^3 - k}{30} \tag{2}$$

Fraction (Polynomial(Integer))

This formula is valid for any positive integer k. For instance, if we replace k by 10, we obtain the number we computed earlier.

```
eval(sum4, k = 10)
```

$$25333 \tag{3}$$

Fraction (Polynomial(Integer))

You can compute a formula for the sum of the first k n^{th} powers in a similar fashion. Just replace the 4 in the definition of **sum4** by any expression not involving k. FriCAS computes these formulas using Bernoulli polynomials; we use the rest of this section to describe this method.

First consider this function of t and x.

```
f := t*exp(x*t) / (exp(t) - 1)
```

$$\frac{t\,e^{t\,x}}{e^t - 1} \tag{4}$$

Expression (Integer)

Since the expressions involved get quite large, we tell FriCAS to show us only terms of degree up to 5.

```
)set streams calculate 5
```

If we look at the Taylor expansion of f(x, t) about t = 0, we see that the coefficients of the powers of t are polynomials in x.

```
ff := taylor(f,t = 0)
```

$$1+\frac{2\,x-1}{2}\,t+\frac{6\,x^2-6\,x+1}{12}\,t^2+\frac{2\,x^3-3\,x^2+x}{12}\,t^3 + \frac{30\,x^4-60\,x^3+30\,x^2-1}{720}\,t^4+\frac{6\,x^5-15\,x^4+10\,x^3-x}{720}\,t^5+O\left(t^6\right) \tag{5}$$

UnivariateTaylorSeries (Expression (Integer), t, 0)

In fact, the n^{th} coefficient in this series is essentially the n^{th} Bernoulli polynomial: the n^{th} coefficient of the series is $\frac{1}{n!}B_n(x)$, where $B_n(x)$ is the n^{th} Bernoulli polynomial. Thus, to obtain the n^{th} Bernoulli polynomial, we multiply the n^{th} coefficient of the series ff by n!. For example, the sixth Bernoulli polynomial is this.

```
factorial(6) * coefficient(ff,6)
```

$$\frac{42\,x^6-126\,x^5+105\,x^4-21\,x^2+1}{42} \tag{6}$$

Expression (Integer)

We derive some properties of the function f(x,t). First we compute f(x + 1,t)- f(x,t).

```
g := eval(f, x = x + 1) - f
```

$$\frac{t\,e^{t\,x+t}-t\,e^{t\,x}}{e^t-1} \tag{7}$$

Expression (Integer)

If we normalize g, we see that it has a particularly simple form.

```
normalize(g)
```

$$t\,e^{t\,x} \tag{8}$$

Expression (Integer)

From this it follows that the n^{th} coefficient in the Taylor expansion of g(x,t) at t = 0 is $\frac{1}{(n-1)!}\,x^{n-1}$. If you want to check this, evaluate the next expression.

```
taylor(g,t = 0)
```

$$t+x\,t^2+\frac{x^2}{2}\,t^3+\frac{x^3}{6}\,t^4+\frac{x^4}{24}\,t^5+O\left(t^6\right) \tag{9}$$

UnivariateTaylorSeries (Expression(Integer), t, 0)

However, since `g(x,t) = f(x+1,t)-f(x,t)`, it follows that the n^{th} coefficient is $\frac{1}{n!}\left(B_n(x+1)-B_n(x)\right)$. Equating coefficients, we see that $\frac{1}{(n-1)!}\,x^{n-1} = \frac{1}{n!}\left(B_n(x+1)-B_n(x)\right)$ and, therefore, $x^{n-1} = \frac{1}{n}\left(B_n(x+1)-B_n(x)\right)$. Let's apply this formula repeatedly, letting `x` vary between two integers `a` and `b`, with `a < b`:

$$\begin{array}{rcl} a^{n-1} & = & \frac{1}{n}(B_n(a+1)-B_n(a)) \\ (a+1)^{n-1} & = & \frac{1}{n}(B_n(a+2)-B_n(a+1)) \\ (a+2)^{n-1} & = & \frac{1}{n}(B_n(a+3)-B_n(a+2)) \\ & \vdots & \\ (b-1)^{n-1} & = & \frac{1}{n}(B_n(b)-B_n(b-1)) \\ b^{n-1} & = & \frac{1}{n}(B_n(b+1)-B_n(b)) \end{array}$$

When we add these equations we find that the sum of the left-hand sides is $\sum_{m=a}^{b} m^{n-1}$,the sum of the $(n-1)^{\text{st}}$ powers from `a` to `b`. The sum of the right-hand sides is a "telescoping series." After cancellation, the sum is simply $\frac{1}{n}\left(B_n(b+1)-B_n(a)\right)$.

Replacing `n` by `n + 1`, we have shown that

$$\sum_{m=a}^{b} m^n = \frac{1}{n+1}\left(B_{n+1}(b+1)-B_{n+1}(a)\right).$$

Let's use this to obtain the formula for the sum of fourth powers. First we obtain the Bernoulli polynomial B_5.

```
B5 := factorial(5) * coefficient(ff,5)
```

$$\frac{6\,x^5-15\,x^4+10\,x^3-x}{6} \tag{10}$$

Expression (Integer)

To find the sum of the first k 4th powers, we multiply `1/5` by $B_5(k+1)-B_5(1)$.

```
1/5 * (eval(B5, x = k + 1) - eval(B5, x = 1))
```

$$\frac{6\,k^5+15\,k^4+10\,k^3-k}{30} \tag{11}$$

Expression (Integer)

This is the same formula that we obtained via `sum(m^4, m = 1..k)`.

```
sum4
```

$$\frac{6\,k^5+15\,k^4+10\,k^3-k}{30} \tag{12}$$

Fraction (Polynomial(Integer))

At this point you may want to do the same computation, but with an exponent other than `4`. For example, you might try to find a formula for the sum of the first k 20th powers.

8.10 Solution of Differential Equations

In this section we discuss FriCAS's facilities for solving differential equations in closed-form and in series.

FriCAS provides facilities for closed-form solution of single differential equations of the following kinds:

- linear ordinary differential equations, and
- non-linear first order ordinary differential equations when integrating factors can be found just by integration.

For a discussion of the solution of systems of linear and polynomial equations, see Section 8.5 on page 270.

8.10.1 Closed-Form Solutions of Linear Differential Equations

A *differential equation* is an equation involving an unknown *function* and one or more of its derivatives. The equation is called *ordinary* if derivatives with respect to only one dependent variable appear in the equation (it is called *partial* otherwise). The package **ElementaryFunctionODESolver** provides the top-level operation solve for finding closed-form solutions of ordinary differential equations.

To solve a differential equation, you must first create an operator for the unknown function. We let `y` be the unknown function in terms of `x`.

```
y := operator 'y
```

$$y \tag{1}$$

BasicOperator

You then type the equation using D to create the derivatives of the unknown function `y(x)` where `x` is any symbol you choose (the so-called *dependent variable*). This is how you enter the equation `y'' + y' + y = 0`.

```
deq := D(y x, x, 2) + D(y x, x) + y x = 0
```

$$y''(x) + y'(x) + y(x) = 0 \tag{2}$$

Equation(Expression(Integer))

The simplest way to invoke the solve command is with three arguments.

- the differential equation,
- the operator representing the unknown function,
- the dependent variable.

So, to solve the above equation, we enter this.

```
solve(deq, y, x)
```

$$\left[particular = 0,\ basis = \left[\cos\left(\frac{x\sqrt{3}}{2}\right) e^{-\frac{x}{2}},\ e^{-\frac{x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right)\right]\right] \tag{3}$$

Union(Record(particular : Expression(Integer), basis : List(Expression(Integer))), ...)

Since linear ordinary differential equations have infinitely many solutions, solve returns a *particular solution* f_p and a basis $f_1, \ldots, f_n$ for the solutions of the corresponding homogeneous equation. Any expression of the form $f_p + c_1 f_1 + \ldots c_n f_n$ where the c_i do not involve the dependent variable is also a solution. This is similar to what you get when you solve systems of linear algebraic equations.

A way to select a unique solution is to specify *initial conditions*: choose a value `a` for the dependent variable and specify the values of the unknown function and its derivatives at `a`. If the number of initial conditions is equal to the order of the equation, then the solution is unique (if it exists in closed form!) and solve tries to find it. To specify initial conditions to solve, use an **Equation** of the form `x = a` for the third parameter instead of the dependent variable, and add a fourth parameter consisting of the list of values `y(a), y'(a), ....`

To find the solution of `y'' + y = 0` satisfying `y(0) = y'(0)= 1`, do this.

```
deq := D(y x, x, 2) + y x
```

$$y''(x) + y(x) \tag{4}$$

Expression(Integer)

You can omit the `= 0` when you enter the equation to be solved.

```
solve(deq, y, x = 0, [1, 1])
```

$$\sin(x) + \cos(x) \tag{5}$$

Union(Expression(Integer), ...)

FriCAS is not limited to linear differential equations with constant coefficients. It can also find solutions when the coefficients are rational or algebraic functions of the dependent variable. Furthermore, FriCAS is not limited by the order of the equation. FriCAS can solve the following third order equations with polynomial coefficients.

```
deq := x^3 * D(y x, x, 3) + x^2 * D(y x, x, 2) - 2 * x * D(y x, x) + 2 * y x = 2 * x^4
```

$$x^3\, y'''(x) + x^2\, y''(x) - 2\, x\, y'(x) + 2\, y(x) = 2\, x^4 \tag{6}$$

Equation(Expression(Integer))

```
solve(deq, y, x)
```

$$\left[particular = \frac{x^5 - 10\,x^3 + 20\,x^2 + 4}{15\,x},\ basis = \left[\frac{2\,x^3 - 3\,x^2 + 1}{x},\ \frac{x^3 - 1}{x},\ \frac{x^3 - 3\,x^2 - 1}{x}\right]\right] \tag{7}$$

Union(Record(particular : Expression(Integer), basis : List(Expression(Integer))), ...)

Here we are solving a homogeneous equation.

```
deq := (x^9+x^3) * D(y x, x, 3) + 18 * x^8 * D(y x, x, 2) - 90 * x * D(y x, x) - 30 * _
    (11 * x^6 - 3) * y x
```

$$\left(x^9+x^3\right)\,y'''(x)+18\,x^8\,y''(x)-90\,x\,y'(x)+\left(-330\,x^6+90\right)\,y(x) \tag{8}$$

Expression(Integer)

```
solve(deq, y, x)
```

$$\left[particular=0,\ basis=\left[\frac{x}{x^6+1},\ \frac{x\,e^{-\sqrt{91}\,\log(x)}}{x^6+1},\ \frac{x\,e^{\sqrt{91}\,\log(x)}}{x^6+1}\right]\right] \tag{9}$$

Union(Record(particular : Expression(Integer), basis : List(Expression(Integer))), ...)

On the other hand, and in contrast with the operation integrate, it can happen that FriCAS finds no solution and that some closed-form solution still exists. While it is mathematically complicated to describe exactly when the solutions are guaranteed to be found, the following statements are correct and form good guidelines for linear ordinary differential equations:

- If the coefficients are constants, FriCAS finds a complete basis of solutions (i,e, all solutions).
- If the coefficients are rational functions in the dependent variable, FriCAS at least finds all solutions that do not involve algebraic functions.

Note that this last statement does not mean that FriCAS does not find the solutions that are algebraic functions. It means that it is not guaranteed that the algebraic function solutions will be found. This is an example where all the algebraic solutions are found.

```
deq := (x^2 + 1) * D(y x, x, 2) + 3 * x * D(y x, x) + y x = 0
```

$$\left(x^2+1\right)\,y''(x)+3\,x\,y'(x)+y(x)=0 \tag{10}$$

Equation(Expression(Integer))

```
solve(deq, y, x)
```

$$\left[particular=0,\ basis=\left[\frac{1}{\sqrt{x^2+1}},\ \frac{\log\left(\sqrt{x^2+1}-x\right)}{\sqrt{x^2+1}}\right]\right] \tag{11}$$

Union(Record(particular : Expression(Integer), basis : List(Expression(Integer))), ...)

8.10.2 Closed-Form Solutions of Non-Linear Differential Equations

This is an example that shows how to solve a non-linear first order ordinary differential equation manually when an integrating factor can be found just by integration. At the end, we show you how to solve it directly.

Let's solve the differential equation `y' = y / (x + y log y)`. Using the notation `m(x, y)+ n(x, y)y' = 0`, we have `m = -y` and `n = x + y log y`.

```
m := -y
```

$$-y \tag{1}$$

Polynomial(Integer)

```
n := x + y * log y
```

$$y\log(y) + x \tag{2}$$

Expression(Integer)

We first check for exactness, that is, does `dm/dy = dn/dx`?

```
D(m, y) - D(n, x)
```

$$-2 \tag{3}$$

Expression(Integer)

This is not zero, so the equation is not exact. Therefore we must look for an integrating factor: a function `μ(x,y)` such that `d(μ m)/dy = d(μ n)/dx`. Normally, we first search for `μ(x,y)` depending only on `x` or only on `y`. Let's search for such a `μ(x)` first.

```
mu := operator 'mu
```

$$mu \tag{4}$$

BasicOperator

```
a := D(mu(x) * m, y) - D(mu(x) * n, x)
```

$$(-y\log(y) - x)\,mu'(x) - 2\,\mathrm{mu}(x) \tag{5}$$

Expression(Integer)

If the above is zero for a function μ that does *not* depend on `y`, then `μ(x)` is an integrating factor.

```
solve(a = 0, mu, x)
```

$$\left[particular = 0,\ basis = \left[\frac{1}{y^2\log(y)^2 + 2\,x\,y\,\log(y) + x^2}\right]\right] \tag{6}$$

Union(Record(particular : Expression(Integer), basis : List(Expression(Integer))), ...)

The solution depends on `y`, so there is no integrating factor that depends on `x` only. Let's look for one that depends on `y` only.

```
b := D(mu(y) * m, y) - D(mu(y) * n, x)
```

$$-y\,mu'(y) - 2\,\mathrm{mu}(y) \tag{7}$$

Expression(Integer)

```
sb := solve(b = 0, mu, y)
```

$$\left[particular = 0,\ basis = \left[\frac{1}{y^2}\right]\right] \tag{8}$$

Union(Record(particular : Expression(Integer), basis : List(Expression(Integer))), ...)

We've found one! The above μ(y) is an integrating factor. We must multiply our initial equation (that is, m and n) by the integrating factor.

```
intFactor := sb.basis.1
```

$$\frac{1}{y^2} \tag{9}$$

Expression(Integer)

```
m := intFactor * m
```

$$-\frac{1}{y} \tag{10}$$

Expression(Integer)

```
n := intFactor * n
```

$$\frac{y\,\log(y) + x}{y^2} \tag{11}$$

Expression(Integer)

Let's check for exactness.

```
D(m, y) - D(n, x)
```

$$0 \tag{12}$$

Expression(Integer)

We must solve the exact equation, that is, find a function s(x,y) such that ds/dx = m and ds/dy = n. We start by writing s(x, y)= h(y)+ integrate(m, x) where h(y) is an unknown function of y. This guarantees that ds/dx = m.

```
h := operator 'h
```

$$h \tag{13}$$

BasicOperator

```
sol := h y + integrate(m, x)
```

$$\frac{y\,h(y) - x}{y} \tag{14}$$

Expression (Integer)

All we want is to find `h(y)` such that `ds/dy = n`.

```
dsol := D(sol, y)
```

$$\frac{y^2\,h'(y)+x}{y^2} \tag{15}$$

Expression (Integer)

```
nsol := solve(dsol = n, h, y)
```

$$\left[particular=\frac{\log(y)^2}{2},\ basis=[1]\right] \tag{16}$$

Union(Record(particular : Expression (Integer), basis : List (Expression (Integer))), ...)

The above particular solution is the `h(y)` we want, so we just replace `h(y)` by it in the implicit solution.

```
eval(sol, h y = nsol.particular)
```

$$\frac{y\log(y)^2-2\,x}{2\,y} \tag{17}$$

Expression (Integer)

A first integral of the initial equation is obtained by setting this result equal to an arbitrary constant.

Now that we've seen how to solve the equation "by hand," we show you how to do it with the solve operation. First define `y` to be an operator.

```
y := operator 'y
```

$$y \tag{18}$$

BasicOperator

Next we create the differential equation.

```
deq := D(y x, x) = y(x) / (x + y(x) * log y x)
```

$$y'(x)=\frac{y(x)}{y(x)\,\log(y(x))+x} \tag{19}$$

Equation(Expression (Integer))

Finally, we solve it.

```
solve(deq, y, x)
```

$$\frac{y(x)\log(y(x))^2-2\,x}{2\,y(x)} \tag{20}$$

Union(Expression(Integer), ...)

8.10.3 Power Series Solutions of Differential Equations

The command to solve differential equations in power series around a particular initial point with specific initial conditions is called seriesSolve. It can take a variety of parameters, so we illustrate its use with some examples.

Since the coefficients of some solutions are quite large, we reset the default to compute only seven terms.

```
)set streams calculate 7
```

You can solve a single nonlinear equation of any order. For example, we solve `y''' = sin(y'')* exp(y)+ cos(x)` subject to `y(0) = 1, y'(0)= 0, y''(0)= 0`.

We first tell FriCAS that the symbol `'y` denotes a new operator.

```
y := operator 'y
```

$$y \tag{1}$$

BasicOperator

Enter the differential equation using `y` like any system function.

```
eq := D(y(x), x, 3) - sin(D(y(x), x, 2))*exp(y(x)) = cos(x)
```

$$y'''(x) - e^{y(x)} \sin\left(y''(x)\right) = \cos(x) \tag{2}$$

Equation(Expression(Integer))

Solve it around `x = 0` with the initial conditions `y(0) = 1, y'(0)= y''(0)= 0`.

```
seriesSolve(eq, y, x = 0, [1, 0, 0])
```

```
   Compiling function %JJ with type List(UnivariateTaylorSeries(Expression(Integer
      ),x,0)) -> UnivariateTaylorSeries(Expression(Integer),x,0)
```

$$1 + \frac{1}{6}\,x^3 + \frac{e}{24}\,x^4 + \frac{e^2-1}{120}\,x^5 + \frac{e^3-2\,e}{720}\,x^6 + \frac{e^4-8\,e^2+4\,e+1}{5040}\,x^7 + O\left(x^8\right) \tag{3}$$

UnivariateTaylorSeries (Expression(Integer), x, 0)

You can also solve a system of nonlinear first order equations. For example, we solve a system that has `tan(t)` and `sec(t)` as solutions.

We tell FriCAS that `x` is also an operator.

```
x := operator 'x
```

```
   Compiled code for %JJ has been cleared.
```

$$x \tag{4}$$

BasicOperator

Enter the two equations forming our system.

```
eq1 := D(x(t), t) = 1 + x(t)^2
```

$$x'(t) = x(t)^2 + 1 \tag{5}$$

Equation(Expression(Integer))

```
eq2 := D(y(t), t) = x(t) * y(t)
```

$$y'(t) = x(t)\,y(t) \tag{6}$$

Equation(Expression(Integer))

Solve the system around `t = 0` with the initial conditions `x(0) = 0` and `y(0) = 1`. Notice that since we give the unknowns in the order `[x, y]`, the answer is a list of two series in the order `[series for x(t), series for y(t)]`.

```
seriesSolve([eq2, eq1], [x, y], t = 0, [y(0) = 1, x(0) = 0])
```

```
   Compiling function %JP with type List(UnivariateTaylorSeries(Expression(Integer
      ),t,0)) -> UnivariateTaylorSeries(Expression(Integer),t,0)
   Compiling function %JQ with type List(UnivariateTaylorSeries(Expression(Integer
      ),t,0)) -> UnivariateTaylorSeries(Expression(Integer),t,0)
```

$$\left[t + \frac{1}{3}\,t^3 + \frac{2}{15}\,t^5 + \frac{17}{315}\,t^7 + O\left(t^8\right),\ 1 + \frac{1}{2}\,t^2 + \frac{5}{24}\,t^4 + \frac{61}{720}\,t^6 + O\left(t^8\right)\right] \tag{7}$$

List(UnivariateTaylorSeries(Expression(Integer), t, 0))

The order in which we give the equations and the initial conditions has no effect on the order of the solution.

8.11 Finite Fields

A *finite field* (also called a *Galois field*) is a finite algebraic structure where one can add, multiply and divide under the same laws (for example, commutativity, associativity or distributivity) as apply to the rational, real or complex numbers. Unlike those three fields, for any finite field there exists a positive prime integer p, called the characteristic, such that $p\,x = 0$ for any element x in the finite field. In fact, the number of elements in a finite field is a power of the characteristic and for each prime p and positive integer n there exists exactly one finite field with p^n elements, up to isomorphism.[1]

When `n = 1`, the field has p elements and is called a *prime field*, discussed in the next section. There are several ways of implementing extensions of finite fields, and FriCAS provides quite a bit of freedom to allow you to choose the one that is best for your application. Moreover, we provide operations for converting among the different representations of extensions and different extensions of a single field. Finally, note that you usually need to package-call operations from finite fields if the operations do

[1] For more information about the algebraic structure and properties of finite fields, see, for example, S. Lang, *Algebra*, Second Edition, New York: Addison-Wesley Publishing Company, Inc., 1984, ISBN 0 201 05487 6; or R. Lidl, H. Niederreiter, *Finite Fields*, Encyclopedia of Mathematics and Its Applications, Vol. 20, Cambridge: Cambridge Univ. Press, 1983, ISBN 0 521 30240 4.

not take as an argument an object of the field. See Section 2.9 on page 89 for more information on package-calling.

8.11.1 Modular Arithmetic and Prime Fields

Let n be a positive integer. It is well known that you can get the same result if you perform addition, subtraction or multiplication of integers and then take the remainder on dividing by n as if you had first done such remaindering on the operands, performed the arithmetic and then (if necessary) done remaindering again. This allows us to speak of arithmetic *modulo* n or, more simply *mod* n. In FriCAS, you use **IntegerMod** to do such arithmetic.

```
(a,b) : IntegerMod 12
```

```
(a, b) := (16, 7)
```

$$7 \tag{2}$$

IntegerMod(12)

```
[a - b, a * b]
```

$$[9,\ 4] \tag{3}$$

List (IntegerMod(12))

If n is not prime, there is only a limited notion of reciprocals and division.

```
a / b
```

```
   There are 10 exposed and 16 unexposed library operations named / having 2
      argument(s) but none was determined to be applicable. Use HyperDoc Browse,
      or issue
                                    )display op /
      to learn more about the available operations. Perhaps package-calling the
      operation or using coercions on the arguments will allow you to apply the
      operation.
   Cannot find a definition or applicable library operation named / with argument
      type(s)
                                    IntegerMod(12)
                                    IntegerMod(12)

      Perhaps you should use "@" to indicate the required return type, or "$" to
      specify which version of the function you need.
```

```
recip a
```

$$\text{"failed"} \tag{4}$$

Union(" failed ", ...)

Here 7 and 12 are relatively prime, so 7 has a multiplicative inverse modulo 12.

```
recip b
```

$$7 \tag{5}$$

Union(IntegerMod(12), ...)

If we take n to be a prime number p, then taking inverses and, therefore, division are generally defined. Use **PrimeField** instead of **IntegerMod** for n prime.

```
c : PrimeField 11 := 8
```

$$8 \tag{6}$$

PrimeField(11)

```
inv c
```

$$7 \tag{7}$$

PrimeField(11)

You can also use `1/c` and `c^(-1)` for the inverse of c.

```
9/c
```

$$8 \tag{8}$$

PrimeField(11)

PrimeField (abbreviation **PF**) checks if its argument is prime when you try to use an operation from it. If you know the argument is prime (particularly if it is large), **InnerPrimeField** (abbreviation **IPF**) assumes the argument has already been verified to be prime. If you do use a number that is not prime, you will eventually get an error message, most likely a division by zero message. For computer science applications, the most important finite fields are **PrimeField 2** and its extensions.

In the following examples, we work with the finite field with $p = 101$ elements.

```
GF101 := PF 101
```

$$\text{PrimeField}(101) \tag{9}$$

Type

Like many domains in FriCAS, finite fields provide an operation for returning a random element of the domain.

```
x := random()$GF101
```

$$21 \tag{10}$$

PrimeField(101)

```
y : GF101 := 37
```

$$37 \tag{11}$$

PrimeField(101)

```
z := x/y
```

$$77 \tag{12}$$

PrimeField(101)

```
z * y - x
```

$$0 \tag{13}$$

PrimeField(101)

The element 2 is a *primitive element* of this field,

```
pe := primitiveElement()$GF101
```

$$2 \tag{14}$$

PrimeField(101)

in the sense that its powers enumerate all nonzero elements.

```
[pe^i for i in 0..99]
```

$$[1, 2, 4, 8, 16, 32, 64, 27, 54, 7, 14, 28, 56, 11, 22, 44, 88, 75, 49, 98, 95, 89, 77, 53, 5, 10, 20, 40, 80, 59, 17, 34, 68, 35, 70, 39, 78, 55, 9, 18, 36, 72, 43, 86, 71, 41, 82, 63, 25, 50, 100, 99, 97, 93, 85, 69, 37, 74, 47, 94, 87, 73, 45, 90, 79, 57, 13, 26, 52, 3, 6, 12, 24, 48, 96, 91, 81, 61, 21, 42, 84, 67, 33, 66, 31, 62, 23, 46, 92, 83, 65, 29, 58, 15, 30, 60, 19, 38, 76, 51] \tag{15}$$

List (PrimeField(101))

If every nonzero element is a power of a primitive element, how do you determine what the exponent is? Use discreteLog.

```
ex := discreteLog(y)
```

$$56 \tag{16}$$

PositiveInteger

```
pe ^ ex
```

$$37 \tag{17}$$

PrimeField(101)

The order of a nonzero element x is the smallest positive integer t such $x^t = 1$.

```
order y
```

$$25 \tag{18}$$

PositiveInteger

The order of a primitive element is the defining $p - 1$.

```
order pe
```

```
                                100                                    (19)
                                                                PositiveInteger
```

8.11.2 Extensions of Finite Fields

When you want to work with an extension of a finite field in FriCAS, you have three choices to make:

1. Do you want to generate an extension of the prime field (for example, **PrimeField 2**) or an extension of a given field?
2. Do you want to use a representation that is particularly efficient for multiplication, exponentiation and addition but uses a lot of computer memory (a representation that models the cyclic group structure of the multiplicative group of the field extension and uses a Zech logarithm table), one that uses a normal basis for the vector space structure of the field extension, or one that performs arithmetic modulo an irreducible polynomial? The cyclic group representation is only usable up to "medium" (relative to your machine's performance) sized fields. If the field is large and the normal basis is relatively simple, the normal basis representation is more efficient for exponentiation than the irreducible polynomial representation.
3. Do you want to provide a polynomial explicitly, a root of which "generates" the extension in one of the three senses in (2), or do you wish to have the polynomial generated for you?

This illustrates one of the most important features of FriCAS: you can choose exactly the right data-type and representation to suit your application best.

We first tell you what domain constructors to use for each case above, and then give some examples.

Constructors that automatically generate extensions of the prime field:
FiniteField
FiniteFieldCyclicGroup
FiniteFieldNormalBasis

Constructors that generate extensions of an arbitrary field:
FiniteFieldExtension
FiniteFieldExtensionByPolynomial
FiniteFieldCyclicGroupExtension
FiniteFieldCyclicGroupExtensionByPolynomial
FiniteFieldNormalBasisExtension
FiniteFieldNormalBasisExtensionByPolynomial

Constructors that use a cyclic group representation:
FiniteFieldCyclicGroup
FiniteFieldCyclicGroupExtension
FiniteFieldCyclicGroupExtensionByPolynomial

Constructors that use a normal basis representation:
FiniteFieldNormalBasis
FiniteFieldNormalBasisExtension
FiniteFieldNormalBasisExtensionByPolynomial

Constructors that use an irreducible modulus polynomial representation:
FiniteField
FiniteFieldExtension
FiniteFieldExtensionByPolynomial

Constructors that generate a polynomial for you:
FiniteField
FiniteFieldExtension
FiniteFieldCyclicGroup
FiniteFieldCyclicGroupExtension
FiniteFieldNormalBasis
FiniteFieldNormalBasisExtension

Constructors for which you provide a polynomial:
FiniteFieldExtensionByPolynomial
FiniteFieldCyclicGroupExtensionByPolynomial
FiniteFieldNormalBasisExtensionByPolynomial

These constructors are discussed in the following sections where we collect together descriptions of extension fields that have the same underlying representation.[2]

If you don't really care about all this detail, just use **FiniteField**. As your knowledge of your application and its FriCAS implementation grows, you can come back and choose an alternative constructor that may improve the efficiency of your code. Note that the exported operations are almost the same for all constructors of finite field extensions and include the operations exported by **PrimeField**.

8.11.3 Irreducible Modulus Polynomial Representations

All finite field extension constructors discussed in this section use a representation that performs arithmetic with univariate (one-variable) polynomials modulo an irreducible polynomial. This polynomial may be given explicitly by you or automatically generated. The ground field may be the prime field or one you specify. See Section 8.11.2 on page 307 for general information about finite field extensions.

For **FiniteField** (abbreviation **FF**) you provide a prime number p and an extension degree n. This degree can be 1. FriCAS uses the prime field **PrimeField(p)**, here **PrimeField 2**, and it chooses an irreducible polynomial of degree n, here 12, over the ground field.

```
GF4096 := FF(2,12);
```
Type

The objects in the generated field extension are polynomials of degree at most $n-1$ with coefficients in the prime field. The polynomial indeterminate is automatically chosen by FriCAS and is typically something like `%A` or `%D`. These (strange) variables are *only* for output display; there are several ways to construct elements of this field.

The operation index enumerates the elements of the field extension and accepts as argument the integers from 1 to p^n. The expression `index(p)` always gives the indeterminate.

```
a := index(2)$GF4096
```

[2] For more information on the implementation aspects of finite fields, see J. Grabmeier, A. Scheerhorn, *Finite Fields in AXIOM,* Technical Report, IBM Heidelberg Scientific Center, 1992.

$$\%JR \tag{2}$$

FiniteField (2, 12)

You can build polynomials in a and calculate in GF4096.

```
b := a^12 - a^5 + a
```

$$\%JR^5 + \%JR^3 + \%JR + 1 \tag{3}$$

FiniteField (2, 12)

```
b ^ 1000
```

$$\%JR^{10} + \%JR^9 + \%JR^7 + \%JR^5 + \%JR^4 + \%JR^3 + \%JR \tag{4}$$

FiniteField (2, 12)

```
c := a/b
```

$$\%JR^{11} + \%JR^8 + \%JR^7 + \%JR^5 + \%JR^4 + \%JR^3 + \%JR^2 \tag{5}$$

FiniteField (2, 12)

Among the available operations are norm and trace.

```
norm c
```

$$1 \tag{6}$$

PrimeField(2)

```
trace c
```

$$0 \tag{7}$$

PrimeField(2)

Since any nonzero element is a power of a primitive element, how do we discover what the exponent is? The operation discreteLog calculates the exponent and, if it is called with only one argument, always refers to the primitive element returned by primitiveElement.

```
dL := discreteLog a
```

$$1729 \tag{8}$$

PositiveInteger

```
g ^ dL
```

$$g^{1729} \tag{9}$$

Polynomial(Integer)

FiniteFieldExtension (abbreviation **FFX**) is similar to **FiniteField** except that the ground-field for **FiniteFieldExtension** is arbitrary and chosen by you. In case you select the prime field as ground field, there is essentially no difference between the constructed two finite field extensions.

```
GF16 := FF(2,4);
```

Type

```
GF4096 := FFX(GF16,3);
```

Type

```
r := (random()$GF4096) ^ 20
```

$$(\%JS^2 + 1)\ \%JT^2 + (\%JS + 1)\ \%JT + \%JS + 1 \tag{12}$$

FiniteFieldExtension (FiniteField (2, 4), 3)

```
norm(r)
```

$$\%JS^2 + \%JS + 1 \tag{13}$$

FiniteField (2, 4)

FiniteFieldExtensionByPolynomial (abbreviation **FFP**) is similar to **FiniteField** and **FiniteFieldExtension** but is more general.

```
GF4 := FF(2,2);
```

Type

```
f := nextIrreduciblePoly(random(6)$FFPOLY(GF4))$FFPOLY(GF4)
```

$$?^6 + \%JU\ ?^5 + ?^4 + ?^3 + (\%JU + 1)\ ?^2 + (\%JU + 1)\ ? + \%JU \tag{15}$$

Union(SparseUnivariatePolynomial(FiniteField (2, 2)), ...)

For **FFP** you choose both the ground field and the irreducible polynomial used in the representation. The degree of the extension is the degree of the polynomial.

```
GF4096 := FFP(GF4,f);
```

Type

```
discreteLog random()$GF4096
```

$$140 \tag{17}$$

PositiveInteger

8.11.4 Cyclic Group Representations

In every finite field there exist elements whose powers are all the nonzero elements of the field. Such an element is called a *primitive element*.

In **FiniteFieldCyclicGroup** (abbreviation **FFCG**) the nonzero elements are represented by the powers of a fixed primitive element of the field (that is, a generator of its cyclic multiplicative group).

Multiplication (and hence exponentiation) using this representation is easy. To do addition, we consider our primitive element as the root of a primitive polynomial (an irreducible polynomial whose roots are all primitive). See Section 8.11.7 on page 317 for examples of how to compute such a polynomial.

To use **FiniteFieldCyclicGroup** you provide a prime number and an extension degree.

```
GF81 := FFCG(3,4);
```

Type

FriCAS uses the prime field, here **PrimeField 3**, as the ground field and it chooses a primitive polynomial of degree n, here 4, over the prime field.

```
a := primitiveElement()$GF81
```

$$\%JW^{1} \tag{2}$$

FiniteFieldCyclicGroup (3, 4)

You can calculate in GF81.

```
b  := a^12 - a^5 + a
```

$$\%JW^{72} \tag{3}$$

FiniteFieldCyclicGroup (3, 4)

In this representation of finite fields the discrete logarithm of an element can be seen directly in its output form.

```
b
```

$$\%JW^{72} \tag{4}$$

FiniteFieldCyclicGroup (3, 4)

```
discreteLog b
```

$$72 \tag{5}$$

PositiveInteger

FiniteFieldCyclicGroupExtension (abbreviation **FFCGX**) is similar to **FiniteFieldCyclicGroup** except that the ground field for **FiniteFieldCyclicGroupExtension** is arbitrary and chosen by you. In case you select the prime field as ground field, there is essentially no difference between the constructed two finite field extensions.

```
GF9 := FF(3,2);
```

Type

```
GF729 := FFCGX(GF9,3);
```

Type

```
r := (random()$GF729) ^ 20
```

$$\%JY^{668} \tag{8}$$

```
                                                FiniteFieldCyclicGroupExtension ( FiniteField (3, 2), 3)
```

```
trace(r)
```

$$2\,\%JX \tag{9}$$

```
                                                                    FiniteField (3, 2)
```

FiniteFieldCyclicGroupExtensionByPolynomial (abbreviation **FFCGP**) is similar to **FiniteFieldCyclicGroup** and **FiniteFieldCyclicGroupExtension** but is more general. For **FiniteFieldCyclicGroupExtensionByPolynomial** you choose both the ground field and the irreducible polynomial used in the representation. The degree of the extension is the degree of the polynomial.

```
GF3  := PrimeField 3;
```

```
                                                                                    Type
```

We use a utility operation to generate an irreducible primitive polynomial (see Section 8.11.7 on page 317). The polynomial has one variable that is "anonymous": it displays as a question mark.

```
f := createPrimitivePoly(4)$FFPOLY(GF3)
```

$$?^4 + ? + 2 \tag{11}$$

```
                                                SparseUnivariatePolynomial(PrimeField(3))
```

```
GF81 := FFCGP(GF3,f);
```

```
                                                                                    Type
```

Let's look at a random element from this field.

```
random()$GF81
```

$$\%JW^{55} \tag{13}$$

```
                                FiniteFieldCyclicGroupExtensionByPolynomial(PrimeField(3), ?^4+?+2)
```

8.11.5 Normal Basis Representations

Let K be a finite extension of degree n of the finite field F and let F have q elements. An element x of K is said to be *normal* over F if the elements

$$1, x^q, x^{q^2}, \ldots, x^{q^{n-1}}$$

form a basis of K as a vector space over F. Such a basis is called a *normal basis*.[3]

If x is normal over F, its minimal polynomial is also said to be *normal* over F. There exist normal bases for all finite extensions of arbitrary finite fields.

In **FiniteFieldNormalBasis** (abbreviation **FFNB**), the elements of the finite field are represented by coordinate vectors with respect to a normal basis.

You provide a prime p and an extension degree n.

[3] This agrees with the general definition of a normal basis because the n distinct powers of the automorphism $x \mapsto x^q$ constitute the Galois group of K/F.

```
K := FFNB(3,8)
```

$$\text{FiniteFieldNormalBasis}(3,8) \tag{1}$$

Type

FriCAS uses the prime field **PrimeField(p)**, here **PrimeField 3**, and it chooses a normal polynomial of degree n, here 8, over the ground field. The remainder class of the indeterminate is used as the normal element. The polynomial indeterminate is automatically chosen by FriCAS and is typically something like `%A` or `%D`. These (strange) variables are only for output display; there are several ways to construct elements of this field. The output of the basis elements is something like $\%A^{q^i}$.

```
a := normalElement()$K
```

$$\%JZ \tag{2}$$

FiniteFieldNormalBasis (3, 8)

You can calculate in K using a.

```
b  := a^12 - a^5 + a
```

$$2\,\%JZ^{q^7} + \%JZ^{q^5} + \%JZ^q \tag{3}$$

FiniteFieldNormalBasis (3, 8)

FiniteFieldNormalBasisExtension (abbreviation **FFNBX**) is similar to **FiniteFieldNormalBasis** except that the groundfield for **FiniteFieldNormalBasisExtension** is arbitrary and chosen by you. In case you select the prime field as ground field, there is essentially no difference between the constructed two finite field extensions.

```
GF9 := FFNB(3,2);
```

Type

```
GF729 := FFNBX(GF9,3);
```

Type

```
r := random()$GF729
```

$$(\%KA^q + 2\,\%KA)\ \%KB^{q^2} + \%KB^q + \%KB \tag{6}$$

FiniteFieldNormalBasisExtension (FiniteFieldNormalBasis (3, 2), 3)

```
r + r^3 + r^9 + r^27
```

$$(2\,\%KA^q + 2\,\%KA)\ \%KB^{q^2} + (2\,\%KA^q + \%KA)\ \%KB^q + (\%KA^q + 2\,\%KA)\ \%KB \tag{7}$$

FiniteFieldNormalBasisExtension (FiniteFieldNormalBasis (3, 2), 3)

FiniteFieldNormalBasisExtensionByPolynomial (abbreviation **FFNBP**) is similar to **FiniteFieldNormalBasis** and **FiniteFieldNormalBasisExtension** but is more general. For **FiniteFieldNormalBasisExtensionByPolynomial** you choose both the ground field and the irreducible polynomial used in the representation. The degree of the extension is the degree of the polynomial.

```
GF3 := PrimeField 3;
```

Type

We use a utility operation to generate an irreducible normal polynomial (see Section 8.11.7 on page 317). The polynomial has one variable that is "anonymous": it displays as a question mark.

```
f := createNormalPoly(4)$FFPOLY(GF3)
```

$$?^4 + 2\,?^3 + 2 \tag{9}$$

SparseUnivariatePolynomial(PrimeField(3))

```
GF81 := FFNBP(GF3,f);
```

Type

Let's look at a random element from this field.

```
r := random()$GF81
```

$$2\,\%KC^{q^3} \tag{11}$$

FiniteFieldNormalBasisExtensionByPolynomial(PrimeField(3), ?^4+2*?^3+2)

```
r * r^3 * r^9 * r^27
```

$$2\,\%KC^{q^3} + 2\,\%KC^{q^2} + 2\,\%KC^{q} + 2\,\%KC \tag{12}$$

FiniteFieldNormalBasisExtensionByPolynomial(PrimeField(3), ?^4+2*?^3+2)

```
norm r
```

$$2 \tag{13}$$

PrimeField(3)

8.11.6 Conversion Operations for Finite Fields

Let K be a finite field.

```
K := PrimeField 3
```

$$\text{PrimeField}(3) \tag{1}$$

Type

An extension field K_m of degree m over K is a subfield of an extension field K_n of degree n over K if and only if m divides n.

$$\begin{array}{c} K_n \\ | \\ K_m \\ | \\ \text{K} \end{array} \iff m|n$$

FiniteFieldHomomorphisms provides conversion operations between different extensions of one fixed finite ground field and between different representations of these finite fields. Let's choose m and n,

```
(m,n) := (4,8)
```

$$8 \tag{2}$$

PositiveInteger

build the field extensions,

```
Km := FiniteFieldExtension(K,m)
```

$$\text{FiniteFieldExtension}(\text{PrimeField}(3), 4) \tag{3}$$

Type

and pick two random elements from the smaller field.

```
Kn := FiniteFieldExtension(K,n)
```

$$\text{FiniteFieldExtension}(\text{PrimeField}(3), 8) \tag{4}$$

Type

```
a1 := random()$Km
```

$$2\,\%KD^2 + \%KD + 1 \tag{5}$$

FiniteFieldExtension (PrimeField(3), 4)

```
b1 := random()$Km
```

$$2\,\%KD^3 + 2\,\%KD^2 + \%KD \tag{6}$$

FiniteFieldExtension (PrimeField(3), 4)

Since m divides n, K_m is a subfield of K_n.

```
a2 := a1 :: Kn
```

$$\%KE^6 + \%KE^4 + 1 \tag{7}$$

FiniteFieldExtension (PrimeField(3), 8)

Therefore we can convert the elements of K_m into elements of K_n.

```
b2 := b1 :: Kn
```

$$\%KE^6 + \%KE^4 + 2\,\%KE^2 \tag{8}$$

FiniteFieldExtension (PrimeField(3), 8)

To check this, let's do some arithmetic.

```
a1+b1 - ((a2+b2) :: Km)
```

$$0 \tag{9}$$

FiniteFieldExtension (PrimeField(3), 4)

```
a1*b1 - ((a2*b2) :: Km)
```

$$0 \tag{10}$$

FiniteFieldExtension (PrimeField(3), 4)

There are also conversions available for the situation, when K_m and K_n are represented in different ways (see Section 8.11.2 on page 307). For example let's choose K_m where the representation is 0 plus the cyclic multiplicative group and K_n with a normal basis representation.

```
Km := FFCGX(K,m)
```

$$\text{FiniteFieldCyclicGroupExtension}(\text{PrimeField}(3), 4) \tag{11}$$

Type

```
Kn := FFNBX(K,n)
```

$$\text{FiniteFieldNormalBasisExtension}(\text{PrimeField}(3), 8) \tag{12}$$

Type

```
(a1,b1) := (random()$Km,random()$Km)
```

$$0 \tag{13}$$

FiniteFieldCyclicGroupExtension (PrimeField(3), 4)

```
a2 := a1 :: Kn
```

$$\%KF^{q^7} + \%KF^{q^4} + \%KF^{q^3} + \%KF \tag{14}$$

FiniteFieldNormalBasisExtension (PrimeField(3), 8)

```
b2 := b1 :: Kn
```

$$0 \tag{15}$$

FiniteFieldNormalBasisExtension (PrimeField(3), 8)

Check the arithmetic again.

```
a1+b1 - ((a2+b2) :: Km)
```

$$0 \tag{16}$$

FiniteFieldCyclicGroupExtension (PrimeField(3), 4)

```
a1*b1 - ((a2*b2) :: Km)
```

$$0 \tag{17}$$

FiniteFieldCyclicGroupExtension (PrimeField(3), 4)

8.11.7 Utility Operations for Finite Fields

FiniteFieldPolynomialPackage (abbreviation **FFPOLY**) provides operations for generating, counting and testing polynomials over finite fields. Let's start with a couple of definitions:

- A polynomial is *primitive* if its roots are primitive elements in an extension of the coefficient field of degree equal to the degree of the polynomial.
- A polynomial is *normal* over its coefficient field if its roots are linearly independent elements in an extension of the coefficient field of degree equal to the degree of the polynomial.

In what follows, many of the generated polynomials have one "anonymous" variable. This indeterminate is displayed as a question mark ("?").

To fix ideas, let's use the field with five elements for the first few examples.

```
GF5 := PF 5;
```

Type

You can generate irreducible polynomials of any (positive) degree (within the storage capabilities of the computer and your ability to wait) by using createIrreduciblePoly.

```
f := createIrreduciblePoly(8)$FFPOLY(GF5)
```

$$?^8 + ?^4 + 2 \tag{2}$$

SparseUnivariatePolynomial(PrimeField(5))

Does this polynomial have other important properties? Use primitive? to test whether it is a primitive polynomial.

```
primitive?(f)$FFPOLY(GF5)
```

$$\texttt{false} \tag{3}$$

Boolean

Use normal? to test whether it is a normal polynomial.

```
normal?(f)$FFPOLY(GF5)
```

$$\texttt{false} \tag{4}$$

Boolean

Note that this is actually a trivial case, because a normal polynomial of degree n must have a nonzero term of degree $n-1$. We will refer back to this later.

To get a primitive polynomial of degree 8 just issue this.

```
p := createPrimitivePoly(8)$FFPOLY(GF5)
```

$$?^8 + ?^3 + ?^2 + ? + 2 \tag{5}$$

SparseUnivariatePolynomial(PrimeField(5))

```
primitive?(p)$FFPOLY(GF5)
```

$$\texttt{true} \tag{6}$$

Boolean

This polynomial is not normal,

```
normal?(p)$FFPOLY(GF5)
```

$$\texttt{false} \tag{7}$$

Boolean

but if you want a normal one simply write this.

```
n := createNormalPoly(8)$FFPOLY(GF5)
```

$$?^8 + 4?^7 + ?^3 + 1 \tag{8}$$

SparseUnivariatePolynomial(PrimeField(5))

This polynomial is not primitive!

```
primitive?(n)$FFPOLY(GF5)
```

$$\texttt{false} \tag{9}$$

Boolean

This could have been seen directly, as the constant term is 1 here, which is not a primitive element up to the factor (-1) raised to the degree of the polynomial.[4]

What about polynomials that are both primitive and normal? The existence of such a polynomial is by no means obvious. [5] If you really need one use either createPrimitiveNormalPoly or createNormalPrimitivePoly.

```
createPrimitiveNormalPoly(8)$FFPOLY(GF5)
```

$$?^8 + 4?^7 + 2?^5 + 2 \tag{10}$$

SparseUnivariatePolynomial(PrimeField(5))

If you want to obtain additional polynomials of the various types above as given by the **create...** operations above, you can use the **next...** operations. For instance, nextIrreduciblePoly yields the next monic irreducible polynomial with the same degree as the input polynomial. By "next" we mean "next

[4] Cf. Lidl, R. & Niederreiter, H., *Finite Fields,* Encycl. of Math. 20, (Addison-Wesley, 1983), p.90, Th. 3.18.

[5] The existence of such polynomials is proved in Lenstra, H. W. & Schoof, R. J., *Primitive Normal Bases for Finite Fields,* Math. Comp. 48, 1987, pp. 217-231.

in a natural order using the terms and coefficients." This will become more clear in the following examples.

This is the field with five elements.

```
GF5 := PF 5;
```

Type

Our first example irreducible polynomial, say of degree 3, must be "greater" than this.

```
h := monomial(1,8)$SUP(GF5)
```

$$?^8 \tag{12}$$

SparseUnivariatePolynomial(PrimeField(5))

You can generate it by doing this.

```
nh := nextIrreduciblePoly(h)$FFPOLY(GF5)
```

$$?^8 + 2 \tag{13}$$

Union(SparseUnivariatePolynomial(PrimeField(5)), ...)

Notice that this polynomial is not the same as the one createIrreduciblePoly.

```
createIrreduciblePoly(3)$FFPOLY(GF5)
```

$$?^3 + ? + 1 \tag{14}$$

SparseUnivariatePolynomial(PrimeField(5))

You can step through all irreducible polynomials of degree 8 over the field with 5 elements by repeatedly issuing this.

```
nh := nextIrreduciblePoly(nh)$FFPOLY(GF5)
```

$$?^8 + 3 \tag{15}$$

Union(SparseUnivariatePolynomial(PrimeField(5)), ...)

You could also ask for the total number of these.

```
numberOfIrreduciblePoly(5)$FFPOLY(GF5)
```

$$624 \tag{16}$$

PositiveInteger

We hope that "natural order" on polynomials is now clear: first we compare the number of monomials of two polynomials ("more" is "greater"); then, if necessary, the degrees of these monomials (lexicographically), and lastly their coefficients (also lexicographically, and using the operation lookup if our field is not a prime field). Also note that we make both polynomials monic before looking at the coefficients: multiplying either polynomial by a nonzero constant produces the same result.

The package **FiniteFieldPolynomialPackage** also provides similar operations for primitive and normal polynomials. With the exception of the number of primitive normal polynomials; we're not aware of any known formula for this.

```
numberOfPrimitivePoly(3)$FFPOLY(GF5)
```

$$20 \tag{17}$$

PositiveInteger

Take these,

```
m := monomial(1,1)$SUP(GF5)
```

$$? \tag{18}$$

SparseUnivariatePolynomial(PrimeField(5))

```
f := m^3 + 4*m^2 + m + 2
```

$$?^3 + 4?^2 + ? + 2 \tag{19}$$

SparseUnivariatePolynomial(PrimeField(5))

and then we have:

```
f1 := nextPrimitivePoly(f)$FFPOLY(GF5)
```

$$?^3 + 4?^2 + 4? + 2 \tag{20}$$

Union(SparseUnivariatePolynomial(PrimeField(5)), ...)

What happened?

```
nextPrimitivePoly(f1)$FFPOLY(GF5)
```

$$?^3 + 3? + 3 \tag{21}$$

Union(SparseUnivariatePolynomial(PrimeField(5)), ...)

Well, for the ordering used in nextPrimitivePoly we use as first criterion a comparison of the constant terms of the polynomials. Analogously, in nextNormalPoly we first compare the monomials of degree 1 less than the degree of the polynomials (which is nonzero, by an earlier remark).

```
f := m^3 + m^2 + 4*m + 1
```

$$?^3 + ?^2 + 4? + 1 \tag{22}$$

SparseUnivariatePolynomial(PrimeField(5))

```
f1 := nextNormalPoly(f)$FFPOLY(GF5)
```

$$?^3 + ?^2 + 4? + 3 \tag{23}$$

Union(SparseUnivariatePolynomial(PrimeField(5)), ...)

```
nextNormalPoly(f1)$FFPOLY(GF5)
```

$$?^3 + 2\,?^2 + 1 \tag{24}$$

Union(SparseUnivariatePolynomial(PrimeField(5)), ...)

We don't have to restrict ourselves to prime fields. Let's consider, say, a field with 16 elements.

```
GF16 := FFX(FFX(PF 2,2),2);
```

Type

We can apply any of the operations described above.

```
createIrreduciblePoly(5)$FFPOLY(GF16)
```

$$?^5 + ? + \%JU \tag{26}$$

SparseUnivariatePolynomial(FiniteFieldExtension (FiniteFieldExtension (PrimeField(2), 2), 2))

FriCAS also provides operations for producing random polynomials of a given degree

```
random(5)$FFPOLY(GF16)
```

$$?^5 + ((\%JU + 1)\ \%KH + \%JU)\ ?^4 + (\%JU\ \%KH + \%JU)\ ?^3 + (\%KH + \%JU)\ ?^2 + (\%KH + 1)\ ? + \%KH + 1 \tag{27}$$

SparseUnivariatePolynomial(FiniteFieldExtension (FiniteFieldExtension (PrimeField(2), 2), 2))

or with degree between two given bounds.

```
random(3,9)$FFPOLY(GF16)
```

$$\begin{aligned} &?^8 + ?^7 + \%KH\,?^6 + (\%JU\,\%KH + 1)\ ?^5 + (\%JU + 1)\ \%KH\,?^4 \\ &+ ?^3 + (\%KH + \%JU + 1)\ ? + (\%JU + 1)\ \%KH + \%JU \end{aligned} \tag{28}$$

SparseUnivariatePolynomial(FiniteFieldExtension (FiniteFieldExtension (PrimeField(2), 2), 2))

FiniteFieldPolynomialPackage2 (abbreviation **FFPOLY2**) exports an operation rootOfIrreduciblePoly for finding one root of an irreducible polynomial `f` in an extension field of the coefficient field. The degree of the extension has to be a multiple of the degree of `f`. It is not checked whether `f` actually is irreducible.

To illustrate this operation, we fix a ground field `GF`

```
GF2 := PrimeField 2;
```

Type

and then an extension field.

```
F := FFX(GF2,12)
```

$$\text{FiniteFieldExtension}(\text{PrimeField}(2), 12) \tag{30}$$

Type

We construct an irreducible polynomial over `GF2`.

```
f := createIrreduciblePoly(6)$FFPOLY(GF2)
```

$$?^6 + ? + 1 \tag{31}$$

SparseUnivariatePolynomial(PrimeField(2))

We compute a root of `f`.

```
root := rootOfIrreduciblePoly(f)$FFPOLY2(F,GF2)
```

$$\%JR^{11} + \%JR^8 + \%JR^7 + \%JR^5 + \%JR + 1 \tag{32}$$

FiniteFieldExtension(PrimeField(2), 12)

8.12 Primary Decomposition of Ideals

FriCAS provides a facility for the primary decomposition of polynomial ideals over fields of characteristic zero. The algorithm works in essentially two steps:

1. the problem is solved for 0-dimensional ideals by "generic" projection on the last coordinate
2. a "reduction process" uses localization and ideal quotients to reduce the general case to the 0-dimensional one.

The FriCAS constructor **PolynomialIdeal** represents ideals with coefficients in any field and supports the basic ideal operations, including intersection, sum and quotient. **IdealDecompositionPackage** contains the specific operations for the primary decomposition and the computation of the radical of an ideal with polynomial coefficients in a field of characteristic 0 with an effective algorithm for factoring polynomials.

The following examples illustrate the capabilities of this facility. First consider the ideal generated by x^2+y^2-1 (which defines a circle in the `(x,y)`-plane) and the ideal generated by x^2-y^2 (corresponding to the straight lines `x = y` and `x = -y`.

```
(n,m) : List DMP([x,y],FRAC INT)
```

```
m := [x^2+y^2-1]
```

$$\left[x^2 + y^2 - 1\right] \tag{2}$$

List(DistributedMultivariatePolynomial([x, y], Fraction(Integer)))

```
n := [x^2-y^2]
```

$$\left[x^2 - y^2\right] \tag{3}$$

List(DistributedMultivariatePolynomial([x, y], Fraction(Integer)))

We find the equations defining the intersection of the two loci. This correspond to the sum of the associated ideals.

```
id := ideal m  + ideal n
```

$$\left[x^2-\frac{1}{2}, y^2-\frac{1}{2}\right] \tag{4}$$

PolynomialIdeal(Fraction(Integer), DirectProduct(2, NonNegativeInteger), OrderedVariableList([x, y]), DistributedMultivariatePolynomial([x, y], Fraction(Integer)))

We can check if the locus contains only a finite number of points, that is, if the ideal is zero-dimensional.

```
zeroDim? id
```

$$\texttt{true} \tag{5}$$

Boolean

```
zeroDim?(ideal m)
```

$$\texttt{false} \tag{6}$$

Boolean

```
dimension ideal m
```

$$1 \tag{7}$$

PositiveInteger

We can find polynomial relations among the generators (`f` and `g` are the parametric equations of the knot).

```
(f,g):DMP([x,y],FRAC INT)
```

```
f := x^2-1
```

$$x^2-1 \tag{9}$$

DistributedMultivariatePolynomial([x, y], Fraction(Integer))

```
g := x*(x^2-1)
```

$$x^3-x \tag{10}$$

DistributedMultivariatePolynomial([x, y], Fraction(Integer))

```
relationsIdeal [f,g]
```

$$\left[-\%KJ^2+\%KI^3+\%KI^2\right] \mid \left[\%KI=x^2-1, \%KJ=x^3-x\right] \tag{11}$$

SuchThat(List(Polynomial(Fraction(Integer))), List(Equation(Polynomial(Fraction(Integer)))))

We can compute the primary decomposition of an ideal.

```
l: List DMP([x,y,z],FRAC INT) := [x^2+2*y^2,x*z^2-y*z,z^2-4]
```

$$\left[x^2+2\,y^2,\ x\,z^2-y\,z,\ z^2-4\right] \tag{12}$$

List (DistributedMultivariatePolynomial ([x, y, z], Fraction (Integer)))

```
ld:=primaryDecomp(ideal l)$IdealDecompositionPackage([x,y,z])
```

$$\left[\left[x+\frac{1}{2}\,y,\ y^2,\ z+2\right],\ \left[x-\frac{1}{2}\,y,\ y^2,\ z-2\right]\right] \tag{13}$$

List (PolynomialIdeal (Fraction (Integer), DirectProduct (3, NonNegativeInteger), OrderedVariableList ([x, y, z]), DistributedMultivariatePolynomial ([x, y, z], Fraction (Integer))))

We can intersect back.

```
reduce(intersect,ld)
```

$$\left[x-\frac{1}{4}\,y\,z,\ y^2,\ z^2-4\right] \tag{14}$$

PolynomialIdeal (Fraction (Integer), DirectProduct (3, NonNegativeInteger), OrderedVariableList ([x, y, z]), DistributedMultivariatePolynomial ([x, y, z], Fraction (Integer)))

We can compute the radical of every primary component.

```
reduce(intersect,[radical(ld.i)$IdealDecompositionPackage([x,y,z]) for i in 1..2])
```

$$\left[x,\ y,\ z^2-4\right] \tag{15}$$

PolynomialIdeal (Fraction (Integer), DirectProduct (3, NonNegativeInteger), OrderedVariableList ([x, y, z]), DistributedMultivariatePolynomial ([x, y, z], Fraction (Integer)))

Their intersection is equal to the radical of the ideal of `l`.

```
radical(ideal l)$IdealDecompositionPackage([x,y,z])
```

$$\left[x,\ y,\ z^2-4\right] \tag{16}$$

PolynomialIdeal (Fraction (Integer), DirectProduct (3, NonNegativeInteger), OrderedVariableList ([x, y, z]), DistributedMultivariatePolynomial ([x, y, z], Fraction (Integer)))

8.13 Computation of Galois Groups

As a sample use of FriCAS's algebraic number facilities, we compute the Galois group of the polynomial $p(x) = x^5 - 5x + 12$.

```
p := x^5 - 5*x + 12
```

$$x^5-5\,x+12 \tag{1}$$

Polynomial(Integer)

We would like to construct a polynomial $f(x)$ such that the splitting field of $p(x)$ is generated by one root of $f(x)$. First we construct a polynomial $r = r(x)$ such that one root of $r(x)$ generates the field generated by two roots of the polynomial $p(x)$. (As it will turn out, the field generated by two roots of $p(x)$ is, in fact, the splitting field of $p(x)$.)

From the proof of the primitive element theorem we know that if a and b are algebraic numbers, then the field $\mathbf{Q}(a, b)$ is equal to $\mathbf{Q}(a + kb)$ for an appropriately chosen integer k. In our case, we construct the minimal polynomial of $a_i - a_j$, where a_i and a_j are two roots of $p(x)$. We construct this polynomial using resultant. The main result we need is the following: If $f(x)$ is a polynomial with roots $a_i \ldots a_m$ and $g(x)$ is a polynomial with roots $b_i \ldots b_n$, then the polynomial `h(x) = resultant(f(y), g(x-y), y)` is a polynomial of degree $m * n$ with roots $a_i + b_j, i = 1 \ldots m, j = 1 \ldots n$.

For $f(x)$ we use the polynomial $p(x)$. For $g(x)$ we use the polynomial $-p(-x)$. Thus, the polynomial we first construct is `resultant(p(y), -p(y-x), y)`.

```
q := resultant(eval(p,x,y),-eval(p,x,y-x),y)
```

$$x^{25} - 50\,x^{21} - 2375\,x^{17} + 90000\,x^{15} - 5000\,x^{13} + 2700000\,x^{11} + 250000\,x^{9} + 18000000\,x^{7} + 64000000\,x^{5} \quad (2)$$

Polynomial(Integer)

The roots of $q(x)$ are $a_i - a_j, i \leq 1, j \leq 5$. Of course, there are five pairs (i, j) with $i = j$, so 0 is a 5-fold root of $q(x)$. Let's get rid of this factor.

```
q1 := exquo(q, x^5)
```

$$x^{20} - 50\,x^{16} - 2375\,x^{12} + 90000\,x^{10} - 5000\,x^{8} + 2700000\,x^{6} + 250000\,x^{4} + 18000000\,x^{2} + 64000000 \quad (3)$$

Union(Polynomial(Integer), ...)

Factor the polynomial q1.

```
factoredQ := factor q1
```

$$\left(x^{10} - 10\,x^{8} - 75\,x^{6} + 1500\,x^{4} - 5500\,x^{2} + 16000\right)\left(x^{10} + 10\,x^{8} + 125\,x^{6} + 500\,x^{4} + 2500\,x^{2} + 4000\right) \quad (4)$$

Factored(Polynomial(Integer))

We see that q1 has two irreducible factors, each of degree 10. (The fact that the polynomial q1 has two factors of degree 10 is enough to show that the Galois group of $p(x)$ is the dihedral group of order 10.[6] Note that the type of factoredQ is **FR POLY INT**, that is, **Factored Polynomial Integer**. This is a special data type for recording factorizations of polynomials with integer coefficients (see '`Factored`' on page 405). We can access the individual factors using the operation factorList.

```
r := factorList(factoredQ).1.factor
```

$$x^{10} - 10\,x^{8} - 75\,x^{6} + 1500\,x^{4} - 5500\,x^{2} + 16000 \quad (5)$$

[6] See McKay, Soicher, Computing Galois Groups over the Rationals, Journal of Number Theory 20, 273-281 (1983). We do not assume the results of this paper, however, and we continue with the computation.

Polynomial(Integer)

Consider the polynomial $r = r(x)$. This is the minimal polynomial of the difference of two roots of $p(x)$. Thus, the splitting field of $p(x)$ contains a subfield of degree 10. We show that this subfield is, in fact, the splitting field of $p(x)$ by showing that $p(x)$ factors completely over this field. First we create a symbolic root of the polynomial $r(x)$. (We replaced `x` by `b` in the polynomial `r` so that our symbolic root would be printed as `b`.)

```
beta:AN := rootOf(eval(r,x,b))
```

$$b \tag{6}$$

AlgebraicNumber

We next tell FriCAS to view $p(x)$ as a univariate polynomial in `x` with algebraic number coefficients. This is accomplished with this type declaration.

```
p := p::UP(x,INT)::UP(x,AN)
```

$$x^5 - 5\,x + 12 \tag{7}$$

UnivariatePolynomial(x, AlgebraicNumber)

Factor $p(x)$ over the field $\mathbf{Q}(\beta)$. (This computation will take some time!)

```
algFactors := factor(p,[beta])
```

$$\left(x + \frac{-85\,b^9 - 116\,b^8 + 780\,b^7 + 2640\,b^6 + 14895\,b^5 - 8820\,b^4 - 127050\,b^3 - 327000\,b^2 - 405200\,b + 2062400}{1339200}\right)\left(x + \frac{143\,b^8 - 2100\,b^6 - 10485\,b^4 + 290550\,b^2 + 334800\,b - 960800}{669600}\right)\left(x + \frac{143\,b^8 - 2100\,b^6 - 10485\,b^4 + 290550\,b^2 - 334800\,b - 960800}{669600}\right)\left(x + \frac{-17\,b^8 + 156\,b^6 + 2979\,b^4 - 25410\,b^2 - 14080}{66960}\right)\left(x + \frac{85\,b^9 - 116\,b^8 - 780\,b^7 + 2640\,b^6 - 14895\,b^5 - 8820\,b^4 + 127050\,b^3 - 327000\,b^2 + 405200\,b + 2062400}{1339200}\right) \tag{8}$$

Factored(UnivariatePolynomial(x, AlgebraicNumber))

When factoring over number fields, it is important to specify the field over which the polynomial is to be factored, as polynomials have different factorizations over different fields. When you use the operation factor, the field over which the polynomial is factored is the field generated by

1. the algebraic numbers that appear in the coefficients of the polynomial, and
2. the algebraic numbers that appear in a list passed as an optional second argument of the operation.

In our case, the coefficients of `p` are all rational integers and only `beta` appears in the list, so the field is simply $\mathbf{Q}(\beta)$. It was necessary to give the list `[beta]` as a second argument of the operation because

otherwise the polynomial would have been factored over the field generated by its coefficients, namely the rational numbers.

```
factor(p)
```

$$x^5 - 5\,x + 12 \tag{9}$$

Factored(UnivariatePolynomial (x, AlgebraicNumber))

We have shown that the splitting field of $p(x)$ has degree 10. Since the symmetric group of degree 5 has only one transitive subgroup of order 10, we know that the Galois group of $p(x)$ must be this group, the dihedral group of order 10. Rather than stop here, we explicitly compute the action of the Galois group on the roots of $p(x)$.

First we assign the roots of $p(x)$ as the values of five variables. We can obtain an individual root by negating the constant coefficient of one of the factors of $p(x)$.

```
factor1 := factorList(algFactors).1.factor
```

$$x + \frac{-85\,b^9 - 116\,b^8 + 780\,b^7 + 2640\,b^6 + 14895\,b^5 - 8820\,b^4 - 127050\,b^3 - 327000\,b^2 - 405200\,b + 2062400}{1339200} \tag{10}$$

UnivariatePolynomial (x, AlgebraicNumber)

```
root1 := -coefficient(factor1,0)
```

$$\frac{85\,b^9 + 116\,b^8 - 780\,b^7 - 2640\,b^6 - 14895\,b^5 + 8820\,b^4 + 127050\,b^3 + 327000\,b^2 + 405200\,b - 2062400}{1339200} \tag{11}$$

AlgebraicNumber

We can obtain a list of all the roots in this way.

```
roots := [-coefficient(j.factor, 0) for j in factorList(algFactors)]
```

$$\left[\frac{85\,b^9 + 116\,b^8 - 780\,b^7 - 2640\,b^6 - 14895\,b^5 + 8820\,b^4 + 127050\,b^3 + 327000\,b^2 + 405200\,b - 2062400}{1339200},\right.$$
$$\frac{-143\,b^8 + 2100\,b^6 + 10485\,b^4 - 290550\,b^2 - 334800\,b + 960800}{669600},$$
$$\frac{-143\,b^8 + 2100\,b^6 + 10485\,b^4 - 290550\,b^2 + 334800\,b + 960800}{669600}, \frac{17\,b^8 - 156\,b^6 - 2979\,b^4 + 25410\,b^2 + 14080}{66960},$$
$$\left.\frac{-85\,b^9 + 116\,b^8 + 780\,b^7 - 2640\,b^6 + 14895\,b^5 + 8820\,b^4 - 127050\,b^3 + 327000\,b^2 - 405200\,b - 2062400}{1339200}\right] \tag{12}$$

List (AlgebraicNumber)

The expression

```
- coefficient(j.factor, 0)
```

is the i^{th} root of $p(x)$ and the elements of `roots` are the i^{th} roots of $p(x)$ as `i` ranges from 1 to 5.

Assign the roots as the values of the variables `a1,...,a5`.

```
(a1,a2,a3,a4,a5) := (roots.1,roots.2,roots.3,roots.4,roots.5);
```

AlgebraicNumber

Next we express the roots of $r(x)$ as polynomials in beta. We could obtain these roots by calling the operation factor: factor(r, [beta]) factors r(x) over $\mathbf{Q}(\beta)$. However, this is a lengthy computation and we can obtain the roots of $r(x)$ as differences of the roots a1,...,a5 of $p(x)$. Only ten of these differences are roots of $r(x)$ and the other ten are roots of the other irreducible factor of q1. We can determine if a given value is a root of $r(x)$ by evaluating $r(x)$ at that particular value. (Of course, the order in which factors are returned by the operation factor is unimportant and may change with different implementations of the operation. Therefore, we cannot predict in advance which differences are roots of $r(x)$ and which are not.) Let's look at four examples (two are roots of $r(x)$ and two are not).

```
eval(r,x,a1 - a2)
```

$$0 \tag{14}$$

Polynomial(AlgebraicNumber)

```
eval(r,x,a1 - a3)
```

$$\frac{47905\,b^9 + 66920\,b^8 - 536100\,b^7 - 980400\,b^6 - 3345075\,b^5 - 5787000\,b^4 + 75572250\,b^3 + 161688000\,b^2 - 184600000\,b - 710912000}{4464} \tag{15}$$

Polynomial(AlgebraicNumber)

```
eval(r,x,a1 - a4)
```

$$0 \tag{16}$$

Polynomial(AlgebraicNumber)

```
eval(r,x,a1 - a5)
```

$$\frac{405\,b^8 + 3450\,b^6 - 19875\,b^4 - 198000\,b^2 - 588000}{31} \tag{17}$$

Polynomial(AlgebraicNumber)

Take one of the differences that was a root of $r(x)$ and assign it to the variable bb. For example, if eval(r,x,a1 - a4) returned 0, you would enter this.

```
bb := a1 - a4
```

$$\frac{85\,b^9 - 224\,b^8 - 780\,b^7 + 480\,b^6 - 14895\,b^5 + 68400\,b^4 + 127050\,b^3 - 181200\,b^2 + 405200\,b - 2344000}{1339200} \tag{18}$$

AlgebraicNumber

Of course, if the difference is, in fact, equal to the root beta, you should choose another root of $r(x)$.

Automorphisms of the splitting field are given by mapping a generator of the field, namely beta, to other roots of its minimal polynomial. Let's see what happens when beta is mapped to bb. We compute the images of the roots a1,...,a5 under this automorphism:

```
aa1 := subst(a1,beta = bb)
```

$$\frac{-85\,b^9+116\,b^8+780\,b^7-2640\,b^6+14895\,b^5+8820\,b^4-127050\,b^3+327000\,b^2-405200\,b-2062400}{1339200} \tag{19}$$

AlgebraicNumber

```
aa2 := subst(a2,beta = bb)
```

$$\frac{17\,b^8-156\,b^6-2979\,b^4+25410\,b^2+14080}{66960} \tag{20}$$

AlgebraicNumber

```
aa3 := subst(a3,beta = bb)
```

$$\frac{85\,b^9+116\,b^8-780\,b^7-2640\,b^6-14895\,b^5+8820\,b^4+127050\,b^3+327000\,b^2+405200\,b-2062400}{1339200} \tag{21}$$

AlgebraicNumber

```
aa4 := subst(a4,beta = bb)
```

$$\frac{-143\,b^8+2100\,b^6+10485\,b^4-290550\,b^2+334800\,b+960800}{669600} \tag{22}$$

AlgebraicNumber

```
aa5 := subst(a5,beta = bb)
```

$$\frac{-143\,b^8+2100\,b^6+10485\,b^4-290550\,b^2-334800\,b+960800}{669600} \tag{23}$$

AlgebraicNumber

Of course, the values `aa1,...,aa5` are simply a permutation of the values `a1,...,a5`. Let's find the value of `aa1` (execute as many of the following five commands as necessary).

```
(aa1 = a1) :: Boolean
```

```
false
```
(24)

Boolean

```
(aa1 = a2) :: Boolean
```

```
false
```
(25)

Boolean

```
(aa1 = a3) :: Boolean
```

```
false
```
(26)

Boolean

```
(aa1 = a4) :: Boolean
```

$$\texttt{false} \tag{27}$$

Boolean

```
(aa1 = a5) :: Boolean
```

$$\texttt{true} \tag{28}$$

Boolean

Proceeding in this fashion, you can find the values of `aa2,...aa5`.[7] You have represented the automorphism `beta` $\rightarrow$`bb` as a permutation of the roots `a1,...,a5`. If you wish, you can repeat this computation for all the roots of $r(x)$ and represent the Galois group of $p(x)$ as a subgroup of the symmetric group on five letters.

Here are two other problems that you may attack in a similar fashion:

1. Show that the Galois group of $p(x) = x^4 + 2x^3 - 2x^2 - 3x + 1$ is the dihedral group of order eight. (The splitting field of this polynomial is the Hilbert class field of the quadratic field $\mathbf{Q}(\sqrt{145})$.)
2. Show that the Galois group of $p(x) = x^6 + 108$ has order 6 and is isomorphic to S_3, the symmetric group on three letters. (The splitting field of this polynomial is the splitting field of $x^3 - 2$.)

8.14 Non-Associative Algebras and Modelling Genetic Laws

Many algebraic structures of mathematics and FriCAS have a multiplication operation * that satisfies the associativity law $a * (b * c) = (a * b) * c$ for all a, b and c. The octonions (see '`Octonion`' on page 515) are a well known exception. There are many other interesting non-associative structures, such as the class of Lie algebras.[8] Lie algebras can be used, for example, to analyse Lie symmetry algebras of partial differential equations. In this section we show a different application of non-associative algebras, the modelling of genetic laws.

The FriCAS library contains several constructors for creating non-associative structures, ranging from the categories **Monad**, **NonAssociativeRng**, and **FramedNonAssociativeAlgebra**, to the domains **AlgebraGivenByStructuralConstants** and **GenericNonAssociativeAlgebra**. Furthermore, the package **AlgebraPackage** provides operations for analysing the structure of such algebras.[9]

Mendel's genetic laws are often written in a form like

$$Aa \times Aa = \frac{1}{4}AA + \frac{1}{2}Aa + \frac{1}{4}aa.$$

The implementation of general algebras in FriCAS allows us to use this as the definition for multiplication in an algebra. Hence, it is possible to study questions of genetic inheritance using FriCAS. To demonstrate

[7] Here you should use the Clef line editor. See Section 1.1.1 on page 22 for more information about Clef.
[8] Two FriCAS implementations of Lie algebras are **LieSquareMatrix** and **FreeNilpotentLie**.
[9] The interested reader can learn more about these aspects of the FriCAS library from the paper "Computations in Algebras of Finite Rank," by Johannes Grabmeier and Robert Wisbauer, Technical Report, IBM Heidelberg Scientific Center, 1992.

this more precisely, we discuss one example from a monograph of A. Wörz-Busekros, where you can also find a general setting of this theory.[10]

We assume that there is an infinitely large random mating population. Random mating of two gametes a_i and a_j gives zygotes $a_i a_j$, which produce new gametes. In classical Mendelian segregation we have $a_i a_j = \frac{1}{2}a_i + \frac{1}{2}a_j$. In general, we have

$$a_i a_j = \sum_{k=1}^{n} \gamma_{i,j}^k \; a_k.$$

The segregation rates $\gamma_{i,j}$ are the structural constants of an n-dimensional algebra. This is provided in FriCAS by the constructor **AlgebraGivenByStructuralConstants** (abbreviation **ALGSC**).

Consider two coupled autosomal loci with alleles A,a, B, and b, building four different gametes $a_1 = AB, a_2 = Ab, a_3 = aB$, and $a_4 = ab$. The zygotes $a_i a_j$ produce gametes a_i and a_j with classical Mendelian segregation. Zygote $a_1 a_4$ undergoes transition to $a_2 a_3$ and vice versa with probability $0 \le \theta \le \frac{1}{2}$.

Define a list $[(\gamma_{i,j}^k)1 \le k \le 4]$ of four four-by-four matrices giving the segregation rates. We use the value $1/10$ for θ.

```
segregationRates : List SquareMatrix(4,FRAC INT) := [matrix [ [1, 1/2, 1/2, 9/20], _
    [1/2, 0, 1/20, 0], [1/2, 1/20, 0, 0], [9/20, 0, 0, 0] ], matrix [ [0, 1/2, 0, _
    1/20], [1/2, 1, 9/20, 1/2], [0, 9/20, 0, 0], [1/20, 1/2, 0, 0] ], matrix [ [0, 0, _
    1/2, 1/20], [0, 0, 9/20, 0], [1/2, 9/20, 1, 1/2], [1/20, 0, 1/2, 0] ], matrix [ _
    [0, 0, 0, 9/20], [0, 0, 1/20, 1/2], [0, 1/20, 0, 1/2], [9/20, 1/2, 1/2, 1] ] ]
```

$$\left[\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{9}{20} \\ \frac{1}{2} & 0 & \frac{1}{20} & 0 \\ \frac{1}{2} & \frac{1}{20} & 0 & 0 \\ \frac{9}{20} & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{20} \\ \frac{1}{2} & 1 & \frac{9}{20} & \frac{1}{2} \\ 0 & \frac{9}{20} & 0 & 0 \\ \frac{1}{20} & \frac{1}{2} & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{20} \\ 0 & 0 & \frac{9}{20} & 0 \\ \frac{1}{2} & \frac{9}{20} & 1 & \frac{1}{2} \\ \frac{1}{20} & 0 & \frac{1}{2} & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & \frac{9}{20} \\ 0 & 0 & \frac{1}{20} & \frac{1}{2} \\ 0 & \frac{1}{20} & 0 & \frac{1}{2} \\ \frac{9}{20} & \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}\right] \tag{1}$$

Type: List (SquareMatrix(4, Fraction (Integer)))

Choose the appropriate symbols for the basis of gametes,

```
gametes := ['AB,'Ab,'aB,'ab]
```

$$[AB,\ Ab,\ aB,\ ab] \tag{2}$$

Type: List (OrderedVariableList ([AB, Ab, aB, ab]))

Define the algebra.

```
A := ALGSC(FRAC INT, 4, gametes, segregationRates);
```

Type

What are the probabilities for zygote $a_1 a_4$ to produce the different gametes?

```
a := basis()$A;  a.1*a.4
```

$$\frac{9}{20}\,ab + \frac{1}{20}\,aB + \frac{1}{20}\,Ab + \frac{9}{20}\,AB \tag{4}$$

[10] Wörz-Busekros, A., *Algebras in Genetics*, Springer Lecture Notes in Biomathematics 36, Berlin e.a. (1980). In particular, see example 1.3.

AlgebraGivenByStructuralConstants(Fraction(Integer), 4, [AB, Ab, aB, ab], [[[1, 1/2, 1/2, 9/20], [1/2, 0, 1/20, 0], [1/2, 1/20, 0, 0], [9/20, 0, 0, 0]], [[0, 1/2, 0, 1/20], [1/2, 1, 9/20, 1/2], [0, 9/20, 0, 0], [1/20, 1/2, 0, 0]], [[0, 0, 1/2, 1/20], [0, 0, 9/20, 0], [1/2, 9/20, 1, 1/2], [1/20, 0, 1/2, 0]], [[0, 0, 0, 9/20], [0, 0, 1/20, 1/2], [0, 1/20, 0, 1/2], [9/20, 1/2, 1/2, 1]]])

Elements in this algebra whose coefficients sum to one play a distinguished role. They represent a population with the distribution of gametes reflected by the coefficients with respect to the basis of gametes.

Random mating of different populations x and y is described by their product $x * y$.

This product is commutative only if the gametes are not sex-dependent, as in our example.

```
commutative?()$A
```

$$\texttt{true} \tag{5}$$

Boolean

In general, it is not associative.

```
associative?()$A
```

$$\texttt{false} \tag{6}$$

Boolean

Random mating within a population x is described by $x * x$. The next generation is $(x * x) * (x * x)$.

Use decimal numbers to compare the distributions more easily.

```
x : ALGSC(DECIMAL, 4, gametes, segregationRates) :=  convert [3/10, 1/5, 1/10, 2/5]
```

$$0.4\,ab + 0.1\,aB + 0.2\,Ab + 0.3\,AB \tag{7}$$

AlgebraGivenByStructuralConstants(DecimalExpansion, 4, [AB, Ab, aB, ab], [[[1, 0.5, 0.5, 0.45], [0.5, 0, 0.05, 0], [0.5, 0.05, 0, 0], [0.45, 0, 0, 0]], [[0, 0.5, 0, 0.05], [0.5, 1, 0.45, 0.5], [0, 0.45, 0, 0], [0.05, 0.5, 0, 0]], [[0, 0, 0.5, 0.05], [0, 0, 0.45, 0], [0.5, 0.45, 1, 0.5], [0.05, 0, 0.5, 0]], [[0, 0, 0, 0.45], [0, 0, 0.05, 0.5], [0, 0.05, 0, 0.5], [0.45, 0.5, 0.5, 1]]])

To compute directly the gametic distribution in the fifth generation, we use plenaryPower.

```
plenaryPower(x,5)
```

$$0.36561\,ab + 0.13439\,aB + 0.23439\,Ab + 0.26561\,AB \tag{8}$$

AlgebraGivenByStructuralConstants(DecimalExpansion, 4, [AB, Ab, aB, ab], [[[1, 0.5, 0.5, 0.45], [0.5, 0, 0.05, 0], [0.5, 0.05, 0, 0], [0.45, 0, 0, 0]], [[0, 0.5, 0, 0.05], [0.5, 1, 0.45, 0.5], [0, 0.45, 0, 0], [0.05, 0.5, 0, 0]], [[0, 0, 0.5, 0.05], [0, 0, 0.45, 0], [0.5, 0.45, 1, 0.5], [0.05, 0, 0.5, 0]], [[0, 0, 0, 0.45], [0, 0, 0.05, 0.5], [0, 0.05, 0, 0.5], [0.45, 0.5, 0.5, 1]]])

We now ask two questions: Does this distribution converge to an equilibrium state? What are the distributions that are stable?

This is an invariant of the algebra and it is used to answer the first question. The new indeterminates describe a symbolic distribution.

```
q := leftRankPolynomial()$GCNAALG(FRAC INT, 4, gametes, segregationRates) :: UP(Y, _
    POLY FRAC INT)
```

$$\begin{aligned} & Y^3 + \left(-\frac{29}{20}\,\%x4 - \frac{29}{20}\,\%x3 - \frac{29}{20}\,\%x2 - \frac{29}{20}\,\%x1\right) Y^2 \\ & + \left(\frac{9}{20}\,\%x4^2 + \left(\frac{9}{10}\,\%x3 + \frac{9}{10}\,\%x2 + \frac{9}{10}\,\%x1\right)\%x4 + \frac{9}{20}\,\%x3^2\right. \\ & \left. + \left(\frac{9}{10}\,\%x2 + \frac{9}{10}\,\%x1\right)\%x3 + \frac{9}{20}\,\%x2^2 + \frac{9}{10}\,\%x1\,\%x2 + \frac{9}{20}\,\%x1^2\right) Y \end{aligned} \tag{9}$$

UnivariatePolynomial(Y, Polynomial(Fraction(Integer)))

Because the coefficient $\frac{9}{20}$ has absolute value less than 1, all distributions do converge, by a theorem of this theory.

```
factor(q :: POLY FRAC INT)
```

$$(Y - \%x4 - \%x3 - \%x2 - \%x1)\left(Y - \frac{9}{20}\,\%x4 - \frac{9}{20}\,\%x3 - \frac{9}{20}\,\%x2 - \frac{9}{20}\,\%x1\right) Y \tag{10}$$

Factored(Polynomial(Fraction(Integer)))

The second question is answered by searching for idempotents in the algebra.

```
cI := conditionsForIdempotents()$GCNAALG(FRAC INT, 4, gametes, segregationRates)
```

$$\begin{aligned} & \left[\frac{9}{10}\,\%x1\,\%x4 + \left(\frac{1}{10}\,\%x2 + \%x1\right)\%x3 + \%x1\,\%x2 + \%x1^2 - \%x1,\right. \\ & \left(\%x2 + \frac{1}{10}\,\%x1\right)\%x4 + \frac{9}{10}\,\%x2\,\%x3 + \%x2^2 + (\%x1 - 1)\,\%x2, \\ & \left(\%x3 + \frac{1}{10}\,\%x1\right)\%x4 + \%x3^2 + \left(\frac{9}{10}\,\%x2 + \%x1 - 1\right)\%x3, \\ & \left.\%x4^2 + \left(\%x3 + \%x2 + \frac{9}{10}\,\%x1 - 1\right)\%x4 + \frac{1}{10}\,\%x2\,\%x3\right] \end{aligned} \tag{11}$$

List(Polynomial(Fraction(Integer)))

Solve these equations and look at the first solution.

```
gbs:= groebnerFactorize cI; gbs.1
```

$$\left[\%x4 + \%x3 + \%x2 + \%x1 - 1,\ (\%x2 + \%x1)\,\%x3 + \%x1\,\%x2 + \%x1^2 - \%x1\right] \tag{12}$$

List(Polynomial(Fraction(Integer)))

Further analysis using the package **PolynomialIdeal** shows that there is a two-dimensional variety of equilibrium states and all other solutions are contained in it.

Choose one equilibrium state by setting two indeterminates to concrete values.

```
sol := solve concat(gbs.1,[%x1-1/10,%x2-1/10])
```

$$\left[\left[\%x4 = \frac{2}{5}, \%x3 = \frac{2}{5}, \%x2 = \frac{1}{10}, \%x1 = \frac{1}{10}\right]\right] \tag{13}$$

List (List (Equation(Fraction(Polynomial(Integer)))))

```
e : A := represents reverse (map(rhs, sol.1) :: List FRAC INT)
```

$$\frac{2}{5}\,ab + \frac{2}{5}\,aB + \frac{1}{10}\,Ab + \frac{1}{10}\,AB \tag{14}$$

AlgebraGivenByStructuralConstants(Fraction(Integer), 4, [AB, Ab, aB, ab], [[[1, 1/2, 1/2, 9/20], [1/2, 0, 1/20, 0], [1/2, 1/20, 0, 0], [9/20, 0, 0, 0]], [[0, 1/2, 0, 1/20], [1/2, 1, 9/20, 1/2], [0, 9/20, 0, 0], [1/20, 1/2, 0, 0]], [[0, 0, 1/2, 1/20], [0, 0, 9/20, 0], [1/2, 9/20, 1, 1/2], [1/20, 0, 1/2, 0]], [[0, 0, 0, 9/20], [0, 0, 1/20, 1/2], [0, 1/20, 0, 1/2], [9/20, 1/2, 1/2, 1]]])

Verify the result.

```
e*e-e
```

$$0 \tag{15}$$

AlgebraGivenByStructuralConstants(Fraction(Integer), 4, [AB, Ab, aB, ab], [[[1, 1/2, 1/2, 9/20], [1/2, 0, 1/20, 0], [1/2, 1/20, 0, 0], [9/20, 0, 0, 0]], [[0, 1/2, 0, 1/20], [1/2, 1, 9/20, 1/2], [0, 9/20, 0, 0], [1/20, 1/2, 0, 0]], [[0, 0, 1/2, 1/20], [0, 0, 9/20, 0], [1/2, 9/20, 1, 1/2], [1/20, 0, 1/2, 0]], [[0, 0, 0, 9/20], [0, 0, 1/20, 1/2], [0, 1/20, 0, 1/2], [9/20, 1/2, 1/2, 1]]])

8.15 Matrix Manipulation

This section shows some examples on selecting various (rectangular) submatrices of matrices. In the numerics literature, these operations are usually referred to as slicing. Apart from indexing matrices by two integers for retrieving single elements, it is possible to use lists of integers (**List(Integer)**), segments (**Segment(Integer)**) and list of segments (**List(Segment(Integer))**) to select slices like whole rows, columns or submatrices.

First, we build a simple test matrix to show the above-mentioned manipulations:

```
m := matrix([[11, 12, 13, 14], [21, 22,23, 24], [31, 32, 33, 34]])
```

$$\begin{bmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \end{bmatrix} \tag{1}$$

Matrix(Integer)

Select the top right two by two submatrix by slicing using segments:

```
m(1..2, 3..4)
```

$$\begin{bmatrix} 13 & 14 \\ 23 & 24 \end{bmatrix} \tag{2}$$

Matrix(Integer)

Having a nonzero step size in the segment is also supported:

```
m(1..2, 1..3 by 2)
```

$$\begin{bmatrix} 11 & 13 \\ 21 & 23 \end{bmatrix} \tag{3}$$

Matrix(Integer)

Indexing by lists works as expected, returning all elements having index pairs from the outer product of both lists:

```
m([1,3], [2,4])
```

$$\begin{bmatrix} 12 & 14 \\ 32 & 34 \end{bmatrix} \tag{4}$$

Matrix(Integer)

Selecting single elements by any index type other than **Integer** for both, the row and column index, will not give the respective element but a 1 times 1 matrix containing it:

```
m(1, 2)
```

$$12 \tag{5}$$

PositiveInteger

```
m([1], [2])
```

$$\begin{bmatrix} 12 \end{bmatrix} \tag{6}$$

Matrix(Integer)

```
m(1, [2])
```

$$\begin{bmatrix} 12 \end{bmatrix} \tag{7}$$

Matrix(Integer)

```
m(1, 2..2)
```

$$\begin{bmatrix} 12 \end{bmatrix} \tag{8}$$

Matrix(Integer)

```
m(1, [2..2])
```

$$\begin{bmatrix} 12 \end{bmatrix} \tag{9}$$

Matrix(Integer)

It is possible to use lists of segments to select multiple submatrices which get stacked together forming the result returned:

```
m([1..2], [1, 3..4])
```

$$\begin{bmatrix} 11 & 13 & 14 \\ 21 & 23 & 24 \end{bmatrix} \tag{10}$$

Matrix(Integer)

Use overlapping segments to repeat elements:

```
m([1..2], [3..4, 3..4])
```

$$\begin{bmatrix} 13 & 14 & 13 & 14 \\ 23 & 24 & 23 & 24 \end{bmatrix} \tag{11}$$

Matrix(Integer)

It is even possible to mix any of the valid index constructs in the selection of rows and columns:

```
m(2, [1,4])
```

$$\begin{bmatrix} 21 & 24 \end{bmatrix} \tag{12}$$

Matrix(Integer)

```
m([1,2,3], 2..3)
```

$$\begin{bmatrix} 12 & 13 \\ 22 & 23 \\ 32 & 33 \end{bmatrix} \tag{13}$$

Matrix(Integer)

```
m([1,2], [1..3, 4])
```

$$\begin{bmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \end{bmatrix} \tag{14}$$

Matrix(Integer)

Assignment to a submatrix using slicing syntax is supported, too:

```
m([1..2], [3..3]) := m([1,2], [2])
```

$$\begin{bmatrix} 12 \\ 22 \end{bmatrix} \tag{15}$$

Matrix(Integer)

```
m
```

$$\begin{bmatrix} 11 & 12 & 12 & 14 \\ 21 & 22 & 22 & 24 \\ 31 & 32 & 33 & 34 \end{bmatrix} \tag{16}$$

Matrix(Integer)

Note that assignment currently does not check for overlapping segments, and the last assignment wins. However, overlapping case should be considered undefined anyway.

Another caveat shows up when assigning single elements.

Selecting the entry by any index type other than two times an **Integer** requires assignment of a matrix type:

```
m([2], [2]) := matrix([[4]])
```

$$\begin{bmatrix} 4 \end{bmatrix} \tag{17}$$

Matrix(Integer)

```
m
```

$$\begin{bmatrix} 11 & 12 & 12 & 14 \\ 21 & 4 & 22 & 24 \\ 31 & 32 & 33 & 34 \end{bmatrix} \tag{18}$$

Matrix(Integer)

By using the functions rowSlice and colSlice it is possible to obtain for a given matrix two special slicing objects that when used will select all elements along a column or row respectively (rowSlice varies row index giving a column). The advantage of using these is that no information about the actual matrix size is necessary.

It is easily possible to select the second and fourth columns of a given matrix:

```
r := rowSlice(m)
```

$$1..3 \tag{19}$$

Segment(Integer)

```
m(r, 2)
```

$$\begin{bmatrix} 12 \\ 4 \\ 32 \end{bmatrix} \tag{20}$$

Matrix(Integer)

```
m(r, 4)
```

$$\begin{bmatrix} 14 \\ 24 \\ 34 \end{bmatrix} \tag{21}$$

Matrix(Integer)

Assignment of course works the same way. The following snippet shows simple row operations as used in Gaussian elimination:

```
c := colSlice(m)
```

$$1..4 \tag{22}$$

Segment(Integer)

```
m := m :: Matrix(Fraction(Integer))
```

$$\begin{bmatrix} 11 & 12 & 12 & 14 \\ 21 & 4 & 22 & 24 \\ 31 & 32 & 33 & 34 \end{bmatrix} \tag{23}$$

Matrix(Fraction (Integer))

```
m(2, c) := m(2, c) - m(2,1)/m(1,1) * m(1, c)
```

$$\begin{bmatrix} 0 & -\frac{208}{11} & -\frac{10}{11} & -\frac{30}{11} \end{bmatrix} \tag{24}$$

Matrix(Fraction (Integer))

```
m(3, c) := m(3, c) - m(3,1)/m(1,1) * m(1, c)
```

$$\begin{bmatrix} 0 & -\frac{20}{11} & -\frac{9}{11} & -\frac{60}{11} \end{bmatrix} \tag{25}$$

Matrix(Fraction (Integer))

```
m(3, c) := m(3, c) - m(3,2)/m(2,2) * m(2, c)
```

$$\begin{bmatrix} 0 & 0 & -\frac{19}{26} & -\frac{135}{26} \end{bmatrix} \tag{26}$$

Matrix(Fraction (Integer))

```
m
```

$$\begin{bmatrix} 11 & 12 & 12 & 14 \\ 0 & -\frac{208}{11} & -\frac{10}{11} & -\frac{30}{11} \\ 0 & 0 & -\frac{19}{26} & -\frac{135}{26} \end{bmatrix} \tag{27}$$

Matrix(Fraction (Integer))

Selecting the whole matrix:

```
r := rowSlice(m)
```

$$1..3 \tag{28}$$

Segment(Integer)

```
c := colSlice(m)
```

$$1\,..\,4 \tag{29}$$

Segment(Integer)

```
m(r,c)
```

$$\begin{bmatrix} 11 & 12 & 12 & 14 \\ 0 & -\frac{208}{11} & -\frac{10}{11} & -\frac{30}{11} \\ 0 & 0 & -\frac{19}{26} & -\frac{135}{26} \end{bmatrix} \tag{30}$$

Matrix(Fraction (Integer))

Chapter 9

Some Examples of Domains and Packages

In this chapter we show examples of many of the most commonly used FriCAS domains and packages. The sections are organized by constructor names.

9.1 AssociationList

The **AssociationList** constructor provides a general structure for associative storage. This type provides association lists in which data objects can be saved according to keys of any type. For a given association list, specific types must be chosen for the keys and entries. You can think of the representation of an association list as a list of records with key and entry fields.

Association lists are a form of table and so most of the operations available for **Table** are also available for **AssociationList**. They can also be viewed as lists and can be manipulated accordingly.

This is a **Record** type with age and gender fields.

```
Data := Record(monthsOld : Integer, gender : String)
```

$$\text{Record}(monthsOld : Integer, gender : String) \tag{4}$$

Type

In this expression, `al` is declared to be an association list whose keys are strings and whose entries are the above records.

```
al : AssociationList(String,Data)
```

The table operation is used to create an empty association list.

```
al := table()
```

$$\texttt{table}() \tag{6}$$

AssociationList (String , Record(monthsOld: Integer, gender: String))

You can use assignment syntax to add things to the association list.

```
al."bob" := [407,"male"]$Data
```

$$[monthsOld = 407,\ gender = \texttt{"male"}] \tag{7}$$

Record(monthsOld: Integer, gender: String)

```
al."judith" := [366,"female"]$Data
```

$$[monthsOld = 366,\ gender = \texttt{"female"}] \tag{8}$$

Record(monthsOld: Integer, gender: String)

```
al."katie" := [24,"female"]$Data
```

$$[monthsOld = 24,\ gender = \texttt{"female"}] \tag{9}$$

Record(monthsOld: Integer, gender: String)

Perhaps we should have included a species field.

```
al."smokie" := [200,"female"]$Data
```

$$[monthsOld = 200,\ gender = \texttt{"female"}] \tag{10}$$

Record(monthsOld: Integer, gender: String)

Now look at what is in the association list. Note that the last-added (key, entry) pair is at the beginning of the list.

```
al
```

$$\begin{aligned}&\texttt{table}(\texttt{"smokie"} = [monthsOld = 200,\ gender = \texttt{"female"}], \texttt{"katie"} = [monthsOld = 24,\\ &\quad gender = \texttt{"female"}], \texttt{"judith"} = [monthsOld = 366,\\ &\quad gender = \texttt{"female"}], \texttt{"bob"} = [monthsOld = 407,\ gender = \texttt{"male"}])\end{aligned} \tag{11}$$

AssociationList (String , Record(monthsOld: Integer, gender: String))

You can reset the entry for an existing key.

```
al."katie" := [23,"female"]$Data
```

$$[monthsOld = 23,\ gender = \texttt{"female"}] \tag{12}$$

Record(monthsOld: Integer, gender: String)

Use delete! to destructively remove an element of the association list. Use delete to return a copy of the association list with the element deleted. The second argument is the index of the element to delete.

```
delete!(al,1)
```

$$\texttt{table}(\texttt{"katie"} = [monthsOld = 23,\ gender = \texttt{"female"}], \texttt{"judith"} = [monthsOld = 366,\ gender = \texttt{"female"}], \texttt{"bob"} = [monthsOld = 407,\ gender = \texttt{"male"}]) \tag{13}$$

AssociationList (String , Record(monthsOld: Integer, gender: String))

For more information about tables, see '`Table`' on page 587. For more information about lists, see '`List`' on page 490. Issue the system command `)show AssociationList` to display the full list of operations defined by **AssociationList**.

9.2 BalancedBinaryTree

BalancedBinaryTree(S) is the domain of balanced binary trees with elements of type `S` at the nodes. A binary tree is either empty or else consists of a node having a value and two branches, each branch a binary tree. A balanced binary tree is one that is balanced with respect to its leaves. One with 2^k leaves is perfectly "balanced": the tree has minimum depth, and the left and right branch of every interior node is identical in shape.

Balanced binary trees are useful in algebraic computation for so-called "divide-and-conquer" algorithms. Conceptually, the data for a problem is initially placed at the root of the tree. The original data is then split into two subproblems, one for each subtree. And so on. Eventually, the problem is solved at the leaves of the tree. A solution to the original problem is obtained by some mechanism that can reassemble the pieces. In fact, an implementation of the Chinese Remainder Algorithm using balanced binary trees was first proposed by David Y. Y. Yun at the IBM T. J. Watson Research Center in Yorktown Heights, New York, in 1978. It served as the prototype for polymorphic algorithms in FriCAS.

In what follows, rather than perform a series of computations with a single expression, the expression is reduced modulo a number of integer primes, a computation is done with modular arithmetic for each prime, and the Chinese Remainder Algorithm is used to obtain the answer to the original problem. We illustrate this principle with the computation of $12^2 = 144$.

A list of moduli.

```
lm := [3,5,7,11]
```

$$[3,\ 5,\ 7,\ 11] \tag{4}$$

List (PositiveInteger)

The expression `modTree(n, lm)` creates a balanced binary tree with leaf values `n mod m` for each modulus `m` in `lm`.

```
modTree(12,lm)
```

$$[0,\ 2,\ 5,\ 1] \tag{5}$$

List (Integer)

Operation `modTree` does this using operations on balanced binary trees. We trace its steps. Create a balanced binary tree `t` of zeros with four leaves.

```
t := balancedBinaryTree(#lm, 0)
```

$$[[0,\ 0,\ 0]\ ,\ 0,\ [0,\ 0,\ 0]] \tag{6}$$

BalancedBinaryTree(NonNegativeInteger)

The leaves of the tree are set to the individual moduli.

```
setleaves!(t,lm)
```

$$[[3,\ 0,\ 5]\ ,\ 0,\ [7,\ 0,\ 11]] \tag{7}$$

BalancedBinaryTree(NonNegativeInteger)

Use mapUp! to do a bottom-up traversal of t, setting each interior node to the product of the values at the nodes of its children.

```
mapUp!(t,_*)
```

$$1155 \tag{8}$$

PositiveInteger

The value at the node of every subtree is the product of the moduli of the leaves of the subtree.

```
t
```

$$[[3,\ 15,\ 5]\ ,\ 1155,\ [7,\ 77,\ 11]] \tag{9}$$

BalancedBinaryTree(NonNegativeInteger)

Operation mapDown!(t,a,fn) replaces the value v at each node of t by fn(a,v).

```
mapDown!(t,12,_rem)
```

$$[[0,\ 12,\ 2]\ ,\ 12,\ [5,\ 12,\ 1]] \tag{10}$$

BalancedBinaryTree(NonNegativeInteger)

The operation leaves returns the leaves of the resulting tree. In this case, it returns the list of 12 mod m for each modulus m.

```
leaves %
```

$$[0,\ 2,\ 5,\ 1] \tag{11}$$

List (NonNegativeInteger)

Compute the square of the images of 12 modulo each m.

```
squares := [x^2 rem m for x in % for m in lm]
```

$$[0,\ 4,\ 4,\ 1] \tag{12}$$

List (NonNegativeInteger)

Call the Chinese Remainder Algorithm to get the answer for 12^2.

```
chineseRemainder(%,lm)
```

$$144 \tag{13}$$

PositiveInteger

9.3 BasicOperator

A basic operator is an object that can be symbolically applied to a list of arguments from a set, the result being a kernel over that set or an expression. In addition to this section, please see '`Expression`' on page 400 and '`Kernel`' on page 454 for additional information and examples.

You create an object of type **BasicOperator** by using the operator operation. This first form of this operation has one argument and it must be a symbol. The symbol should be quoted in case the name has been used as an identifier to which a value has been assigned.

A frequent application of **BasicOperator** is the creation of an operator to represent the unknown function when solving a differential equation. Let `y` be the unknown function in terms of `x`.

```
y := operator 'y
```

$$y \tag{4}$$

BasicOperator

This is how you enter the equation `y'' + y' + y = 0`.

```
deq := D(y x, x, 2) + D(y x, x) + y x = 0
```

$$y''(x) + y'(x) + y(x) = 0 \tag{5}$$

Equation(Expression(Integer))

To solve the above equation, enter this.

```
solve(deq, y, x)
```

$$\left[particular = 0,\ basis = \left[\cos\left(\frac{x\sqrt{3}}{2}\right) e^{-\frac{x}{2}},\ e^{-\frac{x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right)\right]\right] \tag{6}$$

Union(Record(particular : Expression(Integer), basis : List(Expression(Integer))), ...)

See Section 8.10 on page 296 for this kind of use of **BasicOperator**.

Use the single argument form of operator (as above) when you intend to use the operator to create functional expressions with an arbitrary number of arguments *Nary* means an arbitrary number of arguments can be used in the functional expressions.

```
nary? y
```

$$\texttt{true} \tag{7}$$

Boolean

```
unary? y
```

false (8)

Boolean

Use the two-argument form when you want to restrict the number of arguments in the functional expressions created with the operator. This operator can only be used to create functional expressions with one argument.

```
opOne := operator('opOne, 1)
```

opOne (9)

BasicOperator

```
nary? opOne
```

false (10)

Boolean

```
unary? opOne
```

true (11)

Boolean

Use arity to learn the number of arguments that can be used. It returns `"false"` if the operator is nary.

```
arity opOne
```

1 (12)

Union(NonNegativeInteger, ...)

Use name to learn the name of an operator.

```
name opOne
```

opOne (13)

Symbol

Use is? to learn if an operator has a particular name.

```
is?(opOne, 'z2)
```

false (14)

Boolean

You can also use a string as the name to be tested against.

```
is?(opOne, "opOne")
```

$$\texttt{true} \tag{15}$$

Boolean

You can attach named properties to an operator. These are rarely used at the top-level of the FriCAS interactive environment but are used with FriCAS library source code. By default, an operator has no properties.

```
properties y
```

$$\texttt{table}() \tag{16}$$

AssociationList (Symbol, None)

The interface for setting and getting properties is somewhat awkward because the property values are stored as values of type **None**. Attach a property by using setProperty.

```
setProperty(y, "use", "unknown function" :: None )
```

$$y \tag{17}$$

BasicOperator

```
properties y
```

$$\texttt{table}(use = \texttt{NONE}) \tag{18}$$

AssociationList (Symbol, None)

We *know* the property value has type **String**.

```
property(y, "use") :: None pretend String
```

$$\texttt{"unknown function"} \tag{19}$$

String

Use deleteProperty! to destructively remove a property.

```
deleteProperty!(y, "use")
```

$$y \tag{20}$$

BasicOperator

```
properties y
```

$$\texttt{table}() \tag{21}$$

AssociationList (Symbol, None)

9.4 BinaryExpansion

All rational numbers have repeating binary expansions. Operations to access the individual bits of a binary expansion can be obtained by converting the value to **RadixExpansion(2)**. More examples of expansions are available in '`DecimalExpansion`' on page 392, '`HexadecimalExpansion`' on page 440, and '`RadixExpansion`' on page 541.

The expansion (of type **BinaryExpansion**) of a rational number is returned by the binary operation.

```
r := binary(22/7)
```

$$11.\overline{001} \tag{4}$$

BinaryExpansion

Arithmetic is exact.

```
r + binary(6/7)
```

$$100 \tag{5}$$

BinaryExpansion

The period of the expansion can be short or long ...

```
[binary(1/i) for i in 102..106]
```

$$\left[0.0\overline{00000101},\ 0.\overline{0000001001111100010001011001011110011100100101010 01},\ 0.000\overline{000100111011},\right. \\ \left. 0.\overline{000000100111},\ 0.\overline{0000001001101010010000111001111101100101011011110 0011}\right] \tag{6}$$

List (BinaryExpansion)

or very long.

```
binary(1/1007)
```

$$0.\overline{0000000001000001000101001001011110000011111100001011111100101100011111010001001110010011001100110010010101011110} \tag{7}$$

BinaryExpansion

These numbers are bona fide algebraic objects.

```
p := binary(1/4)*x^2 + binary(2/3)*x + binary(4/9)
```

$$0.01\,x^2 + 0.\overline{10}\,x + 0.\overline{011100} \tag{8}$$

Polynomial(BinaryExpansion)

```
q := D(p, x)
```

$$0.1\,x + 0.\overline{10} \tag{9}$$

Polynomial(BinaryExpansion)

```
g := gcd(p, q)
```

$$x + 1.\overline{01} \tag{10}$$

Polynomial(BinaryExpansion)

9.5 BinarySearchTree

BinarySearchTree(R) is the domain of binary trees with elements of type R, ordered across the nodes of the tree. A non-empty binary search tree has a value of type R, and right and left binary search subtrees. If a subtree is empty, it is displayed as a period (".").

Define a list of values to be placed across the tree. The resulting tree has 8 at the root; all other elements are in the left subtree.

```
lv := [8,3,5,4,6,2,1,5,7]
```

$$[8,\ 3,\ 5,\ 4,\ 6,\ 2,\ 1,\ 5,\ 7] \tag{4}$$

List (PositiveInteger)

A convenient way to create a binary search tree is to apply the operation binarySearchTree to a list of elements.

```
t := binarySearchTree lv
```

$$[[[1,\ 2,\ .]\ ,\ 3,\ [4,\ 5,\ [5,\ 6,\ 7]]]\ ,\ 8,\ .] \tag{5}$$

BinarySearchTree(PositiveInteger)

Another approach is to first create an empty binary search tree of integers.

```
emptybst := empty()$BSTREE(INT)
```

$$[] \tag{6}$$

BinarySearchTree(Integer)

Insert the value 8. This establishes 8 as the root of the binary search tree. Values inserted later that are less than 8 get stored in the left subtree, others in the right subtree.

```
t1 := insert!(8,emptybst)
```

$$8 \tag{7}$$

BinarySearchTree(Integer)

Insert the value 3. This number becomes the root of the left subtree of t1. For optimal retrieval, it is thus important to insert the middle elements first.

```
insert!(3,t1)
```

$$[3,\ 8,\ .] \tag{8}$$

BinarySearchTree(Integer)

We go back to the original tree `t`. The leaves of the binary search tree are those which have empty left and right subtrees.

```
leaves t
```

$$[1,\ 4,\ 5,\ 7] \tag{9}$$

List (PositiveInteger)

The operation split(k,t) returns a *record* containing the two subtrees: one with all elements "less" than `k`, another with elements "greater" than `k`.

```
split(3,t)
```

$$[less = [1,\ 2,\ .]\ ,\ greater = [[.,\ 3,\ [4,\ 5,\ [5,\ 6,\ 7]]]\ ,\ 8,\ .]] \tag{10}$$

Record(less : BinarySearchTree(PositiveInteger), greater : BinarySearchTree(PositiveInteger))

Define **insertRoot** to insert new elements by creating a new node.

```
insertRoot: (INT,BSTREE INT) -> BSTREE INT
```

The new node puts the inserted value between its "less" tree and "greater" tree.

```
insertRoot(x, t) ==
    a := split(x, t)
    node(a.less, x, a.greater)
```

Function **buildFromRoot** builds a binary search tree from a list of elements `ls` and the empty tree `emptybst`.

```
buildFromRoot ls == reduce(insertRoot,ls,emptybst)
```

Apply this to the reverse of the list `lv`.

```
rt := buildFromRoot reverse lv
```

```
   Compiling function buildFromRoot with type List(PositiveInteger) ->
      BinarySearchTree(Integer)
   Compiling function insertRoot with type (Integer, BinarySearchTree(Integer))
       -> BinarySearchTree(Integer)
```

$$[[[1,\ 2,\ .]\ ,\ 3,\ [4,\ 5,\ [5,\ 6,\ 7]]]\ ,\ 8,\ .] \tag{14}$$

BinarySearchTree(Integer)

Have FriCAS check that these are equal.

```
(t = rt)@Boolean
```

$$\texttt{true} \tag{15}$$

Boolean

9.6 CardinalNumber

The **CardinalNumber** domain can be used for values indicating the cardinality of sets, both finite and infinite.

The non-negative integers have a natural construction as cardinals

```
0 = #{ }, 1 = {0}, 2 = {0, 1}, ..., n = {i | 0 <= i < n}.
```

The fact that `0` acts as a zero for the multiplication of cardinals is equivalent to the axiom of choice.

Cardinal numbers can be created by conversion from non-negative integers.

```
c0 := 0 :: CardinalNumber
```

$$0 \tag{4}$$

CardinalNumber

```
c1 := 1 :: CardinalNumber
```

$$1 \tag{5}$$

CardinalNumber

```
c2 := 2 :: CardinalNumber
```

$$2 \tag{6}$$

CardinalNumber

```
c3 := 3 :: CardinalNumber
```

$$3 \tag{7}$$

CardinalNumber

They can also be obtained as the named cardinal `Aleph(n)`.

```
A0 := Aleph 0
```

$$\aleph_0 \tag{8}$$

CardinalNumber

```
A1 := Aleph 1
```

$$\aleph_1 \tag{9}$$

CardinalNumber

The finite? operation tests whether a value is a finite cardinal, that is, a non-negative integer.

```
finite? c2
```

$$\texttt{true} \tag{10}$$

Boolean

```
finite? A0
```

$$\texttt{false} \tag{11}$$

Boolean

Similarly, the countable? operation determines whether a value is a countable cardinal, that is, finite or `Aleph(0)`.

```
countable? c2
```

$$\texttt{true} \tag{12}$$

Boolean

```
countable? A0
```

$$\texttt{true} \tag{13}$$

Boolean

```
countable? A1
```

$$\texttt{false} \tag{14}$$

Boolean

Arithmetic operations are defined on cardinal numbers as follows: If `x = #X` and `y = #Y` then

`x+y = #(X+Y)` cardinality of the disjoint union
`x-y = #(X-Y)` cardinality of the relative complement
`x*y = #(X*Y)` cardinality of the Cartesian product
`x^y = #(X^Y)` cardinality of the set of maps from `Y` to `X`

Here are some arithmetic examples.

```
[c2 + c2, c2 + A1]
```

$$[4,\ \aleph_1] \tag{15}$$

List (CardinalNumber)

```
[c0*c2, c1*c2, c2*c2, c0*A1, c1*A1, c2*A1, A0*A1]
```

$$[0,\ 2,\ 4,\ 0,\ \aleph_1,\ \aleph_1,\ \aleph_1] \tag{16}$$

List (CardinalNumber)

```
[c2^c0, c2^c1, c2^c2, A1^c0, A1^c1, A1^c2]
```

$$[1,\ 2,\ 4,\ 1,\ \aleph_1,\ \aleph_1] \tag{17}$$

List (CardinalNumber)

Subtraction is a partial operation: it is not defined when subtracting a larger cardinal from a smaller one, nor when subtracting two equal infinite cardinals.

```
[c2-c1, c2-c2, c2-c3, A1-c2, A1-A0, A1-A1]
```

$$[1,\ 0,\ \texttt{"failed"},\ \aleph_1,\ \aleph_1,\ \texttt{"failed"}] \tag{18}$$

List (Union(CardinalNumber, "failed"))

The generalized continuum hypothesis asserts that

```
2^Aleph i = Aleph(i+1)
```

and is independent of the axioms of set theory.[1] The **CardinalNumber** domain provides an operation to assert whether the hypothesis is to be assumed.

```
generalizedContinuumHypothesisAssumed true
```

$$\texttt{true} \tag{19}$$

Boolean

When the generalized continuum hypothesis is assumed, exponentiation to a transfinite power is allowed.

```
[c0^A0, c1^A0, c2^A0, A0^A0, A0^A1, A1^A0, A1^A1]
```

$$[0,\ 1,\ \aleph_1,\ \aleph_1,\ \aleph_2,\ \aleph_1,\ \aleph_2] \tag{20}$$

List (CardinalNumber)

Three commonly encountered cardinal numbers are

$a = \#\mathbf{Z}$	countable infinity
$c = \#\mathbf{R}$	the continuum
$f = \#\{g \mid g : [0,1] \to \mathbf{R}\}$	

In this domain, these values are obtained under the generalized continuum hypothesis in this way.

```
a := Aleph 0
```

$$\aleph_0 \tag{21}$$

CardinalNumber

```
c := 2^a
```

[1] Goedel, *The consistency of the continuum hypothesis,* Ann. Math. Studies, Princeton Univ. Press, 1940.

$$\aleph_1 \tag{22}$$

CardinalNumber

```
f := 2^c
```

$$\aleph_2 \tag{23}$$

CardinalNumber

9.7 CartesianTensor

CartesianTensor(i0,dim,R) provides Cartesian tensors with components belonging to a commutative ring R. Tensors can be described as a generalization of vectors and matrices. This gives a concise *tensor algebra* for multilinear objects supported by the **CartesianTensor** domain. You can form the inner or outer product of any two tensors and you can add or subtract tensors with the same number of components. Additionally, various forms of traces and transpositions are useful.

The **CartesianTensor** constructor allows you to specify the minimum index for subscripting. In what follows we discuss in detail how to manipulate tensors.

Here we construct the domain of Cartesian tensors of dimension 2 over the integers, with indices starting at 1.

```
CT := CARTEN(i0 := 1, 2, Integer)
```

$$\text{CartesianTensor}(1, 2, Integer) \tag{4}$$

Type

Forming tensors

Scalars can be converted to tensors of rank zero.

```
t0: CT := 8
```

$$8 \tag{5}$$

CartesianTensor(1, 2, Integer)

```
rank t0
```

$$0 \tag{6}$$

NonNegativeInteger

Vectors (mathematical direct products, rather than one dimensional array structures) can be converted to tensors of rank one.

```
v: DirectProduct(2, Integer) := directProduct [3,4]
```

$$[3,\ 4] \tag{7}$$

DirectProduct(2, Integer)

```
Tv: CT := v
```

$$[3,\ 4] \tag{8}$$

CartesianTensor(1, 2, Integer)

Matrices can be converted to tensors of rank two.

```
m: SquareMatrix(2, Integer) := matrix [[1,2],[4,5]]
```

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \tag{9}$$

SquareMatrix(2, Integer)

```
Tm: CT := m
```

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \tag{10}$$

CartesianTensor(1, 2, Integer)

```
n: SquareMatrix(2, Integer) := matrix [[2,3],[0,1]]
```

$$\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \tag{11}$$

SquareMatrix(2, Integer)

```
Tn: CT := n
```

$$\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \tag{12}$$

CartesianTensor(1, 2, Integer)

In general, a tensor of rank `k` can be formed by making a list of rank `k-1` tensors or, alternatively, a `k`-deep nested list of lists.

```
t1: CT := [2, 3]
```

$$[2,\ 3] \tag{13}$$

CartesianTensor(1, 2, Integer)

```
rank t1
```

$$1 \tag{14}$$

PositiveInteger

```
t2: CT := [t1, t1]
```

$$\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} \tag{15}$$

CartesianTensor(1, 2, Integer)

```
t3: CT := [t2, t2]
```

$$\left[\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}\right] \tag{16}$$

CartesianTensor(1, 2, Integer)

```
tt: CT := [t3, t3]; tt := [tt, tt]
```

$$\left[\left[\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} \quad \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}\right], \left[\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} \quad \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}\right]\right] \tag{17}$$

CartesianTensor(1, 2, Integer)

```
rank tt
```

$$5 \tag{18}$$

PositiveInteger

Multiplication

Given two tensors of rank k1 and k2, the outer product forms a new tensor of rank k1+k2. Here $T_{mn}(i,j,k,l) = T_m(i,j)\ T_n(k,l)$.

```
Tmn := product(Tm, Tn)
```

$$\left[\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 4 & 6 \\ 0 & 2 \end{bmatrix}\right] \left[\begin{bmatrix} 8 & 12 \\ 0 & 4 \end{bmatrix} \quad \begin{bmatrix} 10 & 15 \\ 0 & 5 \end{bmatrix}\right] \tag{19}$$

CartesianTensor(1, 2, Integer)

The inner product (contract) forms a tensor of rank k1+k2-2. This product generalizes the vector dot product and matrix-vector product by summing component products along two indices. Here we sum along the second index of T_m and the first index of T_v. Here $T_{mv} = \sum_{j=1}^{\dim} T_m(i,j)\ T_v(j)$

```
Tmv := contract(Tm,2,Tv,1)
```

$$[11,\ 32] \tag{20}$$

CartesianTensor(1, 2, Integer)

The multiplication operator * is scalar multiplication or an inner product depending on the ranks of the arguments. If either argument is rank zero it is treated as scalar multiplication. Otherwise, `a*b` is the inner product summing the last index of `a` with the first index of `b`.

```
Tm*Tv
```

$$[11,\ 32] \tag{21}$$

CartesianTensor(1, 2, Integer)

This definition is consistent with the inner product on matrices and vectors.

```
Tmv = m * v
```

$$[11,\ 32] = [11,\ 32] \tag{22}$$

Equation(CartesianTensor(1, 2, Integer))

Selecting Components

For tensors of low rank (that is, four or less), components can be selected by applying the tensor to its indices.

```
t0()
```

$$8 \tag{23}$$

PositiveInteger

```
t1(1+1)
```

$$3 \tag{24}$$

PositiveInteger

```
t2(2,1)
```

$$2 \tag{25}$$

PositiveInteger

```
t3(2,1,2)
```

$$3 \tag{26}$$

PositiveInteger

```
Tmn(2,1,2,1)
```

$$0 \tag{27}$$

NonNegativeInteger

A general indexing mechanism is provided for a list of indices.

```
t0[]
```

$$8 \tag{28}$$

PositiveInteger

```
t1[2]
```

$$3 \tag{29}$$

PositiveInteger

```
t2[2,1]
```

$$2 \tag{30}$$

PositiveInteger

The general mechanism works for tensors of arbitrary rank, but is somewhat less efficient since the intermediate index list must be created.

```
t3[2,1,2]
```

$$3 \tag{31}$$

PositiveInteger

```
Tmn[2,1,2,1]
```

$$0 \tag{32}$$

NonNegativeInteger

Contraction

A "contraction" between two tensors is an inner product, as we have seen above. You can also contract a pair of indices of a single tensor. This corresponds to a "trace" in linear algebra. The expression `contract(t,k1,k2)` forms a new tensor by summing the diagonal given by indices in position `k1` and `k2`. This is the tensor given by $xT_{mn} = \sum_{k=1}^{\dim} T_{mn}(k,k,i,j)$.

```
cTmn := contract(Tmn,1,2)
```

$$\begin{bmatrix} 12 & 18 \\ 0 & 6 \end{bmatrix} \tag{33}$$

CartesianTensor(1, 2, Integer)

Since Tmn is the outer product of matrix m and matrix n, the above is equivalent to this.

```
trace(m) * n
```

$$\begin{bmatrix} 12 & 18 \\ 0 & 6 \end{bmatrix} \tag{34}$$

SquareMatrix(2, Integer)

In this and the next few examples, we show all possible contractions of Tmn and their matrix algebra equivalents.

```
contract(Tmn,1,2) = trace(m) * n
```

$$\begin{bmatrix} 12 & 18 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 12 & 18 \\ 0 & 6 \end{bmatrix} \tag{35}$$

Equation(CartesianTensor(1, 2, Integer))

```
contract(Tmn,1,3) = transpose(m) * n
```

$$\begin{bmatrix} 2 & 7 \\ 4 & 11 \end{bmatrix} = \begin{bmatrix} 2 & 7 \\ 4 & 11 \end{bmatrix} \tag{36}$$

Equation(CartesianTensor(1, 2, Integer))

```
contract(Tmn,1,4) = transpose(m) * transpose(n)
```

$$\begin{bmatrix} 14 & 4 \\ 19 & 5 \end{bmatrix} = \begin{bmatrix} 14 & 4 \\ 19 & 5 \end{bmatrix} \tag{37}$$

Equation(CartesianTensor(1, 2, Integer))

```
contract(Tmn,2,3) = m * n
```

$$\begin{bmatrix} 2 & 5 \\ 8 & 17 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 8 & 17 \end{bmatrix} \tag{38}$$

Equation(CartesianTensor(1, 2, Integer))

```
contract(Tmn,2,4) = m * transpose(n)
```

$$\begin{bmatrix} 8 & 2 \\ 23 & 5 \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 23 & 5 \end{bmatrix} \tag{39}$$

Equation(CartesianTensor(1, 2, Integer))

```
contract(Tmn,3,4) = trace(n) * m
```

$$\begin{bmatrix} 3 & 6 \\ 12 & 15 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 12 & 15 \end{bmatrix} \tag{40}$$

Equation(CartesianTensor(1, 2, Integer))

Transpositions

You can exchange any desired pair of indices using the transpose operation.

Here the indices in positions one and three are exchanged, that is, $tT_{mn}(i,j,k,l) = T_{mn}(k,j,i,l)$.

```
tTmn := transpose(Tmn,1,3)
```

$$\begin{bmatrix}\begin{bmatrix}2 & 3\\ 8 & 12\end{bmatrix} & \begin{bmatrix}4 & 6\\ 10 & 15\end{bmatrix}\\ \begin{bmatrix}0 & 1\\ 0 & 4\end{bmatrix} & \begin{bmatrix}0 & 2\\ 0 & 5\end{bmatrix}\end{bmatrix} \tag{41}$$

CartesianTensor(1, 2, Integer)

If no indices are specified, the first and last index are exchanged.

```
transpose Tmn
```

$$\begin{bmatrix}\begin{bmatrix}2 & 8\\ 0 & 0\end{bmatrix} & \begin{bmatrix}4 & 10\\ 0 & 0\end{bmatrix}\\ \begin{bmatrix}3 & 12\\ 1 & 4\end{bmatrix} & \begin{bmatrix}6 & 15\\ 2 & 5\end{bmatrix}\end{bmatrix} \tag{42}$$

CartesianTensor(1, 2, Integer)

This is consistent with the matrix transpose.

```
transpose Tm = transpose m
```

$$\begin{bmatrix}1 & 4\\ 2 & 5\end{bmatrix} = \begin{bmatrix}1 & 4\\ 2 & 5\end{bmatrix} \tag{43}$$

Equation(CartesianTensor(1, 2, Integer))

If a more complicated reordering of the indices is required, then the reindex operation can be used. This operation allows the indices to be arbitrarily permuted. This defines $rT_{mn}(i,j,k,l) = T_{mn}(i,l,j,k)$.

```
rTmn := reindex(Tmn, [1,4,2,3])
```

$$\begin{bmatrix}\begin{bmatrix}2 & 0\\ 4 & 0\end{bmatrix} & \begin{bmatrix}3 & 1\\ 6 & 2\end{bmatrix}\\ \begin{bmatrix}8 & 0\\ 10 & 0\end{bmatrix} & \begin{bmatrix}12 & 4\\ 15 & 5\end{bmatrix}\end{bmatrix} \tag{44}$$

CartesianTensor(1, 2, Integer)

Arithmetic

Tensors of equal rank can be added or subtracted so arithmetic expressions can be used to produce new tensors.

```
tt := transpose(Tm)*Tn - Tn*transpose(Tm)
```

$$\begin{bmatrix} -6 & -16 \\ 2 & 6 \end{bmatrix} \tag{45}$$

CartesianTensor(1, 2, Integer)

```
Tv*(tt+Tn)
```

$$[-4, -11] \tag{46}$$

CartesianTensor(1, 2, Integer)

```
reindex(product(Tn,Tn),[4,3,2,1])+3*Tn*product(Tm,Tm)
```

$$\begin{bmatrix} \begin{bmatrix} 46 & 84 \\ 174 & 212 \end{bmatrix} & \begin{bmatrix} 57 & 114 \\ 228 & 285 \end{bmatrix} \\ \begin{bmatrix} 18 & 24 \\ 57 & 63 \end{bmatrix} & \begin{bmatrix} 17 & 30 \\ 63 & 76 \end{bmatrix} \end{bmatrix} \tag{47}$$

CartesianTensor(1, 2, Integer)

Specific Tensors

Two specific tensors have properties which depend only on the dimension.

The Kronecker delta satisfies

$$\text{delta}(i,j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

```
delta:  CT := kroneckerDelta()
```

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{48}$$

CartesianTensor(1, 2, Integer)

This can be used to reindex via contraction.

```
contract(Tmn, 2, delta, 1) = reindex(Tmn, [1,3,4,2])
```

$$\begin{bmatrix} \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \\ \begin{bmatrix} 8 & 10 \\ 12 & 15 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 4 & 5 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \\ \begin{bmatrix} 8 & 10 \\ 12 & 15 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 4 & 5 \end{bmatrix} \end{bmatrix} \tag{49}$$

Equation(CartesianTensor(1, 2, Integer))

The Levi Civita symbol determines the sign of a permutation of indices.

```
epsilon:CT := leviCivitaSymbol()
```

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{50}$$

CartesianTensor(1, 2, Integer)

Here we have:

$$\text{epsilon}(i_1,\ldots,i_{\text{dim}}) = \begin{cases} +1 & \text{if } i_1,\ldots,i_{\text{dim}} \text{ is an even permutation of} \\ & i_0,\ldots,i_0+\text{dim}-1 \\ -1 & \text{if } i_1,\ldots,i_{\text{dim}} \text{ is an odd permutation of} \\ & i_0,\ldots,i_0+\text{dim}-1 \\ 0 & \text{if } i_1,\ldots,i_{\text{dim}} \text{ is } \textit{not} \text{ a permutation of} \\ & i_0,\ldots,i_0+\text{dim}-1 \end{cases}$$

This property can be used to form determinants.

```
contract(epsilon*Tm*epsilon, 1,2) = 2 * determinant m
```

$$-6 = -6 \tag{51}$$

Equation(CartesianTensor(1, 2, Integer))

Properties of the CartesianTensor domain

GradedModule(R,E) denotes "E-graded R-module", that is, a collection of R-modules indexed by an abelian monoid E. An element g of G[s] for some specific s in E is said to be an element of G with degree s. Sums are defined in each module G[s] so two elements of G can be added if they have the same degree. Morphisms can be defined and composed by degree to give the mathematical category of graded modules.

GradedAlgebra(R,E) denotes "E-graded R-algebra." A graded algebra is a graded module together with a degree preserving R-bilinear map, called the product.

```
degree(product(a,b))= degree(a) + degree(b)

product(r*a,b) = product(a,r*b) = r*product(a,b)
product(a1+a2,b) = product(a1,b) + product(a2,b)
product(a,b1+b2) = product(a,b1) + product(a,b2)
product(a,product(b,c)) = product(product(a,b),c)
```

The domain **CartesianTensor(i0, dim, R)** belongs to the category **GradedAlgebra(R, NonNegativeInteger)**. The non-negative integer degree is the tensor rank and the graded algebra product is the tensor outer product. The graded module addition captures the notion that only tensors of equal rank can be added.

If V is a vector space of dimension dim over R, then the tensor module T[k](V) is defined as

```
T[0](V) = R
T[k](V) = T[k-1](V) * V
```

where * denotes the R-module tensor product. **CartesianTensor(i0,dim,R)** is the graded algebra in which the degree k module is T[k](V).

Tensor Calculus

It should be noted here that often tensors are used in the context of tensor-valued manifold maps. This leads to the notion of covariant and contravariant bases with tensor component functions transforming in specific ways under a change of coordinates on the manifold. This is no more directly supported by the **CartesianTensor** domain than it is by the **Vector** domain. However, it is possible to have the components implicitly represent component maps by choosing a polynomial or expression type for the components. In this case, it is up to the user to satisfy any constraints which arise on the basis of this interpretation.

9.8 Character

The members of the domain **Character** are values representing letters, numerals and other text elements. For more information on related topics, see '`CharacterClass`' on page 365 and '`String`' on page 579.

Characters can be obtained using **String** notation.

```
chars := [char "a", char "A", char "X", char "8", char "+"]
```

$$[a, A, X, 8, +] \tag{4}$$

List (Character)

Certain characters are available by name. This is the blank character.

```
space()
```

(5)

Character

This is the quote that is used in strings.

```
quote()
```

$$" \tag{6}$$

Character

This is the underscore character which is used to allow quotes and other special characters within strings.

```
underscore()
```

$$_ \tag{7}$$

Character

Characters are represented as integers in a machine-dependent way. The integer value can be obtained using the ord operation. It is always true that `char(ord c)= c` and `ord(char i)= i`, provided that `i` is in the range `0..size()$Character-1`.

```
[ord c for c in chars]
```

[97, 65, 88, 56, 43] (8)

List (Integer)

The lowerCase operation converts an upper case letter to the corresponding lower case letter. If the argument is not an upper case letter, then it is returned unchanged.

```
[upperCase c for c in chars]
```

[A, A, X, 8, +] (9)

List (Character)

Likewise, the upperCase operation converts lower case letters to upper case.

```
[lowerCase c for c in chars]
```

[a, a, x, 8, +] (10)

List (Character)

A number of tests are available to determine whether characters belong to certain families.

```
[alphabetic? c for c in chars]
```

[true, true, true, false, false] (11)

List (Boolean)

```
[upperCase? c for c in chars]
```

[false, true, true, false, false] (12)

List (Boolean)

```
[lowerCase? c for c in chars]
```

[true, false, false, false, false] (13)

List (Boolean)

```
[digit? c for c in chars]
```

[false, false, false, true, false] (14)

List (Boolean)

```
[hexDigit? c for c in chars]
```

[true, true, false, true, false] (15)

List (Boolean)

```
[alphanumeric? c for c in chars]
```

```
[true, true, true, true, false]                                    (16)
```

List (Boolean)

9.9 CharacterClass

The **CharacterClass** domain allows classes of characters to be defined and manipulated efficiently.

Character classes can be created by giving either a string or a list of characters.

```
cl1 := charClass [char "a", char "e", char "i", char "o", char "u", char "y"]
```

```
"aeiouy"                                                           (4)
```

CharacterClass

```
cl2 := charClass "bcdfghjklmnpqrstvwxyz"
```

```
"bcdfghjklmnpqrstvwxyz"                                            (5)
```

CharacterClass

A number of character classes are predefined for convenience.

```
digit()
```

```
"0123456789"                                                       (6)
```

CharacterClass

```
hexDigit()
```

```
"0123456789ABCDEFabcdef"                                           (7)
```

CharacterClass

```
upperCase()
```

```
"ABCDEFGHIJKLMNOPQRSTUVWXYZ"                                       (8)
```

CharacterClass

```
lowerCase()
```

```
"abcdefghijklmnopqrstuvwxyz"                                       (9)
```

CharacterClass

```
alphabetic()
```

```
"ABCDEFGHIJKLMNOPQRSTUVWXYZabcdefghijklmnopqrstuvwxyz"            (10)
```

CharacterClass

```
alphanumeric()
```

"0123456789ABCDEFGHIJKLMNOPQRSTUVWXYZabcdefghijklmnopqrstuvwxyz" (11)

CharacterClass

You can quickly test whether a character belongs to a class.

```
member?(char "a", cl1)
```

true (12)

Boolean

```
member?(char "a", cl2)
```

false (13)

Boolean

Classes have the usual set operations because the **CharacterClass** domain belongs to the category **FiniteSetAggregate(Character)**.

```
intersect(cl1, cl2)
```

"y" (14)

CharacterClass

```
union(cl1,cl2)
```

"abcdefghijklmnopqrstuvwxyz" (15)

CharacterClass

```
difference(cl1,cl2)
```

"aeiou" (16)

CharacterClass

```
intersect(complement(cl1),cl2)
```

"bcdfghjklmnpqrstvwxz" (17)

CharacterClass

You can modify character classes by adding or removing characters.

```
insert!(char "a", cl2)
```

"abcdfghjklmnpqrstvwxyz" (18)

CharacterClass

```
remove!(char "b", cl2)
```

$$\texttt{"acdfghjklmnpqrstvwxyz"} \tag{19}$$

CharacterClass

For more information on related topics, see '`Character`' on page 363 and '`String`' on page 579. Issue the system command `)show CharacterClass` to display the full list of operations defined by **CharacterClass**.

9.10 CliffordAlgebra

CliffordAlgebra(n,K,Q) defines a vector space of dimension 2^n over the field K with a given bilinear form represented by a square matrix `Q`. If $e_1, \ldots, e_n$ is a basis for K^n then

$$\begin{Bmatrix} 1 & \\ e_i & \text{for } 1 \le i \le n \\ e_{i_1}\ e_{i_2} & \text{for } 1 \le i_1 < i_2 \le n \\ \cdots & \\ e_1\ e_2\ \cdots\ e_n & \end{Bmatrix}$$

is a basis for the Clifford algebra. The algebra is defined by the relation

$$\begin{aligned} e_i\ e_i &= Q(e_i) \\ e_i\ e_j &= -e_j\ e_i \quad \text{for } i \neq j \end{aligned}$$

for all v being linear combinations of $e(i)$. Examples of Clifford Algebras are gaussians (complex numbers), quaternions, exterior algebras and spin algebras.

9.10.1 The Complex Numbers as a Clifford Algebra

This is the field over which we will work, rational functions with integer coefficients.

```
K := Fraction Polynomial Integer
```

$$\text{Fraction}(\text{Polynomial}(Integer)) \tag{1}$$

Type

We use this matrix for the quadratic form.

```
m := matrix [[-1]]
```

$$[-1] \tag{2}$$

Matrix(Integer)

We get complex arithmetic by using this domain.

```
C := CliffordAlgebra(1, K, m)
```

$$\text{CliffordAlgebra}\big(1, \text{Fraction}(\text{Polynomial}(Integer)), [-1]\big) \tag{3}$$

Type

Here is `i`, the usual square root of `-1`.

```
i: C := e(1)
```

$$\mathsf{e}_1 \tag{4}$$

CliffordAlgebra (1, Fraction (Polynomial(Integer)), [[−1]])

Here are some examples of the arithmetic.

```
x := a + b * i
```

$$a + b\,\mathsf{e}_1 \tag{5}$$

CliffordAlgebra (1, Fraction (Polynomial(Integer)), [[−1]])

```
y := c + d * i
```

$$c + d\,\mathsf{e}_1 \tag{6}$$

CliffordAlgebra (1, Fraction (Polynomial(Integer)), [[−1]])

See '`Complex`' on page 372 for examples of FriCAS's constructor implementing complex numbers.

```
x * y
```

$$-b\,d + a\,c + (a\,d + b\,c)\,\mathsf{e}_1 \tag{7}$$

CliffordAlgebra (1, Fraction (Polynomial(Integer)), [[−1]])

9.10.2 The Quaternion Numbers as a Clifford Algebra

This is the field over which we will work, rational functions with integer coefficients.

```
K := Fraction Polynomial Integer
```

$$\text{Fraction}(\text{Polynomial}(Integer)) \tag{1}$$

Type

We use this matrix for the quadratic form.

```
m := matrix [[-1,0],[0,-1]]
```

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \tag{2}$$

Matrix(Integer)

The resulting domain is the quaternions.

```
H  := CliffordAlgebra(2, K, m)
```

$$\text{CliffordAlgebra}\left(2, \text{Fraction}(\text{Polynomial}(Integer)), \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right) \tag{3}$$

Type

We use Hamilton's notation for `i`,`j`,`k`.

```
i: H  := e(1)
```

$$\mathsf{e}_1 \tag{4}$$

CliffordAlgebra (2, Fraction (Polynomial(Integer)), [[−1, 0], [0, −1]])

```
j: H  := e(2)
```

$$\mathsf{e}_2 \tag{5}$$

CliffordAlgebra (2, Fraction (Polynomial(Integer)), [[−1, 0], [0, −1]])

```
k: H  := i * j
```

$$\mathsf{e}_1\,\mathsf{e}_2 \tag{6}$$

CliffordAlgebra (2, Fraction (Polynomial(Integer)), [[−1, 0], [0, −1]])

```
x := a + b * i + c * j + d * k
```

$$a + b\,\mathsf{e}_1 + c\,\mathsf{e}_2 + d\,\mathsf{e}_1\,\mathsf{e}_2 \tag{7}$$

CliffordAlgebra (2, Fraction (Polynomial(Integer)), [[−1, 0], [0, −1]])

```
y := e + f * i + g * j + h * k
```

$$e + f\,\mathsf{e}_1 + g\,\mathsf{e}_2 + h\,\mathsf{e}_1\,\mathsf{e}_2 \tag{8}$$

CliffordAlgebra (2, Fraction (Polynomial(Integer)), [[−1, 0], [0, −1]])

```
x + y
```

$$e + a + (f + b)\,\mathsf{e}_1 + (g + c)\,\mathsf{e}_2 + (h + d)\,\mathsf{e}_1\,\mathsf{e}_2 \tag{9}$$

CliffordAlgebra (2, Fraction (Polynomial(Integer)), [[−1, 0], [0, −1]])

```
x * y
```

$$-d\,h - c\,g - b\,f + a\,e + (c\,h - d\,g + a\,f + b\,e)\,\mathsf{e}_1 + (-b\,h + a\,g + d\,f + c\,e)\,\mathsf{e}_2 + (a\,h + b\,g - c\,f + d\,e)\,\mathsf{e}_1\,\mathsf{e}_2 \tag{10}$$

CliffordAlgebra (2, Fraction(Polynomial(Integer)), [[−1, 0], [0, −1]])

See 'Quaternion' on page 539 for examples of FriCAS's constructor implementing quaternions.

```
y * x
```

$$-dh-cg-bf+ae+(-ch+dg+af+be)\,\mathtt{e}_1+(bh+ag-df+ce)\,\mathtt{e}_2+(ah-bg+cf+de)\,\mathtt{e}_1\,\mathtt{e}_2 \quad (11)$$

CliffordAlgebra (2, Fraction(Polynomial(Integer)), [[−1, 0], [0, −1]])

9.10.3 The Exterior Algebra on a Three Space

This is the field over which we will work, rational functions with integer coefficients.

```
K := Fraction Polynomial Integer
```

$$\mathrm{Fraction}(\mathrm{Polynomial}(Integer)) \quad (1)$$

Type

If we choose the three by three zero quadratic form, we obtain the exterior algebra on `e(1),e(2),e(3)`.

```
Ext := CliffordAlgebra(3, K, 0)
```

$$\mathrm{CliffordAlgebra}\left(3, \mathrm{Fraction}(\mathrm{Polynomial}(Integer)), \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\right) \quad (2)$$

Type

This is a three dimensional vector algebra. We define `i`, `j`, `k` as the unit vectors.

```
i: Ext := e(1)
```

$$\mathtt{e}_1 \quad (3)$$

CliffordAlgebra (3, Fraction(Polynomial(Integer)), [[0, 0, 0], [0, 0, 0], [0, 0, 0]])

```
j: Ext := e(2)
```

$$\mathtt{e}_2 \quad (4)$$

CliffordAlgebra (3, Fraction(Polynomial(Integer)), [[0, 0, 0], [0, 0, 0], [0, 0, 0]])

```
k: Ext := e(3)
```

$$\mathtt{e}_3 \quad (5)$$

CliffordAlgebra (3, Fraction(Polynomial(Integer)), [[0, 0, 0], [0, 0, 0], [0, 0, 0]])

Now it is possible to do arithmetic.

```
x := x1*i + x2*j + x3*k
```

$$x1\,\mathbf{e}_1 + x2\,\mathbf{e}_2 + x3\,\mathbf{e}_3 \tag{6}$$

CliffordAlgebra (3, Fraction(Polynomial(Integer)), [[0, 0, 0], [0, 0, 0], [0, 0, 0]])

```
y := y1*i + y2*j + y3*k
```

$$y1\,\mathbf{e}_1 + y2\,\mathbf{e}_2 + y3\,\mathbf{e}_3 \tag{7}$$

CliffordAlgebra (3, Fraction(Polynomial(Integer)), [[0, 0, 0], [0, 0, 0], [0, 0, 0]])

```
x + y
```

$$(y1 + x1)\,\mathbf{e}_1 + (y2 + x2)\,\mathbf{e}_2 + (y3 + x3)\,\mathbf{e}_3 \tag{8}$$

CliffordAlgebra (3, Fraction(Polynomial(Integer)), [[0, 0, 0], [0, 0, 0], [0, 0, 0]])

```
x * y + y * x
```

$$0 \tag{9}$$

CliffordAlgebra (3, Fraction(Polynomial(Integer)), [[0, 0, 0], [0, 0, 0], [0, 0, 0]])

On an `n` space, a grade `p` form has a dual `n-p` form. In particular, in three space the dual of a grade two element identifies `e1*e2→e3, e2*e3→e1, e3*e1→e2`.

```
dual2 a == coefficient(a,[2,3]) * i + coefficient(a,[3,1]) * j + coefficient(a,[1,2]) _
    * k
```

The vector cross product is then given by this.

```
dual2(x*y)
```

```
   Compiling function dual2 with type CliffordAlgebra(3,Fraction(Polynomial(
      Integer)),[[0,0,0],[0,0,0],[0,0,0]]) -> CliffordAlgebra(3,Fraction(
      Polynomial(Integer)),[[0,0,0],[0,0,0],[0,0,0]])
```

$$(x2\,y3 - x3\,y2)\,\mathbf{e}_1 + (-x1\,y3 + x3\,y1)\,\mathbf{e}_2 + (x1\,y2 - x2\,y1)\,\mathbf{e}_3 \tag{11}$$

CliffordAlgebra (3, Fraction(Polynomial(Integer)), [[0, 0, 0], [0, 0, 0], [0, 0, 0]])

9.10.4 The Dirac Spin Algebra

In this section we will work over the field of rational numbers.

```
K := Fraction Integer
```

$$\text{Fraction}(Integer) \tag{1}$$

Type

We define the quadratic form to be the Minkowski space-time metric.

```
g := matrix [[1,0,0,0], [0,-1,0,0], [0,0,-1,0], [0,0,0,-1]]
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{2}$$

Matrix(Integer)

We obtain the Dirac spin algebra used in Relativistic Quantum Field Theory.

```
D := CliffordAlgebra(4, K, g)
```

$$\text{CliffordAlgebra}\left(4, \text{Fraction}(Integer), \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}\right) \tag{3}$$

Type

The usual notation for the basis is γ with a superscript. For FriCAS input we will use `gam(i)`:

```
gam := [e(i)$D for i in 1..4]
```

$$[\mathsf{e}_1,\ \mathsf{e}_2,\ \mathsf{e}_3,\ \mathsf{e}_4] \tag{4}$$

List(CliffordAlgebra (4, Fraction(Integer), [[1, 0, 0, 0], [0, −1, 0, 0], [0, 0, −1, 0], [0, 0, 0, −1]]))

There are various contraction identities of the form

```
g(l,t)*gam(l)*gam(m)*gam(n)*gam(r)*gam(s)*gam(t) =
      2*(gam(s)gam(m)gam(n)gam(r) + gam(r)*gam(n)*gam(m)*gam(s))
```

where a sum over `l` and `t` is implied. Verify this identity for particular values of `m,n,r,s`.

```
m := 1; n:= 2; r := 3; s := 4;
```

PositiveInteger

```
lhs := reduce(+, [reduce(+, [ g(l,t)*gam(l)*gam(m)*gam(n)*gam(r)*gam(s)*gam(t) for l _
    in 1..4]) for t in 1..4])
```

$$-4\,\mathsf{e}_1\,\mathsf{e}_2\,\mathsf{e}_3\,\mathsf{e}_4 \tag{6}$$

CliffordAlgebra (4, Fraction(Integer), [[1, 0, 0, 0], [0, −1, 0, 0], [0, 0, −1, 0], [0, 0, 0, −1]])

```
rhs := 2*(gam s * gam m*gam n*gam r + gam r*gam n*gam m*gam s)
```

$$-4\,\mathsf{e}_1\,\mathsf{e}_2\,\mathsf{e}_3\,\mathsf{e}_4 \tag{7}$$

CliffordAlgebra (4, Fraction(Integer), [[1, 0, 0, 0], [0, −1, 0, 0], [0, 0, −1, 0], [0, 0, 0, −1]])

9.11 Complex

The **Complex** constructor implements complex objects over a commutative ring `R`. Typically, the ring `R` is **Integer**, **Fraction Integer**, **Float** or **DoubleFloat**. `R` can also be a symbolic type, like

Polynomial Integer. For more information about the numerical and graphical aspects of complex numbers, see Section 8.1 on page 251.

Complex objects are created by the complex operation.

```
a := complex(4/3,5/2)
```

$$\frac{4}{3}+\frac{5}{2}\,i \tag{4}$$

Complex(Fraction(Integer))

```
b := complex(4/3,-5/2)
```

$$\frac{4}{3}-\frac{5}{2}\,i \tag{5}$$

Complex(Fraction(Integer))

The standard arithmetic operations are available.

```
a + b
```

$$\frac{8}{3} \tag{6}$$

Complex(Fraction(Integer))

```
a - b
```

$$5\,i \tag{7}$$

Complex(Fraction(Integer))

```
a * b
```

$$\frac{289}{36} \tag{8}$$

Complex(Fraction(Integer))

If R is a field, you can also divide the complex objects.

```
a / b
```

$$-\frac{161}{289}+\frac{240}{289}\,i \tag{9}$$

Complex(Fraction(Integer))

Use a conversion (Section 2.7 on page 84) to view the last object as a fraction of complex integers.

```
% :: Fraction Complex Integer
```

$$\frac{-15+8\,i}{15+8\,i} \tag{10}$$

Fraction(Complex(Integer))

The predefined macro `%i` is defined to be `complex(0,1)`.

```
3.4 + 6.7 * %i
```

$$3.4 + 6.7\,i \tag{11}$$

Complex(Float)

You can also compute the conjugate and norm of a complex number.

```
conjugate a
```

$$\frac{4}{3} - \frac{5}{2}\,i \tag{12}$$

Complex(Fraction(Integer))

```
norm a
```

$$\frac{289}{36} \tag{13}$$

Fraction(Integer)

The real and imag operations are provided to extract the real and imaginary parts, respectively.

```
real a
```

$$\frac{4}{3} \tag{14}$$

Fraction(Integer)

```
imag a
```

$$\frac{5}{2} \tag{15}$$

Fraction(Integer)

The domain **Complex Integer** is also called the Gaussian integers. If `R` is the integers (or, more generally, a **EuclideanDomain**), you can compute greatest common divisors.

```
gcd(13 - 13*%i,31 + 27*%i)
```

$$5 + i \tag{16}$$

Complex(Integer)

You can also compute least common multiples.

```
lcm(13 - 13*%i,31 + 27*%i)
```

$$143 - 39\,i \tag{17}$$

Complex(Integer)

You can factor Gaussian integers.

```
factor(13 - 13*%i)
```

$$-(1+i)\,(2+3\,i)\,(3+2\,i) \tag{18}$$

Factored(Complex(Integer))

```
factor complex(2,0)
```

$$-i\,(1+i)^2 \tag{19}$$

Factored(Complex(Integer))

9.12 ContinuedFraction

Continued fractions have been a fascinating and useful tool in mathematics for well over three hundred years. FriCAS implements continued fractions for fractions of any Euclidean domain. In practice, this usually means rational numbers. In this section we demonstrate some of the operations available for manipulating both finite and infinite continued fractions. It may be helpful if you review '`Stream`' on page 577 to remind yourself of some of the operations with streams.

The **ContinuedFraction** domain is a field and therefore you can add, subtract, multiply and divide the fractions. The continuedFraction operation converts its fractional argument to a continued fraction.

```
c := continuedFraction(314159/100000)
```

$$3 + \frac{1|}{|7} + \frac{1|}{|15} + \frac{1|}{|1} + \frac{1|}{|25} + \frac{1|}{|1} + \frac{1|}{|7} + \frac{1|}{|4} \tag{4}$$

ContinuedFraction(Integer)

This display is a compact form of the bulkier

$$3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{25 + \cfrac{1}{1 + \cfrac{1}{7 + \cfrac{1}{4}}}}}}}$$

You can write any rational number in a similar form. The fraction will be finite and you can always take the "numerators" to be `1`. That is, any rational number can be written as a simple, finite continued

fraction of the form

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots a_{n-1} + \cfrac{1}{a_n}}}}$$

The a_i are called partial quotients and the operation partialQuotients creates a stream of them.

```
partialQuotients c
```

$$[3,\ 7,\ 15,\ 1,\ 25,\ 1,\ 7,\ \ldots] \tag{5}$$

Stream(Integer)

By considering more and more of the fraction, you get the convergents. For example, the first convergent is a_1, the second is $a_1 + 1/a_2$ and so on.

```
convergents c
```

$$\left[3,\ \frac{22}{7},\ \frac{333}{106},\ \frac{355}{113},\ \frac{9208}{2931},\ \frac{9563}{3044},\ \frac{76149}{24239},\ \ldots\right] \tag{6}$$

Stream(Fraction(Integer))

Since this is a finite continued fraction, the last convergent is the original rational number, in reduced form. The result of approximants is always an infinite stream, though it may just repeat the "last" value.

```
approximants c
```

$$\left[3,\ \frac{22}{7},\ \frac{333}{106},\ \frac{355}{113},\ \frac{9208}{2931},\ \frac{9563}{3044},\ \frac{76149}{24239},\ \ldots\right] \tag{7}$$

Stream(Fraction(Integer))

Inverting c only changes the partial quotients of its fraction by inserting a 0 at the beginning of the list.

```
pq := partialQuotients(1/c)
```

$$[0,\ 3,\ 7,\ 15,\ 1,\ 25,\ 1,\ \ldots] \tag{8}$$

Stream(Integer)

Do this to recover the original continued fraction from this list of partial quotients. The three-argument form of the continuedFraction operation takes an element which is the whole part of the fraction, a stream of elements which are the numerators of the fraction, and a stream of elements which are the denominators of the fraction.

```
continuedFraction(first pq,repeating [1],rest pq)
```

$$\frac{1|}{|3} + \frac{1|}{|7} + \frac{1|}{|15} + \frac{1|}{|1} + \frac{1|}{|25} + \frac{1|}{|1} + \frac{1|}{|7} + \ldots \tag{9}$$

ContinuedFraction(Integer)

The streams need not be finite for continuedFraction. Can you guess which irrational number has the following continued fraction? See the end of this section for the answer.

```
z:=continuedFraction(3,repeating [1],repeating [3,6])
```

$$3+\frac{1|}{|3}+\frac{1|}{|6}+\frac{1|}{|3}+\frac{1|}{|6}+\frac{1|}{|3}+\frac{1|}{|6}+\frac{1|}{|3}+\ldots \tag{10}$$

ContinuedFraction(Integer)

In 1737 Euler discovered the infinite continued fraction expansion

$$\frac{e-1}{2}=\cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{10+\cfrac{1}{14+\cdots}}}}$$

We use this expansion to compute rational and floating point approximations of e.[2]

By looking at the above expansion, we see that the whole part is 0 and the numerators are all equal to 1. This constructs the stream of denominators.

```
dens : Stream Integer := cons(1, stream((x +-> x + 4), 6))
```

$$[1,\ 6,\ 10,\ 14,\ 18,\ 22,\ 26,\ \ldots] \tag{11}$$

Stream(Integer)

Therefore this is the continued fraction expansion for $(e-1)/2$.

```
cf := continuedFraction(0,repeating [1],dens)
```

$$\frac{1|}{|1}+\frac{1|}{|6}+\frac{1|}{|10}+\frac{1|}{|14}+\frac{1|}{|18}+\frac{1|}{|22}+\frac{1|}{|26}+\ldots \tag{12}$$

ContinuedFraction(Integer)

These are the rational number convergents.

```
ccf := convergents cf
```

$$\left[0,\ 1,\ \frac{6}{7},\ \frac{61}{71},\ \frac{860}{1001},\ \frac{15541}{18089},\ \frac{342762}{398959},\ \ldots\right] \tag{13}$$

Stream(Fraction(Integer))

You can get rational convergents for e by multiplying by 2 and adding 1.

```
eConvergents := [2*e + 1 for e in ccf]
```

[2]For this and other interesting expansions, see C. D. Olds, *Continued Fractions,* New Mathematical Library, (New York: Random House, 1963), pp. 134–139.

$$\left[1,\ 3,\ \frac{19}{7},\ \frac{193}{71},\ \frac{2721}{1001},\ \frac{49171}{18089},\ \frac{1084483}{398959},\ \ldots\right] \tag{14}$$

Stream(Fraction(Integer))

You can also compute the floating point approximations to these convergents.

```
eConvergents :: Stream Float
```

$$\begin{array}{l}[1.0,\ 3.0,\ 2.7142857142857142857,\ 2.7183098591549295775,\\ \ 2.7182817182817182817,\ 2.7182818287356957267,\ 2.7182818284585634113,\ \ldots]\end{array} \tag{15}$$

Stream(Float)

Compare this to the value of e computed by the exp operation in **Float**.

```
exp 1.0
```

$$2.7182818284590452354 \tag{16}$$

Float

In about 1658, Lord Brouncker established the following expansion for $4/\pi$.

$$1+\cfrac{1}{2+\cfrac{9}{2+\cfrac{25}{2+\cfrac{49}{2+\cfrac{81}{2+\cdots}}}}}$$

Let's use this expansion to compute rational and floating point approximations for π.

```
cf := continuedFraction(1,[(2*i+1)^2 for i in 0..],repeating [2])
```

$$1+\frac{1|}{|2}+\frac{9|}{|2}+\frac{25|}{|2}+\frac{49|}{|2}+\frac{81|}{|2}+\frac{121|}{|2}+\frac{169|}{|2}+\ldots \tag{17}$$

ContinuedFraction(Integer)

```
ccf := convergents cf
```

$$\left[1,\ \frac{3}{2},\ \frac{15}{13},\ \frac{105}{76},\ \frac{315}{263},\ \frac{3465}{2578},\ \frac{45045}{36979},\ \ldots\right] \tag{18}$$

Stream(Fraction(Integer))

```
piConvergents := [4/p for p in ccf]
```

$$\left[4,\ \frac{8}{3},\ \frac{52}{15},\ \frac{304}{105},\ \frac{1052}{315},\ \frac{10312}{3465},\ \frac{147916}{45045},\ \ldots\right] \tag{19}$$

Stream(Fraction(Integer))

As you can see, the values are converging to $\pi = 3.14159265358979323846...$, but not very quickly.

```
piConvergents :: Stream Float
```

$$[4.0,\ 2.6666666666666666667,\ 3.4666666666666666667,\ 2.8952380952380952381,\ 3.3396825396825396825,\ 2.9760461760461760462,\ 3.2837384837384837385,\ \ldots] \tag{20}$$

Stream(Float)

You need not restrict yourself to continued fractions of integers. Here is an expansion for a quotient of Gaussian integers.

```
continuedFraction((- 122 + 597*%i)/(4 - 4*%i))
```

$$-90 + 59\,i + \cfrac{1|}{|1 - 2\,i} + \cfrac{1|}{|-1 + 2\,i} \tag{21}$$

ContinuedFraction(Complex(Integer))

This is an expansion for a quotient of polynomials in one variable with rational number coefficients.

```
r : Fraction UnivariatePolynomial(x,Fraction Integer)
```

```
r := ((x - 1) * (x - 2)) / ((x-3) * (x-4))
```

$$\frac{x^2 - 3\,x + 2}{x^2 - 7\,x + 12} \tag{23}$$

Fraction(UnivariatePolynomial(x, Fraction(Integer)))

```
continuedFraction r
```

$$1 + \cfrac{1|}{|\frac{1}{4}\,x - \frac{9}{8}} + \cfrac{1|}{|\frac{16}{3}\,x - \frac{40}{3}} \tag{24}$$

ContinuedFraction(UnivariatePolynomial(x, Fraction(Integer)))

To conclude this section, we give you evidence that

```
z
```

$$3 + \cfrac{1|}{|3} + \cfrac{1|}{|6} + \cfrac{1|}{|3} + \cfrac{1|}{|6} + \cfrac{1|}{|3} + \cfrac{1|}{|6} + \cfrac{1|}{|3} + \ldots \tag{25}$$

ContinuedFraction(Integer)

is the expansion of $\sqrt{11}$.

```
[i*i for i in convergents(z) :: Stream Float]
```

$$[9.0,\ 11.111111111111111111,\ 10.99445983379501385,\ 11.000277777777777778,\ 10.999860763987997 86,\ 11.000000697929731039,\ 10.999999965015834446,\ \ldots] \tag{26}$$

Stream(Float)

```
continuedFraction sqrt 11.0
```

$$3+\frac{1|}{|3}+\frac{1|}{|6}+\frac{1|}{|3}+\frac{1|}{|6}+\frac{1|}{|3}+\frac{1|}{|6}+\frac{1|}{|3}+\ldots \tag{27}$$

ContinuedFraction(Integer)

9.13 CycleIndicators

This section is based upon the paper J. H. Redfield, "The Theory of Group-Reduced Distributions", American J. Math.,49 (1927) 433-455, and is an application of group theory to enumeration problems. It is a development of the work by P. A. MacMahon on the application of symmetric functions and Hammond operators to combinatorial theory.

The theory is based upon the power sum symmetric functions s_i which are the sum of the i^{th} powers of the variables. The cycle index of a permutation is an expression that specifies the sizes of the cycles of a permutation, and may be represented as a partition. A partition of a non-negative integer `n` is a collection of positive integers called its parts whose sum is `n`. For example, the partition $(3^2\ 2\ 1^2)$ will be used to represent $s_3^2 s_2 s_1^2$ and will indicate that the permutation has two cycles of length 3, one of length 2 and two of length 1. The cycle index of a permutation group is the sum of the cycle indices of its permutations divided by the number of permutations. The cycle indices of certain groups are provided. We first expose something from the library.

```
)expose EVALCYC
```

```
   EvaluateCycleIndicators is now explicitly exposed in frame initial
```

The operation complete returns the cycle index of the symmetric group of order `n` for argument `n`. Alternatively, it is the n^{th} complete homogeneous symmetric function expressed in terms of power sum symmetric functions.

```
complete 1
```

$$(1) \tag{4}$$

SymmetricPolynomial(Fraction(Integer))

```
complete 2
```

$$\frac{1}{2}\ (2)+\frac{1}{2}\ \left(1^2\right) \tag{5}$$

SymmetricPolynomial(Fraction(Integer))

```
complete 3
```

$$\frac{1}{3}\ (3)+\frac{1}{2}\ (2\ \ 1)+\frac{1}{6}\ \left(1^3\right) \tag{6}$$

SymmetricPolynomial(Fraction(Integer))

```
complete 7
```

$$\frac{1}{7}\,(7)+\frac{1}{6}\,(6\ \ 1)+\frac{1}{10}\,(5\ \ 2)+\frac{1}{10}\,(5\ \ 1^2)+\frac{1}{12}\,(4\ \ 3)+\frac{1}{8}\,(4\ \ 2\ \ 1)+\frac{1}{24}\,(4\ \ 1^3)+\frac{1}{18}\,(3^2\,1) \\ +\frac{1}{24}\,(3\ \ 2^2)+\frac{1}{12}\,(3\ \ 2\ \ 1^2)+\frac{1}{72}\,(3\ \ 1^4)+\frac{1}{48}\,(2^3\,1)+\frac{1}{48}\,(2^2\,1^3)+\frac{1}{240}\,(2\ \ 1^5)+\frac{1}{5040}\,(1^7) \tag{7}$$

SymmetricPolynomial(Fraction(Integer))

The operation elementary computes the n^{th} elementary symmetric function for argument `n`.

```
elementary 7
```

$$\frac{1}{7}\,(7)-\frac{1}{6}\,(6\ \ 1)-\frac{1}{10}\,(5\ \ 2)+\frac{1}{10}\,(5\ \ 1^2)-\frac{1}{12}\,(4\ \ 3)+\frac{1}{8}\,(4\ \ 2\ \ 1)-\frac{1}{24}\,(4\ \ 1^3)+\frac{1}{18}\,(3^2\,1) \\ +\frac{1}{24}\,(3\ \ 2^2)-\frac{1}{12}\,(3\ \ 2\ \ 1^2)+\frac{1}{72}\,(3\ \ 1^4)-\frac{1}{48}\,(2^3\,1)+\frac{1}{48}\,(2^2\,1^3)-\frac{1}{240}\,(2\ \ 1^5)+\frac{1}{5040}\,(1^7) \tag{8}$$

SymmetricPolynomial(Fraction(Integer))

The operation alternating returns the cycle index of the alternating group having an even number of even parts in each cycle partition.

```
alternating 7
```

$$\frac{2}{7}\,(7)+\frac{1}{5}\,(5\ \ 1^2)+\frac{1}{4}\,(4\ \ 2\ \ 1)+\frac{1}{9}\,(3^2\,1)+\frac{1}{12}\,(3\ \ 2^2)+\frac{1}{36}\,(3\ \ 1^4)+\frac{1}{24}\,(2^2\,1^3)+\frac{1}{2520}\,(1^7) \tag{9}$$

SymmetricPolynomial(Fraction(Integer))

The operation cyclic returns the cycle index of the cyclic group.

```
cyclic 7
```

$$\frac{6}{7}\,(7)+\frac{1}{7}\,(1^7) \tag{10}$$

SymmetricPolynomial(Fraction(Integer))

The operation dihedral is the cycle index of the dihedral group.

```
dihedral 7
```

$$\frac{3}{7}\,(7)+\frac{1}{2}\,(2^3\,1)+\frac{1}{14}\,(1^7) \tag{11}$$

SymmetricPolynomial(Fraction(Integer))

The operation graphs for argument `n` returns the cycle index of the group of permutations on the edges of the complete graph with `n` nodes induced by applying the symmetric group to the nodes.

```
graphs 5
```

$$\frac{1}{6}\,(6\ \ 3\ \ 1)+\frac{1}{5}\,(5^2)+\frac{1}{4}\,(4^2\,2)+\frac{1}{6}\,(3^3\,1)+\frac{1}{8}\,(2^4\,1^2)+\frac{1}{12}\,(2^3\,1^4)+\frac{1}{120}\,(1^{10}) \tag{12}$$

SymmetricPolynomial(Fraction(Integer))

The cycle index of a direct product of two groups is the product of the cycle indices of the groups. Redfield provided two operations on two cycle indices which will be called "cup" and "cap" here. The cup of two cycle indices is a kind of scalar product that combines monomials for permutations with the same cycles. The cap operation provides the sum of the coefficients of the result of the cup operation which will be an integer that enumerates what Redfield called group-reduced distributions.

We can, for example, represent `complete 2 * complete 2` as the set of objects `a a b b` and `complete 2 * complete 1 * complete 1` as `c c d e.`

This integer is the number of different sets of four pairs.

```
cap(complete 2^2, complete 2*complete 1^2)
```

$$4 \tag{13}$$

Fraction (Integer)

For example,

```
a a b b     a a b b     a a b b   a a b b
c c d e     c d c e     c e c d   d e c c
```

This integer is the number of different sets of four pairs no two pairs being equal.

```
cap(elementary 2^2, complete 2*complete 1^2)
```

$$2 \tag{14}$$

Fraction (Integer)

For example,

```
a a b b     a a b b
c d c e     c e c d
```

In this case the configurations enumerated are easily constructed, however the theory merely enumerates them providing little help in actually constructing them. Here are the number of 6-pairs, first from `a a a b b c,` second from `d d e e f g.`

```
cap(complete 3*complete 2*complete 1,complete 2^2*complete 1^2)
```

$$24 \tag{15}$$

Fraction (Integer)

Here it is again, but with no equal pairs.

```
cap(elementary 3*elementary 2*elementary 1,complete 2^2*complete 1^2)
```

$$8 \tag{16}$$

Fraction (Integer)

```
cap(complete 3*complete 2*complete 1,elementary 2^2*elementary 1^2)
```

$$8 \tag{17}$$

Fraction (Integer)

The number of 6-triples, first from `a a a b b c`, second from `d d e e f g`, third from `h h i i j j`.

```
eval(cup(complete 3*complete 2*complete 1, cup(complete 2^2*complete 1^2,complete _
    2^3)))
```

$$1500 \tag{18}$$

Fraction (Integer)

The cycle index of vertices of a square is dihedral 4.

```
square:=dihedral 4
```

$$\frac{1}{4}\,(4) + \frac{3}{8}\,(2^2) + \frac{1}{4}\,(2\ \ 1^2) + \frac{1}{8}\,(1^4) \tag{19}$$

SymmetricPolynomial(Fraction(Integer))

The number of different squares with 2 red vertices and 2 blue vertices.

```
cap(complete 2^2,square)
```

$$2 \tag{20}$$

Fraction (Integer)

The number of necklaces with 3 red beads, 2 blue beads and 2 green beads.

```
cap(complete 3*complete 2^2,dihedral 7)
```

$$18 \tag{21}$$

Fraction (Integer)

The number of graphs with 5 nodes and 7 edges.

```
cap(graphs 5,complete 7*complete 3)
```

$$4 \tag{22}$$

Fraction (Integer)

The cycle index of rotations of vertices of a cube.

```
s(x) == powerSum(x)
```

```
cube:=(1/24)*(s 1^8+9*s 2^4 + 8*s 3^2*s 1^2+6*s 4^2)
```

```
   Compiling function s with type PositiveInteger -> SymmetricPolynomial(Fraction(
      Integer))
```

$$\frac{1}{4}\left(4^2\right)+\frac{1}{3}\left(3^2\,1^2\right)+\frac{3}{8}\left(2^4\right)+\frac{1}{24}\left(1^8\right) \tag{24}$$

SymmetricPolynomial(Fraction(Integer))

The number of cubes with 4 red vertices and 4 blue vertices.

```
cap(complete 4^2,cube)
```

$$7 \tag{25}$$

Fraction(Integer)

The number of labeled graphs with degree sequence 2 2 2 1 1 with no loops or multiple edges.

```
cap(complete 2^3*complete 1^2,wreath(elementary 4,elementary 2))
```

$$7 \tag{26}$$

Fraction(Integer)

Again, but with loops allowed but not multiple edges.

```
cap(complete 2^3*complete 1^2,wreath(elementary 4,complete 2))
```

$$17 \tag{27}$$

Fraction(Integer)

Again, but with multiple edges allowed, but not loops

```
cap(complete 2^3*complete 1^2,wreath(complete 4,elementary 2))
```

$$10 \tag{28}$$

Fraction(Integer)

Again, but with both multiple edges and loops allowed

```
cap(complete 2^3*complete 1^2,wreath(complete 4,complete 2))
```

$$23 \tag{29}$$

Fraction(Integer)

Having constructed a cycle index for a configuration we are at liberty to evaluate the s_i components any way we please. For example we can produce enumerating generating functions. This is done by providing a function `f` on an integer `i` to the value required of s_i, and then evaluating `eval(f, cycleindex)`.

```
x: ULS(FRAC INT,'x,0) := 'x
```

$$x \tag{30}$$

UnivariateLaurentSeries (Fraction (Integer) , x, 0)

```
ZeroOrOne: INT -> ULS(FRAC INT, 'x, 0)
```

```
Integers: INT -> ULS(FRAC INT, 'x, 0)
```

For the integers 0 and 1, or two colors.

```
ZeroOrOne n == 1+x^n
```

```
ZeroOrOne 5
```

```
   Compiling function ZeroOrOne with type Integer -> UnivariateLaurentSeries(
      Fraction(Integer),x,0)
```

$$1 + x^5 \tag{34}$$

UnivariateLaurentSeries (Fraction (Integer) , x, 0)

For the integers `0, 1, 2, ...` we have this.

```
Integers n == 1/(1-x^n)
```

```
Integers 5
```

```
   Compiling function Integers with type Integer -> UnivariateLaurentSeries(
      Fraction(Integer),x,0)
```

$$1 + x^5 + O\left(x^8\right) \tag{36}$$

UnivariateLaurentSeries (Fraction (Integer) , x, 0)

The coefficient of x^n is the number of graphs with 5 nodes and `n` edges.

```
eval(ZeroOrOne, graphs 5)
```

$$1 + x + 2\,x^2 + 4\,x^3 + 6\,x^4 + 6\,x^5 + 6\,x^6 + 4\,x^7 + O\left(x^8\right) \tag{37}$$

UnivariateLaurentSeries (Fraction (Integer) , x, 0)

The coefficient of x^n is the number of necklaces with `n` red beads and `n-8` green beads.

```
eval(ZeroOrOne,dihedral 8)
```

$$1 + x + 4\,x^2 + 5\,x^3 + 8\,x^4 + 5\,x^5 + 4\,x^6 + x^7 + O\left(x^8\right) \tag{38}$$

UnivariateLaurentSeries (Fraction (Integer) , x, 0)

The coefficient of x^n is the number of partitions of `n` into 4 or fewer parts.

```
eval(Integers,complete 4)
```

$$1 + x + 2\,x^2 + 3\,x^3 + 5\,x^4 + 6\,x^5 + 9\,x^6 + 11\,x^7 + O\left(x^8\right) \tag{39}$$

UnivariateLaurentSeries (Fraction (Integer), x, 0)

The coefficient of x^n is the number of partitions of `n` into 4 boxes containing ordered distinct parts.

```
eval(Integers,elementary 4)
```

$$x^6 + x^7 + 2\,x^8 + 3\,x^9 + 5\,x^{10} + 6\,x^{11} + 9\,x^{12} + 11\,x^{13} + O\left(x^{14}\right) \tag{40}$$

UnivariateLaurentSeries (Fraction (Integer), x, 0)

The coefficient of x^n is the number of different cubes with `n` red vertices and `8-n` green ones.

```
eval(ZeroOrOne,cube)
```

$$1 + x + 3\,x^2 + 3\,x^3 + 7\,x^4 + 3\,x^5 + 3\,x^6 + x^7 + O\left(x^8\right) \tag{41}$$

UnivariateLaurentSeries (Fraction (Integer), x, 0)

The coefficient of x^n is the number of different cubes with integers on the vertices whose sum is `n.`

```
eval(Integers,cube)
```

$$1 + x + 4\,x^2 + 7\,x^3 + 21\,x^4 + 37\,x^5 + 85\,x^6 + 151\,x^7 + O\left(x^8\right) \tag{42}$$

UnivariateLaurentSeries (Fraction (Integer), x, 0)

The coefficient of x^n is the number of graphs with 5 nodes and with integers on the edges whose sum is `n.` In other words, the enumeration is of multigraphs with 5 nodes and `n` edges.

```
eval(Integers,graphs 5)
```

$$1 + x + 3\,x^2 + 7\,x^3 + 17\,x^4 + 35\,x^5 + 76\,x^6 + 149\,x^7 + O\left(x^8\right) \tag{43}$$

UnivariateLaurentSeries (Fraction (Integer), x, 0)

Graphs with 15 nodes enumerated with respect to number of edges.

```
eval(ZeroOrOne ,graphs 15)
```

$$1 + x + 2\,x^2 + 5\,x^3 + 11\,x^4 + 26\,x^5 + 68\,x^6 + 177\,x^7 + O\left(x^8\right) \tag{44}$$

UnivariateLaurentSeries (Fraction (Integer), x, 0)

Necklaces with 7 green beads, 8 white beads, 5 yellow beads and 10 red beads.

```
cap(dihedral 30,complete 7*complete 8*complete 5*complete 10)
```

$$49958972383320 \tag{45}$$

Fraction (Integer)

The operation SFunction is the S-function or Schur function of a partition written as a descending list of integers expressed in terms of power sum symmetric functions. In this case the argument partition represents a tableau shape. For example `3,2,2,1` represents a tableau with three boxes in the first row, two boxes in the second and third rows, and one box in the fourth row. `SFunction [3,2,2,1]` counts

the number of different tableaux of shape `3, 2, 2, 1` filled with objects with an ascending order in the columns and a non-descending order in the rows.

```
sf3221:= SFunction [3,2,2,1]
```

$$\frac{1}{12}\,(6\;\;2)-\frac{1}{12}\,(6\;\;1^2)-\frac{1}{16}\,(4^2)+\frac{1}{12}\,(4\;\;3\;\;1)+\frac{1}{24}\,(4\;\;1^4)$$
$$-\frac{1}{36}\,(3^2\,2)+\frac{1}{36}\,(3^2\,1^2)-\frac{1}{24}\,(3\;\;2^2\,1)-\frac{1}{36}\,(3\;\;2\;\;1^3)-\frac{1}{72}\,(3\;\;1^5)$$
$$-\frac{1}{192}\,(2^4)+\frac{1}{48}\,(2^3\,1^2)+\frac{1}{96}\,(2^2\,1^4)-\frac{1}{144}\,(2\;\;1^6)+\frac{1}{576}\,(1^8) \tag{46}$$

SymmetricPolynomial(Fraction(Integer))

This is the number filled with `a a b b c c d d.`

```
cap(sf3221,complete 2^4)
```

$$3 \tag{47}$$

Fraction(Integer)

The configurations enumerated above are:

```
a a b    a a c    a a d
b c      b b      b b
c d      c d      c c
d        d        d
```

This is the number of tableaux filled with `1..8.`

```
cap(sf3221, powerSum 1^8)
```

$$70 \tag{48}$$

Fraction(Integer)

The coefficient of x^n is the number of column strict reverse plane partitions of `n` of shape `3 2 2 1.`

```
eval(Integers, sf3221)
```

$$x^9+3\,x^{10}+7\,x^{11}+14\,x^{12}+27\,x^{13}+47\,x^{14}+O\left(x^{15}\right) \tag{49}$$

UnivariateLaurentSeries(Fraction(Integer), x, 0)

The smallest is

```
0 0 0
1 1
2 2
3
```

9.14 DeRhamComplex

The domain constructor **DeRhamComplex** creates the class of differential forms of arbitrary degree over a coefficient ring. The De Rham complex constructor takes two arguments: a ring, `coefRing,` and a list of coordinate variables.

This is the ring of coefficients.

```
coefRing := Integer
```

$$Integer \tag{4}$$

Type

These are the coordinate variables.

```
lv : List Symbol := [x,y,z]
```

$$[x,\ y,\ z] \tag{5}$$

List (Symbol)

This is the De Rham complex of Euclidean three-space using coordinates `x, y` and `z.`

```
der := DERHAM(coefRing,lv)
```

$$\text{DeRhamComplex}(Integer, [x,\ y,\ z]) \tag{6}$$

Type

This complex allows us to describe differential forms having expressions of integers as coefficients. These coefficients can involve any number of variables, for example, `f(x,t,r,y,u,z).` As we've chosen to work with ordinary Euclidean three-space, expressions involving these forms are treated as functions of `x, y` and `z` with the additional arguments `t, r` and `u` regarded as symbolic constants. Here are some examples of coefficients.

```
R := Expression coefRing
```

$$\text{Expression}(Integer) \tag{7}$$

Type

```
f : R := x^2*y*z-5*x^3*y^2*z^5
```

$$-5\,x^3\,y^2\,z^5 + x^2\,y\,z \tag{8}$$

Expression (Integer)

```
g : R := z^2*y*cos(z)-7*sin(x^3*y^2)*z^2
```

$$-7\,z^2\,\sin\left(x^3\,y^2\right) + y\,z^2\,\cos(z) \tag{9}$$

Expression (Integer)

```
h : R :=x*y*z-2*x^3*y*z^2
```

$$-2\,x^3\,y\,z^2+x\,y\,z \tag{10}$$

Expression (Integer)

We now define the multiplicative basis elements for the exterior algebra over R.

```
dx : der := generator(1)
```

$$dx \tag{11}$$

DeRhamComplex(Integer, [x, y, z])

```
dy : der := generator(2)
```

$$dy \tag{12}$$

DeRhamComplex(Integer, [x, y, z])

```
dz : der := generator(3)
```

$$dz \tag{13}$$

DeRhamComplex(Integer, [x, y, z])

This is an alternative way to give the above assignments.

```
[dx,dy,dz] := [generator(i)$der for i in 1..3]
```

$$[dx,\ dy,\ dz] \tag{14}$$

List (DeRhamComplex(Integer, [x, y, z]))

Now we define some one-forms.

```
alpha : der := f*dx + g*dy + h*dz
```

$$\left(-2\,x^3\,y\,z^2+x\,y\,z\right)\,dz+\left(-7\,z^2\,\sin\left(x^3\,y^2\right)+y\,z^2\,\cos(z)\right)\,dy+\left(-5\,x^3\,y^2\,z^5+x^2\,y\,z\right)\,dx \tag{15}$$

DeRhamComplex(Integer, [x, y, z])

```
beta  : der := cos(tan(x*y*z)+x*y*z)*dx + x*dy
```

$$x\,dy+\cos(\tan(x\,y\,z)+x\,y\,z)\,dx \tag{16}$$

DeRhamComplex(Integer, [x, y, z])

A well-known theorem states that the composition of exteriorDifferential with itself is the zero map for continuous forms. Let's verify this theorem for alpha.

```
exteriorDifferential alpha;
```

DeRhamComplex(Integer, [x, y, z])

We suppressed the lengthy output of the last expression, but nevertheless, the composition is zero.

```
exteriorDifferential %
```

$$0 \tag{18}$$

DeRhamComplex(Integer, [x, y, z])

Now we check that exteriorDifferential is a "graded derivation" `D`, that is, `D` satisfies:

$$D(ab) = D(a)b + (-1)^{degree(a)} a D(b)$$

```
gamma := alpha * beta
```

$$\begin{aligned}&\left(2\,x^4\,y\,z^2 - x^2\,y\,z\right)\,dy\,dz + \left(2\,x^3\,y\,z^2 - x\,y\,z\right)\,\cos(\tan(x\,y\,z) + x\,y\,z)\,dx\,dz \\ &+ \left(\left(7\,z^2\,\sin\left(x^3\,y^2\right) - y\,z^2\,\cos(z)\right)\,\cos(\tan(x\,y\,z) + x\,y\,z) - 5\,x^4\,y^2\,z^5 + x^3\,y\,z\right)\,dx\,dy\end{aligned} \tag{19}$$

DeRhamComplex(Integer, [x, y, z])

We try this for the one-forms `alpha` and `beta`.

```
exteriorDifferential(gamma) - (exteriorDifferential(alpha)*beta - alpha * _
    exteriorDifferential(beta))
```

$$0 \tag{20}$$

DeRhamComplex(Integer, [x, y, z])

Now we define some "basic operators" (see '`Operator`' on page 519).

```
a : BOP := operator('a)
```

$$a \tag{21}$$

BasicOperator

```
b : BOP := operator('b)
```

$$b \tag{22}$$

BasicOperator

```
c : BOP := operator('c)
```

$$c \tag{23}$$

BasicOperator

We also define some indeterminate one- and two-forms using these operators.

```
sigma := a(x,y,z) * dx + b(x,y,z) * dy + c(x,y,z) * dz
```

$$c(x,y,z)\,dz + b(x,y,z)\,dy + a(x,y,z)\,dx \tag{24}$$

DeRhamComplex(Integer, [x, y, z])

```
theta  := a(x,y,z) * dx * dy + b(x,y,z) * dx * dz + c(x,y,z) * dy * dz
```

$$c(x,y,z)\,dy\,dz + b(x,y,z)\,dx\,dz + a(x,y,z)\,dx\,dy \tag{25}$$

DeRhamComplex(Integer, [x, y, z])

This allows us to get formal definitions for the "gradient" ...

```
totalDifferential(a(x,y,z))$der
```

$$a_{,3}(x,y,z)\,dz + a_{,2}(x,y,z)\,dy + a_{,1}(x,y,z)\,dx \tag{26}$$

DeRhamComplex(Integer, [x, y, z])

the "curl" ...

```
exteriorDifferential sigma
```

$$(c_{,2}(x,y,z) - b_{,3}(x,y,z))\;dy\,dz + (c_{,1}(x,y,z) - a_{,3}(x,y,z))\;dx\,dz + (b_{,1}(x,y,z) - a_{,2}(x,y,z))\;dx\,dy \tag{27}$$

DeRhamComplex(Integer, [x, y, z])

and the "divergence."

```
exteriorDifferential theta
```

$$(c_{,1}(x,y,z) - b_{,2}(x,y,z) + a_{,3}(x,y,z))\;dx\,dy\,dz \tag{28}$$

DeRhamComplex(Integer, [x, y, z])

Note that the De Rham complex is an algebra with unity. This element `1` is the basis for elements for zero-forms, that is, functions in our space.

```
one : der := 1
```

$$1 \tag{29}$$

DeRhamComplex(Integer, [x, y, z])

To convert a function to a function lying in the De Rham complex, multiply the function by "one."

```
g1 : der := a([x,t,y,u,v,z,e]) * one
```

$$a(x,t,y,u,v,z,e) \tag{30}$$

DeRhamComplex(Integer, [x, y, z])

A current limitation of FriCAS forces you to write functions with more than four arguments using square brackets in this way.

```
h1 : der := a([x,y,x,t,x,z,y,r,u,x]) * one
```

$$a(x, y, x, t, x, z, y, r, u, x) \tag{31}$$

DeRhamComplex(Integer, [x, y, z])

Now note how the system keeps track of where your coordinate functions are located in expressions.

```
exteriorDifferential g1
```

$$a_{,6}(x, t, y, u, v, z, e)\, dz + a_{,3}(x, t, y, u, v, z, e)\, dy + a_{,1}(x, t, y, u, v, z, e)\, dx \tag{32}$$

DeRhamComplex(Integer, [x, y, z])

```
exteriorDifferential h1
```

$$\begin{aligned} &a_{,6}(x, y, x, t, x, z, y, r, u, x)\, dz + (a_{,7}(x, y, x, t, x, z, y, r, u, x) + a_{,2}(x, y, x, t, x, z, y, r, u, x))\, dy \\ &\quad + (a_{,10}(x, y, x, t, x, z, y, r, u, x) + a_{,5}(x, y, x, t, x, z, y, r, u, x) \\ &\quad + a_{,3}(x, y, x, t, x, z, y, r, u, x) + a_{,1}(x, y, x, t, x, z, y, r, u, x))\, dx \end{aligned} \tag{33}$$

DeRhamComplex(Integer, [x, y, z])

In this example of Euclidean three-space, the basis for the De Rham complex consists of the eight forms: `1`, `dx`, `dy`, `dz`, `dx*dy`, `dx*dz`, `dy*dz`, and `dx*dy*dz`.

```
coefficient(gamma, dx*dy)
```

$$\left(7\, z^2\, \sin\left(x^3\, y^2\right) - y\, z^2\, \cos(z)\right)\, \cos(\tan(x\, y\, z) + x\, y\, z) - 5\, x^4\, y^2\, z^5 + x^3\, y\, z \tag{34}$$

Expression(Integer)

```
coefficient(gamma, one)
```

$$0 \tag{35}$$

Expression(Integer)

```
coefficient(g1,one)
```

$$a(x, t, y, u, v, z, e) \tag{36}$$

Expression(Integer)

9.15 DecimalExpansion

All rationals have repeating decimal expansions. Operations to access the individual digits of a decimal expansion can be obtained by converting the value to **RadixExpansion(10)**. More examples of expansions are available in '`BinaryExpansion`' on page 348, '`HexadecimalExpansion`' on page 440, and '`RadixExpansion`' on page 541. Issue the system command `)show DecimalExpansion` to display the full list of operations defined by **DecimalExpansion**.

The operation decimal is used to create this expansion of type **DecimalExpansion**.

```
r := decimal(22/7)
```

$$3.\overline{142857} \tag{4}$$

DecimalExpansion

Arithmetic is exact.

```
r + decimal(6/7)
```

$$4 \tag{5}$$

DecimalExpansion

The period of the expansion can be short or long ...

```
[decimal(1/i) for i in 350..354]
```

$$\left[0.00\overline{285714},\ 0.\overline{002849},\ 0.0028\overline{409},\ 0.\overline{00283286118980169971671388101983},\right. \tag{6}$$
$$\left.0.00\overline{28248587570621468926553672316384180790960451977401129943 5}\right]$$

List (DecimalExpansion)

or very long.

```
decimal(1/2049)
```

$$0.\overline{000488042947779404587603709126403123474865788189360663738408979990239141044411908247925817471(7)} \tag{7}$$

DecimalExpansion

These numbers are bona fide algebraic objects.

```
p := decimal(1/4)*x^2 + decimal(2/3)*x + decimal(4/9)
```

$$0.25\,x^2 + 0.\overline{6}\,x + 0.\overline{4} \tag{8}$$

Polynomial(DecimalExpansion)

```
q := differentiate(p, x)
```

$$0.5\,x + 0.\overline{6} \tag{9}$$

Polynomial(DecimalExpansion)

```
g := gcd(p, q)
```

$$x + 1.\overline{3} \tag{10}$$

Polynomial(DecimalExpansion)

9.16 DistributedMultivariatePolynomial

DistributedMultivariatePolynomial and **HomogeneousDistributedMultivariatePolynomial**, abbreviated **DMP** and **HDMP**, respectively, are very similar to **MultivariatePolynomial** except that they are represented and displayed in a non-recursive manner.

```
(d1,d2,d3) : DMP([z,y,x],FRAC INT)
```

The constructor **DMP** orders its monomials lexicographically while **HDMP** orders them by total order refined by reverse lexicographic order.

```
d1 := -4*z + 4*y^2*x + 16*x^2 + 1
```

$$-4\,z+4\,y^2\,x+16\,x^2+1 \tag{5}$$

DistributedMultivariatePolynomial ([z, y, x], Fraction (Integer))

```
d2 := 2*z*y^2 + 4*x + 1
```

$$2\,z\,y^2+4\,x+1 \tag{6}$$

DistributedMultivariatePolynomial ([z, y, x], Fraction (Integer))

```
d3 := 2*z*x^2 - 2*y^2 - x
```

$$2\,z\,x^2-2\,y^2-x \tag{7}$$

DistributedMultivariatePolynomial ([z, y, x], Fraction (Integer))

These constructors are mostly used in Gröbner basis calculations.

```
groebner [d1,d2,d3]
```

$$\left[z-\frac{1568}{2745}\,x^6-\frac{1264}{305}\,x^5+\frac{6}{305}\,x^4+\frac{182}{549}\,x^3-\frac{2047}{610}\,x^2-\frac{103}{2745}\,x-\frac{2857}{10980},\right.$$
$$y^2+\frac{112}{2745}\,x^6-\frac{84}{305}\,x^5-\frac{1264}{305}\,x^4-\frac{13}{549}\,x^3+\frac{84}{305}\,x^2+\frac{1772}{2745}\,x+\frac{2}{2745},$$
$$\left.x^7+\frac{29}{4}\,x^6-\frac{17}{16}\,x^4-\frac{11}{8}\,x^3+\frac{1}{32}\,x^2+\frac{15}{16}\,x+\frac{1}{4}\right] \tag{8}$$

List (DistributedMultivariatePolynomial ([z, y, x], Fraction (Integer)))

```
(n1,n2,n3) : HDMP([z,y,x],FRAC INT)
```

```
(n1,n2,n3) := (d1,d2,d3)
```

$$2\,z\,x^2-2\,y^2-x \tag{10}$$

HomogeneousDistributedMultivariatePolynomial([z, y, x], Fraction (Integer))

Note that we get a different Gröbner basis when we use the **HDMP** polynomials, as expected.

```
groebner [n1,n2,n3]
```

$$\left[y^4+2\,x^3-\frac{3}{2}\,x^2+\frac{1}{2}\,z-\frac{1}{8},\ x^4+\frac{29}{4}\,x^3-\frac{1}{8}\,y^2-\frac{7}{4}\,z\,x-\frac{9}{16}\,x-\frac{1}{4},\ z\,y^2+2\,x+\frac{1}{2},\right.$$
$$\left.y^2\,x+4\,x^2-z+\frac{1}{4},\ z\,x^2-y^2-\frac{1}{2}\,x,\ z^2-4\,y^2+2\,x^2-\frac{1}{4}\,z-\frac{3}{2}\,x\right] \tag{11}$$

List(HomogeneousDistributedMultivariatePolynomial([z, y, x], Fraction(Integer)))

GeneralDistributedMultivariatePolynomial is somewhat more flexible in the sense that as well as accepting a list of variables to specify the variable ordering, it also takes a predicate on exponent vectors to specify the term ordering. With this polynomial type the user can experiment with the effect of using completely arbitrary term orderings. This flexibility is mostly important for algorithms such as Gröbner basis calculations which can be very sensitive to term ordering.

For more information on related topics, see Section 1.9 on page 48, Section 2.7 on page 84, '`Polynomial`' on page 531, '`UnivariatePolynomial`' on page 596, and '`MultivariatePolynomial`' on page 512. Issue the system command `)show DistributedMultivariatePolynomial` to display the full list of operations defined by **DistributedMultivariatePolynomial**.

9.17 DoubleFloat

FriCAS provides two kinds of floating point numbers. The domain **Float** (abbreviation **FLOAT**) implements a model of arbitrary precision floating point numbers. The domain **DoubleFloat** (abbreviation **DFLOAT**) is intended to make available hardware floating point arithmetic in FriCAS. The actual model of floating point **DoubleFloat** that provides is system-dependent. In the past there were wide variety of floating point formats. For example, the IBM system 370 used hexadecimal format such that double precision number had fourteen hexadecimal digits of precision or roughly sixteen decimal digits. All system currently supported by FriCAS use IEEE binary format with 64-bit double having sign bit, 11 exponents bits and 53 significant bits (that adds to 65, but most significant bit is 1 and there is no need to store it).

Arbitrary precision floats allow the user to specify the precision at which arithmetic operations are computed. Although this is an attractive facility, it comes at a cost. Arbitrary-precision floating-point arithmetic typically takes twenty to two hundred times more time than hardware floating point.

The usual arithmetic and elementary functions are available for **DoubleFloat**. Use `)show DoubleFloat` to get a list of operations or the HyperDoc Browse facility to get more extensive documentation about **DoubleFloat**.

By default, floating point numbers that you enter into FriCAS are of type **Float**.

```
2.71828
```

$$2.71828 \tag{4}$$

Float

You must therefore tell FriCAS that you want to use **DoubleFloat** values and operations. The following are some conservative guidelines for getting FriCAS to use **DoubleFloat**.

To get a value of type **DoubleFloat**, use a target with "`@`", ...

```
2.71828@DoubleFloat
```

$$2.71828 \tag{5}$$

DoubleFloat

a conversion, ...

```
2.71828 :: DoubleFloat
```

$$2.71828 \tag{6}$$

DoubleFloat

or an assignment to a declared variable. It is more efficient if you use a target rather than an explicit or implicit conversion.

```
eApprox : DoubleFloat := 2.71828
```

$$2.71828 \tag{7}$$

DoubleFloat

You also need to declare functions that work with **DoubleFloat**.

```
avg : List DoubleFloat -> DoubleFloat
```

```
avg l ==
  empty? l => 0 :: DoubleFloat
  reduce(_+,l) / #l
```

```
avg []
```

```
   Compiling function avg with type List(DoubleFloat) -> DoubleFloat
```

$$0.0 \tag{10}$$

DoubleFloat

```
avg [3.4,9.7,-6.8]
```

$$2.1 \tag{11}$$

DoubleFloat

Use package-calling for operations from **DoubleFloat** unless the arguments themselves are already of type **DoubleFloat**.

```
cos(3.1415926)$DoubleFloat
```

$$-0.9999999999999986 \tag{12}$$

DoubleFloat

```
cos(3.1415926 :: DoubleFloat)
```

$$-0.9999999999999986 \tag{13}$$

DoubleFloat

By far, the most common usage of **DoubleFloat** is for functions to be graphed. For more information about FriCAS's numerical and graphical facilities, see Section 7 on page 195, Section 8.1 on page 251, and '`Float`' on page 419.

9.18 EqTable

The **EqTable** domain provides tables where the keys are compared using eq?. Keys are considered equal only if they are the same instance of a structure. This is useful if the keys are themselves updatable structures. Otherwise, all operations are the same as for type **Table**. See '`Table`' on page 587 for general information about tables. Issue the system command `)show EqTable` to display the full list of operations defined by **EqTable**.

The operation table is here used to create a table where the keys are lists of integers.

```
e: EqTable(List Integer, Integer) := table()
```

table() (4)

EqTable(List(Integer), Integer)

These two lists are equal according to =, but not according to eq?.

```
l1 := [1,2,3]
```

[1, 2, 3] (5)

List(PositiveInteger)

```
l2 := [1,2,3]
```

[1, 2, 3] (6)

List(PositiveInteger)

Because the two lists are not eq?, separate values can be stored under each.

```
e.l1 := 111
```

111 (7)

PositiveInteger

```
e.l2 := 222
```

222 (8)

PositiveInteger

```
e.l1
```

$$111 \tag{9}$$

PositiveInteger

9.19 Equation

The **Equation** domain provides equations as mathematical objects. These are used, for example, as the input to various solve operations.

Equations are created using the equals symbol, =.

```
eq1 := 3*x + 4*y = 5
```

$$4\,y + 3\,x = 5 \tag{4}$$

Equation(Polynomial(Integer))

```
eq2 := 2*x + 2*y = 3
```

$$2\,y + 2\,x = 3 \tag{5}$$

Equation(Polynomial(Integer))

The left- and right-hand sides of an equation are accessible using the operations lhs and rhs.

```
lhs eq1
```

$$4\,y + 3\,x \tag{6}$$

Polynomial(Integer)

```
rhs eq1
```

$$5 \tag{7}$$

Polynomial(Integer)

Arithmetic operations are supported and operate on both sides of the equation.

```
eq1 + eq2
```

$$6\,y + 5\,x = 8 \tag{8}$$

Equation(Polynomial(Integer))

```
eq1 * eq2
```

$$8\,y^2 + 14\,x\,y + 6\,x^2 = 15 \tag{9}$$

Equation(Polynomial(Integer))

```
2*eq2 - eq1
```

$$x = 1 \tag{10}$$

Equation(Polynomial(Integer))

Equations may be created for any type so the arithmetic operations will be defined only when they make sense. For example, exponentiation is not defined for equations involving non-square matrices.

```
eq1^2
```

$$16\,y^2 + 24\,x\,y + 9\,x^2 = 25 \tag{11}$$

Equation(Polynomial(Integer))

Note that an equals symbol is also used to *test* for equality of values in certain contexts. For example, `x+1` and `y` are unequal as polynomials.

```
if x+1 = y then "equal" else "unequal"
```

$$\texttt{"unequal"} \tag{12}$$

String

```
eqpol := x+1 = y
```

$$x + 1 = y \tag{13}$$

Equation(Polynomial(Integer))

If an equation is used where a **Boolean** value is required, then it is evaluated using the equality test from the operand type.

```
if eqpol then "equal" else "unequal"
```

$$\texttt{"unequal"} \tag{14}$$

String

If one wants a **Boolean** value rather than an equation, all one has to do is ask!

```
eqpol::Boolean
```

$$\texttt{false} \tag{15}$$

Boolean

9.20 Exit

A function that does not return directly to its caller has **Exit** as its return type. The operation error is an example of one which does not return to its caller. Instead, it causes a return to top-level.

```
n := 0
```

$$0 \tag{4}$$

NonNegativeInteger

The function **gasp** is given return type **Exit** since it is guaranteed never to return a value to its caller.

```
gasp(): Exit ==
    free n
    n := n + 1
    error "Oh no!"
```

```
   Function declaration gasp : () -> Exit has been added to workspace.
```

The return type of **half** is determined by resolving the types of the two branches of the `if`.

```
half(k) ==
  if odd? k then gasp()
  else k quo 2
```

Because **gasp** has the return type **Exit**, the type of `if` in **half** is resolved to be **Integer**.

```
half 4
```

```
   Compiling function gasp with type () -> Exit
   Compiling function half with type PositiveInteger -> Integer
```

$$2 \tag{7}$$

PositiveInteger

```
half 3
```

```
   Error signalled from user code in function gasp:
      Oh no!
```

```
n
```

$$1 \tag{8}$$

NonNegativeInteger

For functions which return no value at all, use **Void**. See Section 6 on page 149 and '`Void`' on page 605 for more information. Issue the system command `)show Exit` to display the full list of operations defined by **Exit**.

9.21 Expression

Expression is a constructor that creates domains whose objects can have very general symbolic forms. Here are some examples: This is an object of type **Expression Integer**.

```
sin(x) + 3*cos(x)^2
```

$$\sin(x) + 3\cos(x)^2 \tag{4}$$

Expression (Integer)

This is an object of type **Expression Float**.

```
tan(x) - 3.45*x
```

$$\tan(x) - 3.45\,x \tag{5}$$

Expression (Float)

This object contains symbolic function applications, sums, products, square roots, and a quotient.

```
(tan sqrt 7 - sin sqrt 11)^2 / (4 - cos(x - y))
```

$$\frac{-\tan\left(\sqrt{7}\right)^2 + 2\,\sin\left(\sqrt{11}\right)\,\tan\left(\sqrt{7}\right) - \sin\left(\sqrt{11}\right)^2}{\cos(y - x) - 4} \tag{6}$$

Expression (Integer)

As you can see, **Expression** actually takes an argument domain. The *coefficients* of the terms within the expression belong to the argument domain. **Integer** and **Float**, along with **Complex Integer** and **Complex Float** are the most common coefficient domains. The choice of whether to use a **Complex** coefficient domain or not is important since FriCAS can perform some simplifications on real-valued objects

```
log(exp x)@Expression(Integer)
```

$$x \tag{7}$$

Expression (Integer)

... which are not valid on complex ones.

```
log(exp x)@Expression(Complex Integer)
```

$$\log(e^x) \tag{8}$$

Expression (Complex(Integer))

Many potential coefficient domains, such as **AlgebraicNumber**, are not usually used because **Expression** can subsume them.

```
sqrt 3 + sqrt(2 + sqrt(-5))
```

$$\sqrt{\sqrt{-5} + 2} + \sqrt{3} \tag{9}$$

AlgebraicNumber

```
% :: Expression Integer
```

$$\sqrt{\sqrt{-5} + 2} + \sqrt{3} \tag{10}$$

Expression (Integer)

Note that we sometimes talk about "an object of type **Expression**." This is not really correct because we should say, for example, "an object of type **Expression Integer**" or "an object of type **Expression**

Float." By a similar abuse of language, when we refer to an "expression" in this section we will mean an object of type **Expression R** for some domain **R**.

The FriCAS documentation contains many examples of the use of **Expression**. For the rest of this section, we'll give you some pointers to those examples plus give you some idea of how to manipulate expressions.

It is important for you to know that **Expression** creates domains that have category **Field**. Thus you can invert any non-zero expression and you shouldn't expect an operation like factor to give you much information. You can imagine expressions as being represented as quotients of "multivariate" polynomials where the "variables" are kernels (see '`Kernel`' on page 454). A kernel can either be a symbol such as `x` or a symbolic function application like `sin(x + 4)`. The second example is actually a nested kernel since the argument to sin contains the kernel `x`.

```
height mainKernel sin(x + 4)
```

$$2 \tag{11}$$

PositiveInteger

Actually, the argument to sin is an expression, and so the structure of **Expression** is recursive. '`Kernel`' on page 454 demonstrates how to extract the kernels in an expression.

Use the HyperDoc Browse facility to see what operations are applicable to expression. At the time of this writing, there were 262 operations with 147 distinct name in **Expression Integer**. For example, numer and denom extract the numerator and denominator of an expression.

```
e := (sin(x) - 4)^2 / ( 1 - 2*y*sqrt(- y) )
```

$$\frac{-\sin(x)^2 + 8\,\sin(x) - 16}{2\,y\,\sqrt{-y} - 1} \tag{12}$$

Expression (Integer)

```
numer e
```

$$-\sin(x)^2 + 8\,\sin(x) - 16 \tag{13}$$

SparseMultivariatePolynomial (Integer , Kernel(Expression (Integer)))

```
denom e
```

$$2\,y\,\sqrt{-y} - 1 \tag{14}$$

SparseMultivariatePolynomial (Integer , Kernel(Expression (Integer)))

Use D to compute partial derivatives.

```
D(e, x)
```

$$\frac{(4\,y\,\cos(x)\,\sin(x) - 16\,y\,\cos(x))\,\sqrt{-y} - 2\,\cos(x)\,\sin(x) + 8\,\cos(x)}{4\,y\,\sqrt{-y} + 4\,y^3 - 1} \tag{15}$$

Expression (Integer)

See Section 1.12 on page 53 for more examples of expressions and derivatives.

```
D(e, [x, y], [1, 2])
```

$$\frac{\left(\left(-2304\,y^7+960\,y^4\right)\cos(x)\sin(x)+\left(9216\,y^7-3840\,y^4\right)\cos(x)\right)\sqrt{-y}+\left(-960\,y^9+2160\,y^6-180\,y^3-3\right)\cos(x)\sin(x)+\left(3840\,y^{8}\right.}{\left(256\,y^{12}-1792\,y^9+1120\,y^6-112\,y^3+1\right)\sqrt{-y}-1024\,y^{11}+1792\,y^8-448\,y^5+16\,y^2} \tag{16}$$

Expression (Integer)

See Section 1.10 on page 49 and Section 1.11 on page 51 for more examples of expressions and calculus. Differential equations involving expressions are discussed in Section 8.10 on page 296. Chapter 8 has many advanced examples: see Section 8.8 on page 279 for a discussion of FriCAS's integration facilities.

When an expression involves no "symbol kernels" (for example, `x`), it may be possible to numerically evaluate the expression. If you suspect the evaluation will create a complex number, use complexNumeric.

```
complexNumeric(cos(2 - 3*%i))
```

$$-4.1896256909688072301+9.109227893755336598\,i \tag{17}$$

Complex(Float)

If you know it will be real, use numeric.

```
numeric(tan 3.8)
```

$$0.77355609050312607286 \tag{18}$$

Float

The numeric operation will display an error message if the evaluation yields a value with a non-zero imaginary part. Both of these operations have an optional second argument `n` which specifies that the accuracy of the approximation be up to `n` decimal places.

When an expression involves no "symbolic application" kernels, it may be possible to convert it to a polynomial or rational function in the variables that are present.

```
e2 := cos(x^2 - y + 3)
```

$$\cos\left(y-x^2-3\right) \tag{19}$$

Expression (Integer)

```
e3 := asin(e2) - %pi/2
```

$$\frac{2\,\arcsin\left(\cos\left(y-x^2-3\right)\right)-\pi}{2} \tag{20}$$

Expression (Integer)

```
e4 := normalize(e3)
```

$$-y+x^2+3 \tag{21}$$

Expression(Integer)

```
e4 :: Polynomial Integer
```

$$-y+x^2+3 \tag{22}$$

Polynomial(Integer)

This also works for the polynomial types where specific variables and their ordering are given.

```
e4 :: DMP([x, y], Integer)
```

$$x^2-y+3 \tag{23}$$

DistributedMultivariatePolynomial ([x, y], Integer)

Finally, a certain amount of simplification takes place as expressions are constructed.

```
sin %pi
```

$$0 \tag{24}$$

Expression(Integer)

```
cos(%pi / 4)
```

$$\frac{\sqrt{2}}{2} \tag{25}$$

Expression(Integer)

For simplifications that involve multiple terms of the expression, use simplify.

```
tan(x)^6 + 3*tan(x)^4 + 3*tan(x)^2 + 1
```

$$\tan(x)^6+3\tan(x)^4+3\tan(x)^2+1 \tag{26}$$

Expression(Integer)

```
simplify %
```

$$\frac{1}{\cos(x)^6} \tag{27}$$

Expression(Integer)

See Section 6.21 on page 187 for examples of how to write your own rewrite rules for expressions.

9.22 Factored

Factored creates a domain whose objects are kept in factored form as long as possible. Thus certain operations like * (multiplication) and gcd are relatively easy to do. Others, such as addition, require somewhat more work, and the result may not be completely factored unless the argument domain R provides a factor operation. Each object consists of a unit and a list of factors, where each factor consists of a member of R (the *base*), an exponent, and a flag indicating what is known about the base. A flag may be one of "nil", "sqfr", "irred" or "prime", which mean that nothing is known about the base, it is square-free, it is irreducible, or it is prime, respectively. The current restriction to factored objects of integral domains allows simplification to be performed without worrying about multiplication order.

9.22.1 Decomposing Factored Objects

In this section we will work with a factored integer.

```
g := factor(4312)
```

$$2^3\,7^2\,11 \tag{1}$$

Factored(Integer)

Let's begin by decomposing g into pieces. The only possible units for integers are 1 and -1.

```
unit(g)
```

$$1 \tag{2}$$

PositiveInteger

There are three factors.

```
numberOfFactors(g)
```

$$3 \tag{3}$$

PositiveInteger

We can make a list of the bases, ...

```
[i.factor for i in factorList(g)]
```

$$[2,\ 7,\ 11] \tag{4}$$

List (Integer)

and the exponents, ...

```
[i.exponent for i in factorList(g)]
```

$$[3,\ 2,\ 1] \tag{5}$$

List (NonNegativeInteger)

and the flags. You can see that all the bases (factors) are prime.

```
[i.flag for i in factorList(g)]
```

$$\texttt{["prime", "prime", "prime"]} \tag{6}$$

List (Union(" nil", " sqfr", " irred", "prime"))

A useful operation for pulling apart a factored object into a list of records of the components is factorList.

```
factorList(g)
```

$$[[flag = \texttt{"prime"},\ factor = 2,\ exponent = 3],\ [flag = \texttt{"prime"},\ factor = 7,\ exponent = 2],\ [flag = \texttt{"prime"},\ factor = 11,\ exponent = 1]] \tag{7}$$

List (Record(flag : Union(" nil", " sqfr", " irred", "prime"), factor : Integer , exponent: NonNegativeInteger))

If you don't care about the flags, use factors.

```
factors(g)
```

$$[[factor = 2,\ exponent = 3],\ [factor = 7,\ exponent = 2],\ [factor = 11,\ exponent = 1]] \tag{8}$$

List (Record(factor : Integer , exponent: NonNegativeInteger))

Neither of these operations returns the unit.

```
first(%).factor
```

$$2 \tag{9}$$

PositiveInteger

9.22.2 Expanding Factored Objects

Recall that we are working with this factored integer.

```
g := factor(4312)
```

$$2^3\,7^2\,11 \tag{1}$$

Factored(Integer)

To multiply out the factors with their multiplicities, use expand.

```
expand(g)
```

$$4312 \tag{2}$$

PositiveInteger

If you would like, say, the distinct factors multiplied together but with multiplicity one, you could do it this way.

```
reduce(*,[t.factor for t in factors(g)])
```

$$154 \tag{3}$$

PositiveInteger

9.22.3 Arithmetic with Factored Objects

We're still working with this factored integer.

```
g := factor(4312)
```

$$2^3\,7^2\,11 \tag{1}$$

Factored(Integer)

We'll also define this factored integer.

```
f := factor(246960)
```

$$2^4\,3^2\,5\,7^3 \tag{2}$$

Factored(Integer)

Operations involving multiplication and division are particularly easy with factored objects.

```
f * g
```

$$2^7\,3^2\,5\,7^5\,11 \tag{3}$$

Factored(Integer)

```
f^500
```

$$2^{2000}\,3^{1000}\,5^{500}\,7^{1500} \tag{4}$$

Factored(Integer)

```
gcd(f,g)
```

$$2^3\,7^2 \tag{5}$$

Factored(Integer)

```
lcm(f,g)
```

$$2^4\,3^2\,5\,7^3\,11 \tag{6}$$

Factored(Integer)

If we use addition and subtraction things can slow down because we may need to compute greatest common divisors.

```
f + g
```

$$2^3\,7^2\,641 \tag{7}$$

Factored(Integer)

```
f - g
```

$$2^3\,7^2\,619 \tag{8}$$

Factored(Integer)

Test for equality with 0 and 1 by using zero? and one?, respectively.

```
zero?(factor(0))
```

```
true
```
(9)

Boolean

```
zero?(g)
```

```
false
```
(10)

Boolean

```
one?(factor(1))
```

```
true
```
(11)

Boolean

```
one?(f)
```

```
false
```
(12)

Boolean

Another way to get the zero and one factored objects is to use package calling (see Section 2.9 on page 89).

```
0$Factored(Integer)
```

$$0 \tag{13}$$

Factored(Integer)

```
1$Factored(Integer)
```

$$1 \tag{14}$$

Factored(Integer)

9.22.4 Creating New Factored Objects

The map operation is used to iterate across the unit and bases of a factored object. See '`FactoredFunctions2`' on page 411 for a discussion of map.

The following four operations take a base and an exponent and create a factored object. They differ in handling the flag component.

```
nilFactor(24,2)
```

$$24^2 \tag{1}$$

Factored(Integer)

This factor has no associated information.

```
factorList(%).1.flag
```

$$\texttt{"nil"} \tag{2}$$

Union(" nil", ...)

This factor is asserted to be square-free.

```
sqfrFactor(30,2)
```

$$30^2 \tag{3}$$

Factored(Integer)

This factor is asserted to be irreducible.

```
irreducibleFactor(13,10)
```

$$13^{10} \tag{4}$$

Factored(Integer)

This factor is asserted to be prime.

```
primeFactor(11,5)
```

$$11^5 \tag{5}$$

Factored(Integer)

A partial inverse to factorList is makeFR.

```
h := factor(-720)
```

$$-2^4\,3^2\,5 \tag{6}$$

Factored(Integer)

The first argument is the unit and the second is a list of records as returned by factorList.

```
h - makeFR(unit(h),factorList(h))
```

$$0 \tag{7}$$

Factored(Integer)

9.22.5 Factored Objects with Variables

Some of the operations available for polynomials are also available for factored polynomials.

```
p := (4*x*x-12*x+9)*y*y + (4*x*x-12*x+9)*y + 28*x*x - 84*x + 63
```

$$\left(4\,x^2-12\,x+9\right)\,y^2+\left(4\,x^2-12\,x+9\right)\,y+28\,x^2-84\,x+63 \tag{1}$$

Polynomial(Integer)

```
fp := factor(p)
```

$$(2\,x-3)^2\,\left(y^2+y+7\right) \tag{2}$$

Factored(Polynomial(Integer))

You can differentiate with respect to a variable.

```
D(p,x)
```

$$(8\,x-12)\,y^2+(8\,x-12)\,y+56\,x-84 \tag{3}$$

Polynomial(Integer)

```
D(fp,x)
```

$$4\,(2\,x-3)\,\left(y^2+y+7\right) \tag{4}$$

Factored(Polynomial(Integer))

```
numberOfFactors(%)
```

$$2 \tag{5}$$

PositiveInteger

9.23 FactoredFunctions2

The **FactoredFunctions2** package implements one operation, map, for applying an operation to every base in a factored object and to the unit.

```
double(x) == x + x
```

```
f := factor(720)
```

$$2^4\,3^2\,5 \tag{5}$$

Factored(Integer)

Actually, the map operation used in this example comes from **Factored** itself, since **double** takes an integer argument and returns an integer result.

```
map(double,f)
```

```
   Compiling function double with type Integer -> Integer
```

$$2\,4^4\,6^2\,10 \tag{6}$$

Factored(Integer)

If we want to use an operation that returns an object that has a type different from the operation's argument, the map in **Factored** cannot be used and we use the one in **FactoredFunctions2**.

```
makePoly(b) == x + b
```

In fact, the "2" in the name of the package means that we might be using factored objects of two different types.

```
g := map(makePoly,f)
```

```
   Compiling function makePoly with type Integer -> Polynomial(Integer)
```

$$(x+1)\,(x+2)^4\,(x+3)^2\,(x+5) \tag{8}$$

Factored(Polynomial(Integer))

It is important to note that both versions of map destroy any information known about the bases (the fact that they are prime, for instance). The flags for each base are set to "nil" in the object returned by map.

```
factorList(g).1.flag
```

```
"nil"                                                    (9)
```

Union(" nil", ...)

For more information about factored objects and their use, see '`Factored`' on page 405 and Section 8.13 on page 324.

9.24 File

The **File(S)** domain provides a basic interface to read and write values of type S in files. Before working with a file, it must be made accessible to FriCAS with the open operation.

```
ifile:File List Integer:=open("/tmp/jazz1","output")
```

"/tmp/jazz1" (4)

File (List (Integer))

The open function arguments are a **FileName** and a **String** specifying the mode. If a full pathname is not specified, the current default directory is assumed. The mode must be one of "input" or "output". If it is not specified, "input" is assumed. Once the file has been opened, you can read or write data. The operations read! and write! are provided.

```
write!(ifile, [-1,2,3])
```

$$[-1,\ 2,\ 3] \tag{5}$$

List (Integer)

```
write!(ifile, [10,-10,0,111])
```

$$[10,\ -10,\ 0,\ 111] \tag{6}$$

List (Integer)

```
write!(ifile, [7])
```

$$[7] \tag{7}$$

List (Integer)

You can change from writing to reading (or vice versa) by reopening a file.

```
reopen!(ifile, "input")
```

"/tmp/jazz1" (8)

File (List (Integer))

```
read! ifile
```

$$[-1,\ 2,\ 3] \tag{9}$$

List (Integer)

```
read! ifile
```

$$[10,\ -10,\ 0,\ 111] \tag{10}$$

List (Integer)

The read! operation can cause an error if one tries to read more data than is in the file. To guard against this possibility the readIfCan! operation should be used.

```
readIfCan! ifile
```

[7] (11)

Union(List (Integer), ...)

```
readIfCan! ifile
```

"failed" (12)

Union(" failed ", ...)

You can find the current mode of the file, and the file's name.

```
iomode ifile
```

"input" (13)

String

```
name ifile
```

"/tmp/jazz1" (14)

FileName

When you are finished with a file, you should close it.

```
close! ifile
```

"/tmp/jazz1" (15)

File (List (Integer))

```
)system rm /tmp/jazz1
```

A limitation of the underlying LISP system is that not all values can be represented in a file. In particular, delayed values containing compiled functions cannot be saved.

For more information on related topics, see 'TextFile' on page 591, 'KeyedAccessFile' on page 457, 'Library' on page 474, and 'FileName' on page 413. Issue the system command)show File to display the full list of operations defined by **File**.

9.25 FileName

The **FileName** domain provides an interface to the computer's file system. Functions are provided to manipulate file names and to test properties of files.

The simplest way to use file names in the FriCAS interpreter is to rely on conversion to and from strings. The syntax of these strings depends on the operating system.

```
fn: FileName
```

On AIX, this is a proper file syntax:

```
fn := "/spad/src/input/fname.input"
```

$$\texttt{"/spad/src/input/fname.input"} \tag{5}$$

FileName

Although it is very convenient to be able to use string notation for file names in the interpreter, it is desirable to have a portable way of creating and manipulating file names from within programs. A measure of portability is obtained by considering a file name to consist of three parts: the *directory*, the *name*, and the *extension*.

```
directory fn
```

$$\texttt{"/spad/src/input"} \tag{6}$$

String

```
name fn
```

$$\texttt{"fname"} \tag{7}$$

String

```
extension fn
```

$$\texttt{"input"} \tag{8}$$

String

The meaning of these three parts depends on the operating system. For example, on CMS the file `"SPADPROF INPUT M"` would have directory `"M"`, name `"SPADPROF"` and extension `"INPUT"`.

It is possible to create a filename from its parts.

```
fn := filename("/u/smwatt/work", "fname", "input")
```

$$\texttt{"/u/smwatt/work/fname.input"} \tag{9}$$

FileName

When writing programs, it is helpful to refer to directories via variables.

```
objdir := "/tmp"
```

$$\texttt{"/tmp"} \tag{10}$$

String

```
fn := filename(objdir, "table", "spad")
```

$$\texttt{"/tmp/table.spad"} \tag{11}$$

FileName

If the directory or the extension is given as an empty string, then a default is used. On AIX, the defaults are the current directory and no extension.

```
fn := filename("", "letter", "")
```

```
"letter"                                                    (12)
```

FileName

Three tests provide information about names in the file system. The exists? operation tests whether the named file exists.

```
exists? "/etc/passwd"
```

```
true                                                        (13)
```

Boolean

The operation readable? tells whether the named file can be read. If the file does not exist, then it cannot be read.

```
readable? "/etc/passwd"
```

```
true                                                        (14)
```

Boolean

```
readable? "/etc/security/passwd"
```

```
false                                                       (15)
```

Boolean

```
readable? "/etc/passwd"
```

```
true                                                        (16)
```

Boolean

Likewise, the operation writable? tells whether the named file can be written. If the file does not exist, the test is determined by the properties of the directory.

```
writable? "/etc/passwd"
```

```
false                                                       (17)
```

Boolean

```
writable? "/dev/null"
```

```
true                                                        (18)
```

Boolean

```
writable? "/etc/DoesNotExist"
```

$$\texttt{false} \tag{19}$$

Boolean

```
writable? "/tmp/DoesNotExist"
```

$$\texttt{true} \tag{20}$$

Boolean

The new operation constructs the name of a new writable file. The argument sequence is the same as for filename, except that the name part is actually a prefix for a constructed unique name. The resulting file is in the specified directory with the given extension, and the same defaults are used.

```
fn := new(objdir, "xxx", "yy")
```

$$\texttt{"/tmp/xxx20.yy"} \tag{21}$$

FileName

9.26 FlexibleArray

The **FlexibleArray** domain constructor creates one-dimensional arrays of elements of the same type. Flexible arrays are an attempt to provide a data type that has the best features of both one-dimensional arrays (fast, random access to elements) and lists (flexibility). They are implemented by a fixed block of storage. When necessary for expansion, a new, larger block of storage is allocated and the elements from the old storage area are copied into the new block.

Flexible arrays have available most of the operations provided by **OneDimensionalArray** (see '`OneDimensionalArray`' on page 517 and '`Vector`' on page 603). Since flexible arrays are also of category **ExtensibleLinearAggregate**, they have operations concat!, delete!, insert!, merge!, remove!, removeDuplicates!, and select!. In addition, the operations physicalLength and physicalLength! provide user-control over expansion and contraction.

A convenient way to create a flexible array is to apply the operation flexibleArray to a list of values.

```
flexibleArray [i for i in 1..6]
```

$$[1,\ 2,\ 3,\ 4,\ 5,\ 6] \tag{4}$$

FlexibleArray (PositiveInteger)

Create a flexible array of six zeroes.

```
f : FARRAY INT := new(6,0)
```

$$[0,\ 0,\ 0,\ 0,\ 0,\ 0] \tag{5}$$

FlexibleArray (Integer)

For $i = 1 \ldots 6$, set the i^{th} element to i. Display `f`.

```
for i in 1..6 repeat f.i := i; f
```

$$[1,\ 2,\ 3,\ 4,\ 5,\ 6] \tag{6}$$

FlexibleArray (Integer)

Initially, the physical length is the same as the number of elements.

```
physicalLength f
```

$$6 \tag{7}$$

PositiveInteger

Add an element to the end of `f`.

```
concat!(f,11)
```

$$[1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 11] \tag{8}$$

FlexibleArray (Integer)

See that its physical length has grown.

```
physicalLength f
```

$$10 \tag{9}$$

PositiveInteger

Make `f` grow to have room for `15` elements.

```
physicalLength!(f,15)
```

$$[1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 11] \tag{10}$$

FlexibleArray (Integer)

Concatenate the elements of `f` to itself. The physical length allows room for three more values at the end.

```
concat!(f,f)
```

$$[1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 11,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 11] \tag{11}$$

FlexibleArray (Integer)

Use insert! to add an element to the front of a flexible array.

```
insert!(22,f,1)
```

$$[22,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 11,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 11] \tag{12}$$

FlexibleArray (Integer)

Create a second flexible array from `f` consisting of the elements from index 10 forward.

```
g := f(10..)
```

$$[2,\ 3,\ 4,\ 5,\ 6,\ 11] \tag{13}$$

FlexibleArray (Integer)

Insert this array at the front of `f`.

```
insert!(g,f,1)
```

$$[2,\ 3,\ 4,\ 5,\ 6,\ 11,\ 22,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 11,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 11] \tag{14}$$

FlexibleArray (Integer)

Merge the flexible array `f` into `g` after sorting each in place.

```
merge!(sort! f, sort! g)
```

$$[1,\ 1,\ 2,\ 2,\ 2,\ 2,\ 3,\ 3,\ 3,\ 3,\ 4,\ 4,\ 4,\ 4,\ 5,\ 5,\ 5,\ 5,\ 6,\ 6,\ 6,\ 6,\ 11,\ 11,\ 11,\ 11,\ 22] \tag{15}$$

FlexibleArray (Integer)

Remove duplicates in place.

```
removeDuplicates! f
```

$$[1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 11,\ 22] \tag{16}$$

FlexibleArray (Integer)

Remove all odd integers.

```
select!(i +-> even? i,f)
```

$$[2,\ 4,\ 6,\ 22] \tag{17}$$

FlexibleArray (Integer)

All these operations have shrunk the physical length of `f`.

```
physicalLength f
```

$$8 \tag{18}$$

PositiveInteger

To force FriCAS not to shrink flexible arrays call the shrinkable operation with the argument `false`. You must package call this operation. The previous value is returned.

```
shrinkable(false)$FlexibleArray(Integer)
```

```
true                                                                    (19)
                                                                    Boolean
```

9.27 Float

FriCAS provides two kinds of floating point numbers. The domain **Float** (abbreviation **FLOAT**) implements a model of arbitrary precision floating point numbers. The domain **DoubleFloat** (abbreviation **DFLOAT**) is intended to make available hardware floating point arithmetic in FriCAS. The actual model of floating point that **DoubleFloat** provides is system-dependent. For example, on the IBM system 370 FriCAS uses IBM double precision which has fourteen hexadecimal digits of precision or roughly sixteen decimal digits. Arbitrary precision floats allow the user to specify the precision at which arithmetic operations are computed. Although this is an attractive facility, it comes at a cost. Arbitrary-precision floating-point arithmetic typically takes twenty to two hundred times more time than hardware floating point.

For more information about FriCAS's numeric and graphic facilities, see Section 7 on page 195, Section 8.1 on page 251, and '`DoubleFloat`' on page 395.

9.27.1 Introduction to Float

Scientific notation is supported for input and output of floating point numbers. A floating point number is written as a string of digits containing a decimal point optionally followed by the letter "`E`", and then the exponent. We begin by doing some calculations using arbitrary precision floats. The default precision is twenty decimal digits.

```
1.234
```

```
1.234                                                                   (1)
                                                                      Float
```

A decimal base for the exponent is assumed, so the number `1.234E2` denotes $1.234 \cdot 10^2$.

```
1.234E2
```

```
123.4                                                                   (2)
                                                                      Float
```

The normal arithmetic operations are available for floating point numbers.

```
sqrt(1.2 + 2.3 / 3.4 ^ 4.5)
```

```
1.0996972790671286226                                                   (3)
                                                                      Float
```

9.27.2 Conversion Functions

You can use conversion (Section 2.7 on page 84) to go back and forth between **Integer**, **Fraction Integer** and **Float**, as appropriate.

```
i := 3 :: Float
```

$$3.0 \tag{1}$$

Float

```
i :: Integer
```

$$3 \tag{2}$$

Integer

```
i :: Fraction Integer
```

$$3 \tag{3}$$

Fraction (Integer)

Since you are explicitly asking for a conversion, you must take responsibility for any loss of exactness.

```
r := 3/7 :: Float
```

$$0.42857142857142857143 \tag{4}$$

Float

```
r :: Fraction Integer
```

$$\frac{3}{7} \tag{5}$$

Fraction (Integer)

This conversion cannot be performed: use truncate or round if that is what you intend.

```
r :: Integer
```

```
   Cannot convert the value from type Float to Integer .
```

The operations truncate and round truncate ...

```
truncate 3.6
```

$$3.0 \tag{6}$$

Float

and round to the nearest integral **Float** respectively.

```
round 3.6
```

4.0 (7)

Float

```
truncate(-3.6)
```

− 3.0 (8)

Float

```
round(-3.6)
```

− 4.0 (9)

Float

The operation fractionPart computes the fractional part of `x`, that is, `x - truncate x`.

```
fractionPart 3.6
```

0.6 (10)

Float

The operation digits allows the user to set the precision. It returns the previous value it was using.

```
digits 40
```

20 (11)

PositiveInteger

```
sqrt 0.2
```

0.4472135954999579392818347337462552470881 (12)

Float

```
pi()$Float
```

3.1415926535897932384626433832795028841 97 (13)

Float

The precision is only limited by the computer memory available. Calculations at 500 or more digits of precision are not difficult.

```
digits 500
```

40 (14)

PositiveInteger

```
pi()$Float
```

3.14159265358979323846264338327950288419716939937510582097494459230781640628620899862803482534211706798214808651 (15)

Float

Reset digits to its default value.

```
digits 20
```

500 (16)

PositiveInteger

Numbers of type **Float** are represented as a record of two integers, namely, the mantissa and the exponent where the base of the exponent is binary. That is, the floating point number `(m,e)` represents the number $m \cdot 2^e$. A consequence of using a binary base is that decimal numbers can not, in general, be represented exactly.

9.27.3 Output Functions

A number of operations exist for specifying how numbers of type **Float** are to be displayed. By default, spaces are inserted every ten digits in the output for readability.[3]

Output spacing can be modified with the outputSpacing operation. This inserts no spaces and then displays the value of `x`.

```
outputSpacing 0; x := sqrt 0.2
```

0.44721359549995793928 (1)

Float

Issue this to have the spaces inserted every `5` digits.

```
outputSpacing 5; x
```

0.44721359549995793928 (2)

Float

By default, the system displays floats in either fixed format or scientific format, depending on the magnitude of the number.

```
y := x/10^10
```

$0.44721359549995793928E - 10$ (3)

Float

A particular format may be requested with the operations outputFloating and outputFixed.

```
outputFloating(); x
```

[3]Note that you cannot include spaces in the input form of a floating point number, though you can use underscores.

$$0.44721359549995793928E0 \tag{4}$$

Float

```
outputFixed(); y
```

$$0.00000000004472135954999579392 8 \tag{5}$$

Float

Additionally, you can ask for `n` digits to be displayed after the decimal point.

```
outputFloating 2; y
```

$$0.45E-10 \tag{6}$$

Float

```
outputFixed 2; x
```

$$0.45 \tag{7}$$

Float

This resets the output printing to the default behavior.

```
outputGeneral()
```

9.27.4 An Example: Determinant of a Hilbert Matrix

Consider the problem of computing the determinant of a `10` by `10` Hilbert matrix. The $(i,j)^{\text{th}}$ entry of a Hilbert matrix is given by `1/(i+j+1)`.

First do the computation using rational numbers to obtain the exact result.

```
a: Matrix Fraction Integer := matrix [[1/(i+j+1) for j in 0..9] for i in 0..9]
```

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} \\ \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} \\ \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} & \frac{1}{16} \\ \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} & \frac{1}{16} & \frac{1}{17} \\ \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} & \frac{1}{16} & \frac{1}{17} & \frac{1}{18} \\ \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} & \frac{1}{16} & \frac{1}{17} & \frac{1}{18} & \frac{1}{19} \end{bmatrix} \tag{1}$$

Matrix(Fraction (Integer))

This version of determinant uses Gaussian elimination.

```
d:= determinant a
```

$$\frac{1}{46206893947914691316295628839036278726983680000000000} \tag{2}$$

Fraction (Integer)

```
d :: Float
```

$$0.21641792264314918691E-52 \tag{3}$$

Float

Now use hardware floats. Note that a semicolon (;) is used to prevent the display of the matrix.

```
b: Matrix DoubleFloat := matrix [[1/(i+j+1$DoubleFloat) for j in 0..9] for i in 0..9];
```

Matrix(DoubleFloat)

The result given by hardware floats is correct only to four significant digits of precision. In the jargon of numerical analysis, the Hilbert matrix is said to be "ill-conditioned."

```
determinant b
```

$$2.164367794572141e\text{-}53 \tag{5}$$

DoubleFloat

Now repeat the computation at a higher precision using **Float**.

```
digits 40
```

$$20 \tag{6}$$

PositiveInteger

```
c: Matrix Float := matrix [[1/(i+j+1$Float) for j in 0..9] for i in 0..9];
```

Matrix(Float)

```
determinant c
```

$$0.21641792264314918690605949836226174361 59E-52 \tag{8}$$

Float

Reset digits to its default value.

```
digits 20
```

$$40 \tag{9}$$

PositiveInteger

9.28 Fraction

The **Fraction** domain implements quotients. The elements must belong to a domain of category **IntegralDomain**: multiplication must be commutative and the product of two non-zero elements must

not be zero. This allows you to make fractions of most things you would think of, but don't expect to create a fraction of two matrices! The abbreviation for **Fraction** is **FRAC**.

Use / to create a fraction.

```
a := 11/12
```

$$\frac{11}{12} \tag{4}$$

Fraction (Integer)

```
b := 23/24
```

$$\frac{23}{24} \tag{5}$$

Fraction (Integer)

The standard arithmetic operations are available.

```
3 - a*b^2 + a + b/a
```

$$\frac{313271}{76032} \tag{6}$$

Fraction (Integer)

Extract the numerator and denominator by using numer and denom, respectively.

```
numer(a)
```

$$11 \tag{7}$$

PositiveInteger

```
denom(b)
```

$$24 \tag{8}$$

PositiveInteger

Operations like max, min, negative?, positive? and zero? are all available if they are provided for the numerators and denominators. See '`Integer`' on page 441 for examples.

Don't expect a useful answer from factor, gcd or lcm if you apply them to fractions.

```
r := (x^2 + 2*x + 1)/(x^2 - 2*x + 1)
```

$$\frac{x^2 + 2\,x + 1}{x^2 - 2\,x + 1} \tag{9}$$

Fraction (Polynomial(Integer))

Since all non-zero fractions are invertible, these operations have trivial definitions.

```
factor(r)
```

$$\frac{x^2+2\,x+1}{x^2-2\,x+1} \tag{10}$$

Factored(Fraction (Polynomial(Integer)))

Use map to apply factor to the numerator and denominator, which is probably what you mean.

```
map(factor,r)
```

$$\frac{(x+1)^2}{(x-1)^2} \tag{11}$$

Fraction (Factored(Polynomial(Integer)))

Other forms of fractions are available. Use continuedFraction to create a continued fraction.

```
continuedFraction(7/12)
```

$$\frac{1|}{|1}+\frac{1|}{|1}+\frac{1|}{|2}+\frac{1|}{|2} \tag{12}$$

ContinuedFraction(Integer)

Use partialFraction to create a partial fraction. See '`ContinuedFraction`' on page 375 and '`PartialFraction`' on page 528 for additional information and examples.

```
partialFraction(7,12)
```

$$\frac{1}{2^2}+\frac{1}{3} \tag{13}$$

PartialFraction (Integer)

Use conversion to create alternative views of fractions with objects moved in and out of the numerator and denominator.

```
g := 2/3 + 4/5*%i
```

$$\frac{2}{3}+\frac{4}{5}\,i \tag{14}$$

Complex(Fraction(Integer))

Conversion is discussed in detail in Section 2.7 on page 84.

```
g :: FRAC COMPLEX INT
```

$$\frac{10+12\,i}{15} \tag{15}$$

Fraction (Complex(Integer))

9.29 FreeMagma

Initialisations

```
x:Symbol :='x
```

$$x \tag{4}$$

Symbol

```
y:Symbol :='y
```

$$y \tag{5}$$

Symbol

```
z:Symbol :='z
```

$$z \tag{6}$$

Symbol

```
word := FreeMonoid(Symbol)
```

$$\text{FreeMonoid}(Symbol) \tag{7}$$

Type

```
tree := FreeMagma(Symbol)
```

$$\text{FreeMagma}(Symbol) \tag{8}$$

Type

Let's make some trees

```
a:tree := x*x
```

$$[x,\ x] \tag{9}$$

FreeMagma(Symbol)

```
b:tree := y*y
```

$$[y,\ y] \tag{10}$$

FreeMagma(Symbol)

```
c:tree := a*b
```

$$[[x,\ x]\,,\ [y,\ y]] \tag{11}$$

FreeMagma(Symbol)

Query the trees

```
left c
```

$$[x,\ x] \tag{12}$$

FreeMagma(Symbol)

```
right c
```

$$[y,\ y] \tag{13}$$

FreeMagma(Symbol)

```
length c
```

$$4 \tag{14}$$

PositiveInteger

Coerce to the monoid

```
c::word
```

$$x^2\,y^2 \tag{15}$$

FreeMonoid(Symbol)

Check ordering

```
a < b
```

$$\texttt{true} \tag{16}$$

Boolean

```
a < c
```

$$\texttt{true} \tag{17}$$

Boolean

```
b < c
```

$$\texttt{true} \tag{18}$$

Boolean

Navigate the tree

```
first c
```

$$x \tag{19}$$

Symbol

```
rest c
```

$$[x,\ [y,\ y]] \tag{20}$$

FreeMagma(Symbol)

```
rest rest c
```

$$[y,\ y] \tag{21}$$

FreeMagma(Symbol)

Check ordering

```
ax:tree := a*x
```

$$[[x,\ x]\,,\ x] \tag{22}$$

FreeMagma(Symbol)

```
xa:tree := x*a
```

$$[x,\ [x,\ x]] \tag{23}$$

FreeMagma(Symbol)

```
xa < ax
```

$$\texttt{true} \tag{24}$$

Boolean

```
lexico(xa,ax)
```

$$\texttt{false} \tag{25}$$

Boolean

9.30 FullPartialFractionExpansion

The domain **FullPartialFractionExpansion** implements factor-free conversion of quotients to full partial fractions.

Our examples will all involve quotients of univariate polynomials with rational number coefficients.

```
Fx := FRAC UP(x, FRAC INT)
```

$$\text{Fraction}(\text{UnivariatePolynomial}(x, \text{Fraction}(Integer))) \tag{4}$$

Type

Here is a simple-looking rational function.

```
f : Fx := 36 / (x^5-2*x^4-2*x^3+4*x^2+x-2)
```

$$\frac{36}{x^5-2\,x^4-2\,x^3+4\,x^2+x-2} \tag{5}$$

Fraction (UnivariatePolynomial (x, Fraction (Integer)))

We use fullPartialFraction to convert it to an object of type **FullPartialFractionExpansion**.

```
g := fullPartialFraction f
```

$$\frac{4}{x-2}-\frac{4}{x+1}+\sum_{\%A^2-1\,=0}\frac{-3\,\%A-6}{(x-\%A)^2} \tag{6}$$

FullPartialFractionExpansion (Fraction (Integer), UnivariatePolynomial (x, Fraction (Integer)))

Use a coercion to change it back into a quotient.

```
g :: Fx
```

$$\frac{36}{x^5-2\,x^4-2\,x^3+4\,x^2+x-2} \tag{7}$$

Fraction (UnivariatePolynomial (x, Fraction (Integer)))

Full partial fractions differentiate faster than rational functions.

```
g5 := D(g, 5)
```

$$-\frac{480}{(x-2)^6}+\frac{480}{(x+1)^6}+\sum_{\%A^2-1\,=0}\frac{2160\,\%A+4320}{(x-\%A)^7} \tag{8}$$

FullPartialFractionExpansion (Fraction (Integer), UnivariatePolynomial (x, Fraction (Integer)))

```
f5 := D(f, 5)
```

$$\frac{-544320\,x^{10}+4354560\,x^9-14696640\,x^8+28615680\,x^7-40085280\,x^6+46656000\,x^5-39411360\,x^4+18247}{x^{20}-12\,x^{19}+53\,x^{18}-76\,x^{17}-159\,x^{16}+676\,x^{15}-391\,x^{14}-1596\,x^{13}+2527\,x^{12}+1148\,x^{11}-4977\,x^{10}+1372\,x^9+4907\,x^8-344} \tag{9}$$

Fraction (UnivariatePolynomial (x, Fraction (Integer)))

We can check that the two forms represent the same function.

```
g5::Fx - f5
```

$$0 \tag{10}$$

Fraction (UnivariatePolynomial (x, Fraction (Integer)))

Here are some examples that are more complicated.

```
f : Fx := (x^5 * (x-1)) / ((x^2 + x + 1)^2 * (x-2)^3)
```

$$\frac{x^6 - x^5}{x^7 - 4\,x^6 + 3\,x^5 + 9\,x^3 - 6\,x^2 - 4\,x - 8} \tag{11}$$

Fraction (UnivariatePolynomial (x, Fraction (Integer)))

```
g := fullPartialFraction f
```

$$\frac{\frac{1952}{2401}}{x-2} + \frac{\frac{464}{343}}{(x-2)^2} + \frac{\frac{32}{49}}{(x-2)^3} + \sum_{\%A^2+\%A+1\,=0} \frac{-\frac{179}{2401}\,\%A + \frac{135}{2401}}{x - \%A} + \sum_{\%A^2+\%A+1\,=0} \frac{\frac{37}{1029}\,\%A + \frac{20}{1029}}{(x - \%A)^2} \tag{12}$$

FullPartialFractionExpansion (Fraction (Integer), UnivariatePolynomial (x, Fraction (Integer)))

```
g :: Fx - f
```

$$0 \tag{13}$$

Fraction (UnivariatePolynomial (x, Fraction (Integer)))

```
f : Fx := (2*x^7-7*x^5+26*x^3+8*x) / (x^8-5*x^6+6*x^4+4*x^2-8)
```

$$\frac{2\,x^7 - 7\,x^5 + 26\,x^3 + 8\,x}{x^8 - 5\,x^6 + 6\,x^4 + 4\,x^2 - 8} \tag{14}$$

Fraction (UnivariatePolynomial (x, Fraction (Integer)))

```
g := fullPartialFraction f
```

$$\sum_{\%A^2-2\,=0} \frac{\frac{1}{2}}{x - \%A} + \sum_{\%A^2-2\,=0} \frac{1}{(x - \%A)^3} + \sum_{\%A^2+1\,=0} \frac{\frac{1}{2}}{x - \%A} \tag{15}$$

FullPartialFractionExpansion (Fraction (Integer), UnivariatePolynomial (x, Fraction (Integer)))

```
g :: Fx - f
```

$$0 \tag{16}$$

Fraction (UnivariatePolynomial (x, Fraction (Integer)))

```
f:Fx := x^3 / (x^21 + 2*x^20 + 4*x^19 + 7*x^18 + 10*x^17 + 17*x^16 + 22*x^15 + _
    30*x^14 + 36*x^13 + 40*x^12 + 47*x^11 + 46*x^10 + 49*x^9 + 43*x^8 + 38*x^7 + _
    32*x^6 + 23*x^5 + 19*x^4 + 10*x^3 + 7*x^2 + 2*x + 1)
```

$$\frac{x^3}{x^{21} + 2\,x^{20} + 4\,x^{19} + 7\,x^{18} + 10\,x^{17} + 17\,x^{16} + 22\,x^{15} + 30\,x^{14} + 36\,x^{13} + 40\,x^{12} + 47\,x^{11} + 46\,x^{10} + 49\,x^9 + 43\,x^8 + 38\,x^7 + 32\,x^6 +} \tag{17}$$

Fraction (UnivariatePolynomial (x, Fraction (Integer)))

```
g := fullPartialFraction f
```

$$\sum_{\%A^2+1=0} \frac{\frac{1}{2}\,\%A}{x-\%A} + \sum_{\%A^2+\%A+1=0} \frac{\frac{1}{9}\,\%A - \frac{19}{27}}{x-\%A} + \sum_{\%A^2+\%A+1=0} \frac{\frac{1}{27}\,\%A - \frac{1}{27}}{(x-\%A)^2}$$
$$+ \sum_{\%A^5+\%A^2+1=0} \frac{-\frac{96556567040}{912390759099}\,\%A^4 + \frac{420961732891}{912390759099}\,\%A^3 - \frac{59101056149}{912390759099}\,\%A^2 - \frac{373545875923}{912390759099}\,\%A + \frac{529673492498}{912390759099}}{x-\%A}$$
$$+ \sum_{\%A^5+\%A^2+1=0} \frac{-\frac{5580868}{94070601}\,\%A^4 - \frac{2024443}{94070601}\,\%A^3 + \frac{4321919}{94070601}\,\%A^2 - \frac{84614}{1542141}\,\%A - \frac{5070620}{94070601}}{(x-\%A)^2}$$
$$+ \sum_{\%A^5+\%A^2+1=0} \frac{\frac{1610957}{94070601}\,\%A^4 + \frac{2763014}{94070601}\,\%A^3 - \frac{2016775}{94070601}\,\%A^2 + \frac{266953}{94070601}\,\%A + \frac{4529359}{94070601}}{(x-\%A)^3} \tag{18}$$

FullPartialFractionExpansion (Fraction (Integer) , UnivariatePolynomial (x , Fraction (Integer)))

This verification takes much longer than the conversion to partial fractions.

```
g :: Fx - f
```

$$0 \tag{19}$$

Fraction (UnivariatePolynomial (x , Fraction (Integer)))

For more information, see the paper: Bronstein, M and Salvy, B. "Full Partial Fraction Decomposition of Rational Functions," *Proceedings of ISSAC'93, Kiev*, ACM Press. Also see '`PartialFraction`' on page 528 for standard partial fraction decompositions.

9.31 GeneralQuaternion

The domain constructor **GeneralQuaternion** implements general quaternions over commutative rings. For information on related topics, see '`Quaternion`' on page 539, '`Complex`' on page 372 and '`Octonion`' on page 515. You can also issue the system command `)show GeneralQuaternion` to display the full list of operations defined by **GeneralQuaternion**.

To use general quaternions we need to explicitly qualify calls. So first we initialize the domain and assign it to a variable.

```
Q2 := GeneralQuaternion(Fraction(Integer), 2, 3)
```

$$\text{GeneralQuaternion}(\text{Fraction}(Integer), 2, 3) \tag{4}$$

Type

The basic operation for creating quaternions is quatern.

```
i := quatern(0, 1, 0, 0)$Q2
```

$$i \tag{5}$$

GeneralQuaternion(Fraction (Integer) , 2, 3)

In **GeneralQuaternion(Fraction(Integer), a, b)** squaring `i` gives `a`.

```
i^2
```

$$2 \tag{6}$$

GeneralQuaternion(Fraction(Integer), 2, 3)

Similarly for b.

```
( quatern(0, 0, 1, 0)$Q2)^2
```

$$3 \tag{7}$$

GeneralQuaternion(Fraction(Integer), 2, 3)

Yet another quaternion.

```
q := quatern(2/11,-8,3/4,1)$Q2
```

$$\frac{2}{11} - 8\,i + \frac{3}{4}\,j + k \tag{8}$$

GeneralQuaternion(Fraction(Integer), 2, 3)

Because q is over the rationals (and nonzero), you can invert it.

```
iq := inv q
```

$$-\frac{352}{239395} - \frac{15488}{239395}\,i + \frac{1452}{239395}\,j + \frac{1936}{239395}\,k \tag{9}$$

GeneralQuaternion(Fraction(Integer), 2, 3)

Check the inverse.

```
iq*q
```

$$1 \tag{10}$$

GeneralQuaternion(Fraction(Integer), 2, 3)

The usual arithmetic (ring) operations are available

```
q^6
```

$$\frac{13785787472776443}{7256313856} - \frac{172175104091}{1288408}\,i + \frac{516525312273}{41229056}\,j + \frac{172175104091}{10307264}\,k \tag{11}$$

GeneralQuaternion(Fraction(Integer), 2, 3)

```
r := quatern(-2,3,23/9,-89)$Q2; q + r
```

$$-\frac{20}{11} - 5\,i + \frac{119}{36}\,j - 88\,k \tag{12}$$

GeneralQuaternion(Fraction(Integer), 2, 3)

In general, multiplication is not commutative.

```
q * r - r * q
```

$$\frac{2495}{6}\,i + 2836\,j - \frac{817}{18}\,k \tag{13}$$

GeneralQuaternion(Fraction(Integer), 2, 3)

The norm is the quaternion times its conjugate. Norm is rational, but may be negative.

```
norm q
```

$$-\frac{239395}{1936} \tag{14}$$

Fraction(Integer)

```
conjugate q
```

$$\frac{2}{11} + 8\,i - \frac{3}{4}\,j - k \tag{15}$$

GeneralQuaternion(Fraction(Integer), 2, 3)

```
q * %
```

$$-\frac{239395}{1936} \tag{16}$$

GeneralQuaternion(Fraction(Integer), 2, 3)

9.32 GeneralSparseTable

Sometimes when working with tables there is a natural value to use as the entry in all but a few cases. The **GeneralSparseTable** constructor can be used to provide any table type with a default value for entries. See '`Table`' on page 587 for general information about tables. Issue the system command `)show GeneralSparseTable` to display the full list of operations defined by **GeneralSparseTable**.

Suppose we launched a fund-raising campaign to raise fifty thousand dollars. To record the contributions, we want a table with strings as keys (for the names) and integer entries (for the amount). In a data base of cash contributions, unless someone has been explicitly entered, it is reasonable to assume they have made a zero dollar contribution. This creates a keyed access file with default entry 0.

```
patrons: GeneralSparseTable(String, Integer, KeyedAccessFile(Integer), 0) := table() ;
```

GeneralSparseTable(String, Integer, KeyedAccessFile(Integer), 0)

Now `patrons` can be used just as any other table. Here we record two gifts.

```
patrons."Smith" := 10500
```

$$10500 \tag{5}$$

PositiveInteger

```
patrons."Jones" := 22000
```

22000 (6)

PositiveInteger

Now let us look up the size of the contributions from Jones and Stingy.

```
patrons."Jones"
```

22000 (7)

PositiveInteger

```
patrons."Stingy"
```

0 (8)

NonNegativeInteger

Have we met our seventy thousand dollar goal?

```
reduce(+, entries patrons)
```

32500 (9)

PositiveInteger

So the project is cancelled and we can delete the data base:

```
)system rm -r kaf*.sdata
```

9.33 GroebnerFactorizationPackage

Solving systems of polynomial equations with the Gröbner basis algorithm can often be very time consuming because, in general, the algorithm has exponential run-time. These systems, which often come from concrete applications, frequently have symmetries which are not taken advantage of by the algorithm. However, it often happens in this case that the polynomials which occur during the Gröbner calculations are reducible. Since FriCAS has an excellent polynomial factorization algorithm, it is very natural to combine the Gröbner and factorization algorithms.

GroebnerFactorizationPackage exports the groebnerFactorize operation which implements a modified Gröbner basis algorithm. In this algorithm, each polynomial that is to be put into the partial list of the basis is first factored. The remaining calculation is split into as many parts as there are irreducible factors. Call these factors $p_1, \ldots, p_n$. In the branches corresponding to $p_2, \ldots, p_n$, the factor p_1 can be divided out, and so on. This package also contains operations that allow you to specify the polynomials that are not zero on the common roots of the final Gröbner basis.

Here is an example from chemistry. In a theoretical model of the cyclohexan C_6H_{12}, the six carbon atoms each sit in the center of gravity of a tetrahedron that has two hydrogen atoms and two carbon atoms at its corners. We first normalize and set the length of each edge to 1. Hence, the distances of one fixed carbon atom to each of its immediate neighbours is 1. We will denote the distances to the other three carbon atoms by x, y and z.

A. Dress developed a theory to decide whether a set of points and distances between them can be realized in an n-dimensional space. Here, of course, we have $n = 3$.

```
mfzn : SQMATRIX(6,DMP([x,y,z],Fraction INT)) := [[0,1,1,1,1,1], [1,0,1,8/3,x,8/3], -
    [1,1,0,1,8/3,y], [1,8/3,1,0,1,8/3], [1,x,8/3,1,0,1], [1,8/3,y,8/3,1,0]]
```

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & \frac{8}{3} & x & \frac{8}{3} \\ 1 & 1 & 0 & 1 & \frac{8}{3} & y \\ 1 & \frac{8}{3} & 1 & 0 & 1 & \frac{8}{3} \\ 1 & x & \frac{8}{3} & 1 & 0 & 1 \\ 1 & \frac{8}{3} & y & \frac{8}{3} & 1 & 0 \end{bmatrix} \tag{4}$$

SquareMatrix(6, DistributedMultivariatePolynomial ([x, y, z], Fraction(Integer)))

For the cyclohexan, the distances have to satisfy this equation.

```
eq := determinant mfzn
```

$$-x^2y^2+\frac{22}{3}x^2y-\frac{25}{9}x^2+\frac{22}{3}xy^2-\frac{388}{9}xy-\frac{250}{27}x-\frac{25}{9}y^2-\frac{250}{27}y+\frac{14575}{81} \tag{5}$$

DistributedMultivariatePolynomial ([x, y, z], Fraction(Integer))

They also must satisfy the equations given by cyclic shifts of the indeterminates.

```
groebnerFactorize [eq, eval(eq, [x,y,z], [y,z,x]), eval(eq, [x,y,z], [z,x,y])]
```

$$\begin{aligned}&\left[\left[xy+xz-\frac{22}{3}x+yz-\frac{22}{3}y-\frac{22}{3}z+\frac{121}{3},\right.\right.\\ &xz^2-\frac{22}{3}xz+\frac{25}{9}x+yz^2-\frac{22}{3}yz+\frac{25}{9}y-\frac{22}{3}z^2+\frac{388}{9}z+\frac{250}{27},\\ &\left.y^2z^2-\frac{22}{3}y^2z+\frac{25}{9}y^2-\frac{22}{3}yz^2+\frac{388}{9}yz+\frac{250}{27}y+\frac{25}{9}z^2+\frac{250}{27}z-\frac{14575}{81}\right],\left[x+y-\frac{21994}{5625},\right.\\ &y^2-\frac{21994}{5625}y+\frac{4427}{675},z-\frac{463}{87}\Big],\left[x^2-\frac{1}{2}xz-\frac{11}{2}x-\frac{5}{6}z+\frac{265}{18},y-z,z^2-\frac{38}{3}z+\frac{265}{9}\right],\left[x-\frac{25}{9},\right.\\ &\left.y-\frac{11}{3},z-\frac{11}{3}\right],\left[x-\frac{11}{3},y-\frac{11}{3},z-\frac{11}{3}\right],\left[x+\frac{5}{3},y+\frac{5}{3},z+\frac{5}{3}\right],\left[x-\frac{19}{3},y+\frac{5}{3},z+\frac{5}{3}\right]\Big]\end{aligned} \tag{6}$$

List(List(DistributedMultivariatePolynomial ([x, y, z], Fraction(Integer))))

The union of the solutions of this list is the solution of our original problem. If we impose positivity conditions, we get two relevant ideals. One ideal is zero-dimensional, namely $x = y = z = 11/3$, and this determines the "boat" form of the cyclohexan. The other ideal is one-dimensional, which means that we have a solution space given by one parameter. This gives the "chair" form of the cyclohexan. The parameter describes the angle of the "back of the chair."

groebnerFactorize has an optional **Boolean**-valued second argument. When it is `true` partial results are displayed, since it may happen that the calculation does not terminate in a reasonable time. See the source code for **GroebnerFactorizationPackage** in **groebf.spad** for more details about the algorithms used.

9.34 GroupPresentation

The domain **GroupPresentation** implements group presentations.

We first expose it to simplify notation.

```
)expose GroupPresentation
```

```
   GroupPresentation is now explicitly exposed in frame initial
```

We create a cyclic group.

```
c3 := groupPresentation([1], [[1, 1, 1]])
```

$$\texttt{<a | a*a*a>} \tag{4}$$

GroupPresentation

And we convert it to **PermutationGroup** using Todd-Coxeter coset enumeration.

```
toPermutationIfCan(c3)
```

$$< (1\ 2\ 3) > \tag{5}$$

Union(PermutationGroup(Integer), ...)

For nicer input there is package **GroupPresentationFunctions1**.

We first assign generators of free group to variables.

```
fG := FreeGroup(Symbol)
```

$$\text{FreeGroup}(Symbol) \tag{6}$$

Type

```
a := (^$fG)('a, 1)
```

$$a \tag{7}$$

FreeGroup(Symbol)

```
b := (^$fG)('b, 1)
```

$$b \tag{8}$$

FreeGroup(Symbol)

```
c := (^$fG)('c, 1)
```

$$c \tag{9}$$

FreeGroup(Symbol)

Now we give presentation of Mathieu group M12.

```
m12f := [a^11, b^2, c^2, (a*b)^3, (a*c)^3, (b*c)^10, a^2*(b*c)^2*a*(b*c)^(-2)]
```

$$[a^{11},\ b^2,\ c^2,\ ababab,\ acacac,\ bcbcbcbcbcbcbcbcbcbc,\ a^2\,bcbcac^{-1}\,b^{-1}\,c^{-1}\,b^{-1}] \tag{10}$$

List (FreeGroup(Symbol))

```
cP := GroupPresentationFunctions1(Symbol)
```

$$\text{GroupPresentationFunctions1}(Symbol) \tag{11}$$

Type

```
m12pres := convert(['a, 'b, 'c], m12f)$cP
```

$$\begin{array}{l}\texttt{<a b c |\ \ a*a*a*a*a*a*a*a*a*a*a,\ \ b*b,\ \ c*c,\ \ a*b*a*b*a*b,\ \ a*c*a*c*a*c,}\\ \texttt{\ \ b*c*b*c*b*c*b*c*b*c*b*c*b*c*b*c*b*c*b*c,\ \ a*a*b*c*b*c*a*-c*-b*-c*-b>}\end{array} \tag{12}$$

GroupPresentation

And convert it to permutation group. Since Mathieu group has many elements we compute representation on cosets of group generated by `a` and `b`.

```
m12per := toPermutationIfCan(m12pres, [[1], [2]], false)
```

$$\begin{array}{l}<(2\,3\,5\,6\,7\,8\,9\,10\,11\,12\,4)\ (13\,14\,23\,40\,41\,42\,43\,44\,45\,46\,26)\ (15\,38\,50\,51\,28\,22\,25\,52\,53\,54\,24)\ (16\,33\,21\,31\,60\,61\,49\,62\,59\,63\,35)\ (17\,2\\ (3\,13)\ (4\,14)\ (5\,22)\ (6\,19)\ (7\,17)\ (8\,21)\ (9\,16)\ (10\,20)\ (11\,18)\ (12\,15)\ (23\,24)\ (25\,26)\ (27\,28)\ (29\,30)\ (31\,32)\ (34\,35)\ (36\,37)\ (38\,39)\ (40\,62)\\ (1\,2)\ (3\,4)\ (5\,8)\ (6\,10)\ (7\,11)\ (9\,12)\ (13\,22)\ (14\,15)\ (16\,17)\ (18\,19)\ (20\,21)\ (23\,55)\ (24\,56)\ (25\,47)\ (26\,48)\ (27\,49)\ (28\,38)\ (29\,57)\ (30\,80)\ (3\end{array}$$

Union(PermutationGroup(Integer), ...)

To check the result we compute order.

```
order(m12per::PermutationGroup(Integer))
```

$$95040 \tag{14}$$

PositiveInteger

9.35 GuessPolynomialInteger

The package **GuessPolynomialInteger** can guess formulas for sequences of polynomials or rational functions over integers, given the first few terms. Related packages are **GuessInteger** for sequences of rational numbers or rational functions, **GuessAlgebraicNumber** when sequences contain algebraic numbers, **GuessPolynomial** for polynomials and rational functions with general coefficients and **Guess** (general version).

Below we show how to guess a recurrence relation for squares of Hermite polynomials.

We first need to prepare data.

```
hl := [hermiteH(i, x)^2 for i in 0..15];
```

List(Polynomial(Integer))

Now guessing proper:

```
guessPRec(hl, homogeneous == true, maxDegree==3)
```

$$[[f(n): \ -f(n+3)+\left(4x^2-2n-4\right)f(n+2)+\left(\left(-8n-16\right)x^2+4n^2+16n+16\right)f(n+1) \\ +\left(8n^3+32n^2+40n+16\right)f(n)=0,\ f(0)=1,\ f(1)=4x^2,\ f(2)=16x^4-16x^2+4]] \tag{5}$$

List(Expression(Integer))

9.36 Heap

The domain **Heap(S)** implements a priority queue of objects of type S such that the operation extract! removes and returns the maximum element. The implementation represents heaps as flexible arrays (see '`FlexibleArray`' on page 416). The representation and algorithms give complexity of $O(\log(n))$ for insertion and extractions, and $O(n)$ for construction.

Create a heap of six elements.

```
h := heap [-4,9,11,2,7,-7]
```

$$[11,\ 9,\ -4,\ 2,\ 7,\ -7] \tag{4}$$

Heap(Integer)

Use insert! to add an element.

```
insert!(3,h)
```

$$[11,\ 9,\ 3,\ 2,\ 7,\ -7,\ -4] \tag{5}$$

Heap(Integer)

The operation extract! removes and returns the maximum element.

```
extract! h
```

$$11 \tag{6}$$

PositiveInteger

The internal structure of h has been appropriately adjusted.

```
h
```

$$[9,\ 7,\ 3,\ 2,\ -4,\ -7] \tag{7}$$

Heap(Integer)

Now extract! elements repeatedly until none are left, collecting the elements in a list.

```
[extract!(h) while not empty?(h)]
```

$$[9,\ 7,\ 3,\ 2,\ -4,\ -7] \tag{8}$$

List (Integer)

Another way to produce the same result is by defining a **heapsort** function.

```
heapsort(x) == (empty? x => []; cons(extract!(x),heapsort x))
```

Create another sample heap.

```
h1 := heap [17,-4,9,-11,2,7,-7]
```

$$[17,\ 2,\ 9,\ -11,\ -4,\ 7,\ -7] \tag{10}$$

Heap(Integer)

Apply heapsort to present elements in order.

```
heapsort h1
```

```
   Compiling function heapsort with type Heap(Integer) -> List(Integer)
```

$$[17,\ 9,\ 7,\ 2,\ -4,\ -7,\ -11] \tag{11}$$

List (Integer)

9.37 HexadecimalExpansion

All rationals have repeating hexadecimal expansions. The operation hex returns these expansions of type **HexadecimalExpansion**. Operations to access the individual numerals of a hexadecimal expansion can be obtained by converting the value to **RadixExpansion(16)**. More examples of expansions are available in the '`DecimalExpansion`' on page 392, '`BinaryExpansion`' on page 348, and '`RadixExpansion`' on page 541.

Issue the system command `)show HexadecimalExpansion` to display the full list of operations defined by **HexadecimalExpansion**.

This is a hexadecimal expansion of a rational number.

```
r := hex(22/7)
```

$$3.\overline{249} \tag{4}$$

HexadecimalExpansion

Arithmetic is exact.

```
r + hex(6/7)
```

$$4 \tag{5}$$

HexadecimalExpansion

The period of the expansion can be short or long ...

```
[hex(1/i) for i in 350..354]
```

$$\left[0.00\overline{\texttt{BB3EE721A54D88}},\ 0.\overline{00\texttt{BAB6561}},\ 0.00\overline{\texttt{BA2E8}},\right.$$
$$\left.0.\overline{00\texttt{B9A7862A0FF465879D5F}},\ 0.00\overline{\texttt{B92143FA36F5E02E4850FE8DBD78}}\right] \tag{6}$$

List (HexadecimalExpansion)

or very long!

```
hex(1/1007)
```

$$0.\overline{0041149783\texttt{F0BF2C7D13933192AF6980619EE345E91EC2BB9D5CCA5C071E40926E54E8DDAE24196C0B2F8A0AAD60D}\texttt{BA57F5D4C853626221}$$

HexadecimalExpansion

These numbers are bona fide algebraic objects.

```
p := hex(1/4)*x^2 + hex(2/3)*x + hex(4/9)
```

$$0.4\,x^2 + 0.\overline{\texttt{A}}\,x + 0.\overline{\texttt{71C}} \tag{8}$$

Polynomial(HexadecimalExpansion)

```
q := D(p, x)
```

$$0.8\,x + 0.\overline{\texttt{A}} \tag{9}$$

Polynomial(HexadecimalExpansion)

```
g := gcd(p, q)
```

$$x + 1.\overline{5} \tag{10}$$

Polynomial(HexadecimalExpansion)

9.38 Integer

FriCAS provides many operations for manipulating arbitrary precision integers. In this section we will show some of those that come from **Integer** itself plus some that are implemented in other packages. More examples of using integers are in the following sections: 'Some Numbers' in Section 1.5 on page 31, '`IntegerNumberTheoryFunctions`' on page 450, '`DecimalExpansion`' on page 392, '`BinaryExpansion`' on page 348, '`HexadecimalExpansion`' on page 440, and '`RadixExpansion`' on page 541.

9.38.1 Basic Functions

The size of an integer in FriCAS is only limited by the amount of computer storage you have available. The usual arithmetic operations are available.

```
2^(5678 - 4856 + 2 * 17)
```

$$4804810770435008147181540925125924391239526139871682263473855610088084200076308293086342527091141208374307457227 8$$ (1)

PositiveInteger

There are a number of ways of working with the sign of an integer. Let's use this `x` as an example.

```
x := -101
```

$$-101 \tag{2}$$

Integer

First of all, there is the absolute value function.

```
abs(x)
```

$$101 \tag{3}$$

PositiveInteger

The sign operation returns `-1` if its argument is negative, `0` if zero and `1` if positive.

```
sign(x)
```

$$-1 \tag{4}$$

Integer

You can determine if an integer is negative in several other ways.

```
x < 0
```

```
true
```
(5)

Boolean

```
x <= -1
```

```
true
```
(6)

Boolean

```
negative?(x)
```

```
true
```
(7)

Boolean

Similarly, you can find out if it is positive.

```
x > 0
```

```
false
```
(8)

Boolean

```
x >= 1
```

```
false
```
(9)

Boolean

```
positive?(x)
```

```
false
```
(10)

Boolean

This is the recommended way of determining whether an integer is zero.

```
zero?(x)
```

```
false
```
(11)

Boolean

> Use the zero? operation whenever you are testing any mathematical object for equality with zero. This is usually more efficient than using = (think of matrices: it is easier to tell if a matrix is zero by just checking term by term than constructing another "zero" matrix and comparing the two matrices term by term) and also avoids the problem that = is usually used for creating equations.

This is the recommended way of determining whether an integer is equal to one.

```
one?(x)
```

```
false
```
(12)

Boolean

This syntax is used to test equality using =. It says that you want a **Boolean** (`true` or `false`) answer rather than an equation.

```
(x = -101)@Boolean
```

```
true
```
(13)

Boolean

The operations odd? and even? determine whether an integer is odd or even, respectively. They each return a **Boolean** object.

```
odd?(x)
```

```
true
```
(14)

Boolean

```
even?(x)
```

```
false                                                    (15)
                                                         Boolean
```

The operation gcd computes the greatest common divisor of two integers.

```
gcd(56788,43688)
```

```
4                                                        (16)
                                                         PositiveInteger
```

The operation lcm computes their least common multiple.

```
lcm(56788,43688)
```

```
620238536                                                (17)
                                                         PositiveInteger
```

To determine the maximum of two integers, use max.

```
max(678,567)
```

```
678                                                      (18)
                                                         PositiveInteger
```

To determine the minimum, use min.

```
min(678,567)
```

```
567                                                      (19)
                                                         PositiveInteger
```

The reduce operation is used to extend binary operations to more than two arguments. For example, you can use reduce to find the maximum integer in a list or compute the least common multiple of all integers in the list.

```
reduce(max,[2,45,-89,78,100,-45])
```

```
100                                                      (20)
                                                         PositiveInteger
```

```
reduce(min,[2,45,-89,78,100,-45])
```

```
-89                                                      (21)
                                                         Integer
```

```
reduce(gcd,[2,45,-89,78,100,-45])
```

$$1 \tag{22}$$

PositiveInteger

```
reduce(lcm,[2,45,-89,78,100,-45])
```

$$1041300 \tag{23}$$

PositiveInteger

The infix operator "/" is *not* used to compute the quotient of integers. Rather, it is used to create rational numbers as described in '`Fraction`' on page 424.

```
13 / 4
```

$$\frac{13}{4} \tag{24}$$

Fraction (Integer)

The infix operation quo computes the integer quotient.

```
13 quo 4
```

$$3 \tag{25}$$

PositiveInteger

The infix operation rem computes the integer remainder.

```
13 rem 4
```

$$1 \tag{26}$$

PositiveInteger

One integer is evenly divisible by another if the remainder is zero. The operation exquo can also be used. See Section 2.5 on page 79 for an example.

```
zero?(167604736446952 rem 2003644)
```

`true` (27)

Boolean

The operation divide returns a record of the quotient and remainder and thus is more efficient when both are needed.

```
d := divide(13,4)
```

$$[quotient = 3, remainder = 1] \tag{28}$$

Record(quotient: Integer , remainder: Integer)

```
d.quotient
```

$$3 \tag{29}$$

PositiveInteger

Records are discussed in detail in Section 2.4 on page 76.

```
d.remainder
```

$$1 \tag{30}$$

PositiveInteger

9.38.2 Primes and Factorization

Use the operation factor to factor integers. It returns an object of type **Factored Integer**. See '`Factored`' on page 405 for a discussion of the manipulation of factored objects.

```
factor 102400
```

$$2^{12}\,5^2 \tag{1}$$

Factored(Integer)

The operation prime? returns `true` or `false` depending on whether its argument is a prime.

```
prime? 7
```

$$\texttt{true} \tag{2}$$

Boolean

```
prime? 8
```

$$\texttt{false} \tag{3}$$

Boolean

The operation nextPrime returns the least prime number greater than its argument.

```
nextPrime 100
```

$$101 \tag{4}$$

PositiveInteger

The operation prevPrime returns the greatest prime number less than its argument.

```
prevPrime 100
```

$$97 \tag{5}$$

PositiveInteger

To compute all primes between two integers (inclusively), use the operation primes.

```
primes(100,175)
```

$$[101,\ 103,\ 107,\ 109,\ 113,\ 127,\ 131,\ 137,\ 139,\ 149,\ 151,\ 157,\ 163,\ 167,\ 173] \tag{6}$$

List (Integer)

You might sometimes want to see the factorization of an integer when it is considered a *Gaussian integer*. See '`Complex`' on page 372 for more details.

```
factor(2 :: Complex Integer)
```

$$-i\,(1+i)^2 \tag{7}$$

Factored(Complex(Integer))

9.38.3 Some Number Theoretic Functions

FriCAS provides several number theoretic operations for integers. More examples are in '`IntegerNumberTheoryFunctions`' on page 450.

The operation fibonacci computes the Fibonacci numbers. The algorithm has running time $O\left(\log^3(n)\right)$ for argument `n`.

```
[fibonacci(k) for k in 0..]
```

$$[0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ \ldots] \tag{1}$$

Stream(Integer)

The operation legendre computes the Legendre symbol for its two integer arguments where the second one is prime. If you know the second argument to be prime, use jacobi instead where no check is made.

```
[legendre(i,11) for i in 0..10]
```

$$[0,\ 1,\ -1,\ 1,\ 1,\ 1,\ -1,\ -1,\ -1,\ 1,\ -1] \tag{2}$$

List (Integer)

The operation jacobi computes the Jacobi symbol for its two integer arguments. By convention, `0` is returned if the greatest common divisor of the numerator and denominator is not `1`.

```
[jacobi(i,15) for i in 0..9]
```

$$[0,\ 1,\ 1,\ 0,\ 1,\ 0,\ 0,\ -1,\ 1,\ 0] \tag{3}$$

List (Integer)

The operation eulerPhi computes the values of Euler's φ-function where φ`(n)` equals the number of positive integers less than or equal to `n` that are relatively prime to the positive integer `n`.

```
[eulerPhi i for i in 1..]
```

$$[1,\ 1,\ 2,\ 2,\ 4,\ 2,\ 6,\ \ldots] \tag{4}$$

Stream(Integer)

The operation moebiusMu computes the Möbius μ function.

```
[moebiusMu i for i in 1..]
```

$$[1,\ -1,\ -1,\ 0,\ -1,\ 1,\ -1,\ \ldots] \tag{5}$$

Stream(Integer)

Although they have somewhat limited utility, FriCAS provides Roman numerals.

```
a := roman(78)
```

$$LXXVIII \tag{6}$$

RomanNumeral

```
b := roman(87)
```

$$LXXXVII \tag{7}$$

RomanNumeral

```
a + b
```

$$CLXV \tag{8}$$

RomanNumeral

```
a * b
```

$$MMMMMMDCCLXXXVI \tag{9}$$

RomanNumeral

```
b rem a
```

$$IX \tag{10}$$

RomanNumeral

9.39 IntegerLinearDependence

The elements $v_1, \ldots, v_n$ of a module M over a ring R are said to be *linearly dependent over R* if there exist $c_1, \ldots, c_n$ in R, not all 0, such that $c_1 v_1 + \ldots c_n v_n = 0$. If such c_i's exist, they form what is called a *linear dependence relation over R* for the v_i's.

The package **IntegerLinearDependence** provides functions for testing whether some elements of a module over the integers are linearly dependent over the integers, and to find the linear dependence relations, if any. Consider the domain of two by two square matrices with integer entries.

```
M := SQMATRIX(2,INT)
```

$$\text{SquareMatrix}(2, Integer) \tag{4}$$

Type

Now create three such matrices.

```
m1: M := squareMatrix matrix [[1, 2], [0, -1]]
```

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \tag{5}$$

SquareMatrix(2, Integer)

```
m2: M := squareMatrix matrix [[2, 3], [1, -2]]
```

$$\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix} \tag{6}$$

SquareMatrix(2, Integer)

```
m3: M := squareMatrix matrix [[3, 4], [2, -3]]
```

$$\begin{bmatrix} 3 & 4 \\ 2 & -3 \end{bmatrix} \tag{7}$$

SquareMatrix(2, Integer)

This tells you whether m1, m2 and m3 are linearly dependent over the integers.

```
linearlyDependentOverZ? vector [m1, m2, m3]
```

$$\texttt{true} \tag{8}$$

Boolean

Since they are linearly dependent, you can ask for the dependence relation.

```
c := linearDependenceOverZ vector [m1, m2, m3]
```

$$[1, -2, 1] \tag{9}$$

Union(Vector(Integer), ...)

This means that the following linear combination should be 0.

```
c.1 * m1 + c.2 * m2 + c.3 * m3
```

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{10}$$

SquareMatrix(2, Integer)

When a given set of elements are linearly dependent over R, this also means that at least one of them can be rewritten as a linear combination of the others with coefficients in the quotient field of R. To express a given element in terms of other elements, use the operation solveLinearlyOverQ.

```
solveLinearlyOverQ(vector [m1, m3], m2)
```

$$\left[particular = \left[\frac{1}{2}, \frac{1}{2}\right], basis = []\right] \tag{11}$$

Record(particular : Union(Vector(Fraction(Integer)), " failed "), basis : List (Vector(Fraction (Integer))))

9.40 IntegerNumberTheoryFunctions

The **IntegerNumberTheoryFunctions** package contains a variety of operations of interest to number theorists. Many of these operations deal with divisibility properties of integers. (Recall that an integer a divides an integer b if there is an integer c such that b = a * c.)

The operation divisors returns a list of the divisors of an integer.

```
div144 := divisors(144)
```

$$[1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 36, 48, 72, 144] \tag{4}$$

List (Integer)

You can now compute the number of divisors of 144 and the sum of the divisors of 144 by counting and summing the elements of the list we just created.

```
#(div144)
```

$$15 \tag{5}$$

PositiveInteger

```
reduce(+,div144)
```

$$403 \tag{6}$$

PositiveInteger

Of course, you can compute the number of divisors of an integer n, usually denoted d(n), and the sum of the divisors of an integer n, usually denoted σ(n), without ever listing the divisors of n.

In FriCAS, you can simply call the operations numberOfDivisors and sumOfDivisors.

```
numberOfDivisors(144)
```

$$15 \tag{7}$$

PositiveInteger

```
sumOfDivisors(144)
```

$$403 \tag{8}$$

PositiveInteger

The key is that `d(n)` and σ`(n)` are "multiplicative functions." This means that when `n` and `m` are relatively prime, that is, when `n` and `m` have no prime factor in common, then `d(nm) = d(n) d(m)` and σ`(nm) =` σ`(n)` σ`(m)`. Note that these functions are trivial to compute when `n` is a prime power and are computed for general `n` from the prime factorization of `n`. Other examples of multiplicative functions are σ_k`(n)`, the sum of the k^{th} powers of the divisors of `n` and φ`(n)`, the number of integers between 1 and `n` which are prime to `n`. The corresponding FriCAS operations are called sumOfKthPowerDivisors and eulerPhi.

An interesting function is μ`(n)`, the Möbius μ function, defined as follows: μ`(1) = 1`, μ`(n) = 0`, when `n` is divisible by a square, and $\mu = (-1)^k$, when `n` is the product of `k` distinct primes. The corresponding FriCAS operation is moebiusMu. This function occurs in the following theorem:

Theorem (Möbius Inversion Formula):
Let `f(n)` be a function on the positive integers and let `F(n)` be defined by

$$F(n) = \sum_{d|n} f(d)$$

where the sum is taken over the positive divisors of `n`. Then the values of `f(n)` can be recovered from the values of `F(n)`:

$$f(n) = \sum_{d|n} \mu(d) F(\frac{n}{d})$$

where again the sum is taken over the positive divisors of `n`.

When `f(n) = 1`, then `F(n) = d(n)`. Thus, if you sum μ`(d) d(n/d)` over the positive divisors `d` of `n`, you should always get `1`.

```
f1(n) == reduce(+,[moebiusMu(d) * numberOfDivisors(quo(n,d)) for d in divisors(n)])
```

```
f1(200)
```

```
   Compiling function f1 with type PositiveInteger -> Integer
```

$$1 \tag{10}$$

PositiveInteger

```
f1(846)
```

$$1 \tag{11}$$

PositiveInteger

Similarly, when `f(n) = n`, then `F(n) =` σ`(n)`. Thus, if you sum μ`(d)` σ `(n/d)` over the positive divisors `d` of `n`, you should always get `n`.

```
f2(n) == reduce(+,[moebiusMu(d) * sumOfDivisors(quo(n,d)) for d in divisors(n)])
```

```
f2(200)
```

```
   Compiling function f2 with type PositiveInteger -> Integer
```

$$200 \tag{13}$$

PositiveInteger

```
f2(846)
```

$$846 \tag{14}$$

PositiveInteger

The Möbius inversion formula is derived from the multiplication of formal Dirichlet series. A Dirichlet series is an infinite series of the form

$$\sum_{n=1}^{\infty} a(n)n^{-s}$$

When

$$\sum_{n=1}^{\infty} a(n)n^{-s} \cdot \sum_{n=1}^{\infty} b(n)n^{-s} = \sum_{n=1}^{\infty} c(n)n^{-s}$$

then $c(n) = \sum_{d|n} a(d)b(n/d)$. Recall that the Riemann ζ function is defined by

$$\zeta(s) = \prod_{p}(1-p^{-s})^{-1} = \sigma_{n=1}^{\infty} n^{-s}$$

where the product is taken over the set of (positive) primes. Thus,

$$\zeta(s)^{-1} = \prod_{p}(1-p^{-s}) = \sigma_{n=1}^{\infty}\mu(n)n^{-s}$$

Now if $F(n) = \sum_{d|n)f(d)}$, then

$$\sum f(n)n^{-s} \cdot \zeta(s) = \sum F(n)n^{-s}$$

Thus,

$$\zeta(s)^{-1} \cdot \sum F(n)n^{-s} = \sum f(n)n^{-s}$$

and $f(n) = \sum_{d|n} \mu(d)F(n/d)$.

The Fibonacci numbers are defined by `F(1) = F(2) = 1` and `F(n) = F(n-1)+ F(n-2)` for `n = 3,4, ...`. The operation fibonacci computes the n^{th} Fibonacci number.

```
fibonacci(25)
```

$$75025 \tag{15}$$

PositiveInteger

```
[fibonacci(n) for n in 1..15]
```

$$[1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ 233,\ 377,\ 610] \tag{16}$$

List (Integer)

Fibonacci numbers can also be expressed as sums of binomial coefficients.

```
fib(n) == reduce(+,[binomial(n-1-k,k) for k in 0..quo(n-1,2)])
```

```
fib(25)
```

```
   Compiling function fib with type PositiveInteger -> Integer
```

$$75025 \tag{18}$$

PositiveInteger

```
[fib(n) for n in 1..15]
```

$$[1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ 233,\ 377,\ 610] \tag{19}$$

List (Integer)

Quadratic symbols can be computed with the operations legendre and jacobi. The Legendre symbol $\left(\frac{a}{p}\right)$ is defined for integers a and p with p an odd prime number. By definition, $\left(\frac{a}{p}\right)$, when a is a square (mod p), $\left(\frac{a}{p}\right)$, when a is not a square (mod p), and $\left(\frac{a}{p}\right)$, when a is divisible by p. You compute $\left(\frac{a}{p}\right)$ via the command legendre(a,p).

```
legendre(3,5)
```

$$-1 \tag{20}$$

Integer

```
legendre(23,691)
```

$$-1 \tag{21}$$

Integer

The Jacobi symbol $\left(\frac{a}{p}\right)$ is the usual extension of the Legendre symbol, where n is an arbitrary integer. The most important property of the Jacobi symbol is the following: if K is a quadratic field with discriminant d and quadratic character χ, then χ(n) = (d/n). Thus, you can use the Jacobi symbol to compute, say, the class numbers of imaginary quadratic fields from a standard class number formula.

This function computes the class number of the imaginary quadratic field with discriminant d.

```
h(d) == quo(reduce(+, [jacobi(d,k) for k in 1..quo(-d, 2)]), 2 - jacobi(d,2))
```

```
h(-163)
```

```
   Compiling function h with type Integer -> Integer
```

$$1 \tag{23}$$

PositiveInteger

```
h(-499)
```

$$3 \tag{24}$$

PositiveInteger

```
h(-1832)
```

$$26 \tag{25}$$

PositiveInteger

9.41 Kernel

A *kernel* is a symbolic function application (such as `sin(x + y)`) or a symbol (such as `x`). More precisely, a non-symbol kernel over a set S is an operator applied to a given list of arguments from S. The operator has type **BasicOperator** (see '`BasicOperator`' on page 345) and the kernel object is usually part of an expression object (see '`Expression`' on page 400).

Kernels are created implicitly for you when you create expressions.

```
x :: Expression Integer
```

$$x \tag{4}$$

Expression (Integer)

You can directly create a "symbol" kernel by using the kernel operation.

```
kernel x
```

$$x \tag{5}$$

Kernel (Expression (Integer))

This expression has two different kernels.

```
sin(x) + cos(x)
```

$$\sin(x) + \cos(x) \tag{6}$$

Expression (Integer)

The operator kernels returns a list of the kernels in an object of type **Expression**.

```
kernels %
```

$$[\sin(x),\ \cos(x)] \tag{7}$$

List (Kernel (Expression (Integer)))

This expression also has two different kernels.

```
sin(x)^2 + sin(x) + cos(x)
```

$$\sin(x)^2 + \sin(x) + \cos(x) \tag{8}$$

Expression (Integer)

The `sin(x)` kernel is used twice.

```
kernels %
```

$$[\sin(x),\ \cos(x)] \tag{9}$$

List (Kernel (Expression (Integer)))

An expression need not contain any kernels.

```
kernels(1 :: Expression Integer)
```

$$[] \tag{10}$$

List (Kernel (Expression (Integer)))

If one or more kernels are present, one of them is designated the *main* kernel.

```
mainKernel(cos(x) + tan(x))
```

$$\tan(x) \tag{11}$$

Union(Kernel(Expression (Integer)) , ...)

Kernels can be nested. Use height to determine the nesting depth.

```
height kernel x
```

$$1 \tag{12}$$

PositiveInteger

This has height 2 because the `x` has height 1 and then we apply an operator to that.

```
height mainKernel(sin x)
```

$$2 \tag{13}$$

PositiveInteger

```
height mainKernel(sin cos x)
```

$$3 \tag{14}$$

PositiveInteger

```
height mainKernel(sin cos (tan x + sin x))
```

$$4 \tag{15}$$

PositiveInteger

Use the operator operation to extract the operator component of the kernel. The operator has type **BasicOperator**.

```
operator mainKernel(sin cos (tan x + sin x))
```

$$sin \tag{16}$$

BasicOperator

Use the name operation to extract the name of the operator component of the kernel. The name has type **Symbol**. This is really just a shortcut for a two-step process of extracting the operator and then calling name on the operator.

```
name mainKernel(sin cos (tan x + sin x))
```

$$sin \tag{17}$$

Symbol

FriCAS knows about functions such as sin, cos and so on and can make kernels and then expressions using them. To create a kernel and expression using an arbitrary operator, use operator. Now `f` can be used to create symbolic function applications.

```
f := operator 'f
```

$$f \tag{18}$$

BasicOperator

```
e := f(x, y, 10)
```

$$f(x, y, 10) \tag{19}$$

Expression (Integer)

Use the is? operation to learn if the operator component of a kernel is equal to a given operator.

```
is?(e, f)
```

$$\texttt{true} \tag{20}$$

Boolean

You can also use a symbol or a string as the second argument to is?.

```
is?(e, 'f)
```

$$\texttt{true} \tag{21}$$

Boolean

Use the argument operation to get a list containing the argument component of a kernel.

```
argument mainKernel e
```

$$[x,\ y,\ 10] \tag{22}$$

List (Expression (Integer))

Conceptually, an object of type **Expression** can be thought of as a quotient of multivariate polynomials, where the "variables" are kernels. The arguments of the kernels are again expressions and so the structure recurses. See '`Expression`' on page 400 for examples of using kernels to take apart expression objects.

9.42 KeyedAccessFile

The domain **KeyedAccessFile(S)** provides files which can be used as associative tables. Data values are stored in these files and can be retrieved according to their keys. The keys must be strings so this type behaves very much like the **StringTable(S)** domain. The difference is that keyed access files reside in secondary storage while string tables are kept in memory. For more information on table-oriented operations, see the description of **Table**.

Before a keyed access file can be used, it must first be opened. A new file can be created by opening it for output.

```
ey: KeyedAccessFile(Integer) := open("/tmp/editor.year", "output")
```

"/tmp/editor.year" (4)

KeyedAccessFile(Integer)

Just as for vectors, tables or lists, values are saved in a keyed access file by setting elements.

```
ey."Char"     := 1986
```

1986 (5)

PositiveInteger

```
ey."Caviness" := 1985
```

1985 (6)

PositiveInteger

```
ey."Fitch"    := 1984
```

1984 (7)

PositiveInteger

Values are retrieved using application, in any of its syntactic forms.

```
ey."Char"
```

1986 (8)

PositiveInteger

```
ey("Char")
```

1986 (9)

PositiveInteger

```
ey "Char"
```

1986 (10)

PositiveInteger

Attempting to retrieve a non-existent element in this way causes an error. If it is not known whether a key exists, you should use the search operation.

```
search("Char", ey)
```

1986 (11)

Union(Integer, ...)

```
search("Smith", ey)
```

"failed" (12)

Union("failed", ...)

When an entry is no longer needed, it can be removed from the file.

```
remove!("Char", ey)
```

1986 (13)

Union(Integer, ...)

The keys operation returns a list of all the keys for a given file.

```
keys ey
```

["Fitch", "Caviness"] (14)

List(String)

The # operation gives the number of entries.

```
#ey
```

2 (15)

PositiveInteger

The table view of keyed access files provides safe operations. That is, if the FriCAS program is terminated between file operations, the file is left in a consistent, current state. This means, however, that the operations are somewhat costly. For example, after each update the file is closed. Here we add several more items to the file, then check its contents.

```
KE := Record(key: String, entry: Integer)
```

$$\text{Record}(key: String, entry: Integer) \tag{16}$$

Type

```
reopen!(ey, "output")
```

$$\texttt{"/tmp/editor.year"} \tag{17}$$

KeyedAccessFile(Integer)

If many items are to be added to a file at the same time, then it is more efficient to use the write! operation.

```
write!(ey, ["van Hulzen", 1983]$KE)
```

$$[key = \texttt{"van Hulzen"}, entry = 1983] \tag{18}$$

Record(key: String , entry: Integer)

```
write!(ey, ["Calmet", 1982]$KE)
```

$$[key = \texttt{"Calmet"}, entry = 1982] \tag{19}$$

Record(key: String , entry: Integer)

```
write!(ey, ["Wang", 1981]$KE)
```

$$[key = \texttt{"Wang"}, entry = 1981] \tag{20}$$

Record(key: String , entry: Integer)

```
close! ey
```

$$\texttt{"/tmp/editor.year"} \tag{21}$$

KeyedAccessFile(Integer)

The read! operation is also available from the file view, but it returns elements in a random order. It is generally clearer and more efficient to use the keys operation and to extract elements by key.

```
keys ey
```

$$[\texttt{"Wang"}, \texttt{"Calmet"}, \texttt{"van Hulzen"}, \texttt{"Fitch"}, \texttt{"Caviness"}] \tag{22}$$

List (String)

```
members ey
```

$$[1981, 1982, 1983, 1984, 1985] \tag{23}$$

List (Integer)

```
)system rm -r /tmp/editor.year
```

For more information on related topics, see 'File' on page 412, 'TextFile' on page 591, and 'Library' on page 474. Issue the system command)show KeyedAccessFile to display the full list of operations defined by **KeyedAccessFile**.

9.43 LLLReduction

The package **LLLReduction** implements LLL reduction. We show how to use it to find equation satisfied by imaginary part of fifth primitive root of 1.

```
digits(24)
```

$$20 \tag{4}$$

PositiveInteger

```
ii := imag(exp(2.0*%i*%pi/5))
```

$$0.95105651629515357211643 \tag{5}$$

Float

```
lf := [ii^i for i in 0..4]
```

$$[1.0,\ 0.95105651629515357211643,\ 0.90450849718747371205114{\tt 7},\ 0.86023870029448346137950{\tt 6},\ 0.81813562148434214006393{\tt 3}] \tag{6}$$

List (Float)

```
rel := find_relation(lf, 20)$LLLReduction
```

$$[-5,\ 0,\ 20,\ 0,\ -16] \tag{7}$$

List (Integer)

```
pol := reduce(_+, [ci*x^i for ci in rel for i in 0..4])
```

$$-16\,x^4 + 20\,x^2 - 5 \tag{8}$$

Polynomial(Integer)

```
eval(pol, x = ii)
```

$$0.0 \tag{9}$$

Polynomial(Float)

9.44 LazardSetSolvingPackage

The **LazardSetSolvingPackage** package constructor solves polynomial systems by means of Lazard triangular sets. However one condition is relaxed: Regular triangular sets whose saturated ideals have positive dimension are not necessarily normalized.

The decompositions are computed in two steps. First the algorithm of Moreno Maza (implemented in the **RegularTriangularSet** domain constructor) is called. Then the resulting decompositions are converted into lists of square-free regular triangular sets and the redundant components are removed. Moreover, zero-dimensional regular triangular sets are normalized.

Note that the way of understanding triangular decompositions is detailed in the example of the **RegularTriangularSet** constructor.

The **LazardSetSolvingPackage** constructor takes six arguments. The first one, **R**, is the coefficient ring of the polynomials; it must belong to the category **GcdDomain**. The second one, **E**, is the exponent monoid of the polynomials; it must belong to the category **OrderedAbelianMonoidSup**. The third one, **V**, is the ordered set of variables; it must belong to the category **OrderedSet**. The fourth one is the polynomial ring; it must belong to the category **RecursivePolynomialCategory(R,E,V)**. The fifth one is a domain of the category **RegularTriangularSetCategory(R,E,V,P)** and the last one is a domain of the category **SquareFreeRegularTriangularSetCategory(R,E,V,P)**. The abbreviation for **LazardSetSolvingPackage** is `LAZM3PK`.

N.B. For the purpose of solving zero-dimensional algebraic systems, see also **LexTriangularPackage** and **ZeroDimensionalSolvePackage**. These packages are easier to call than `LAZM3PK`. Moreover, the **ZeroDimensionalSolvePackage** package provides operations to compute either the complex roots or the real roots.

We illustrate now the use of the **LazardSetSolvingPackage** package constructor with two examples (Butcher and Vermeer).

Define the coefficient ring.

```
R := Integer
```

$$Integer \tag{4}$$

Type

Define the list of variables,

```
ls : List Symbol := [b1,x,y,z,t,v,u,w]
```

$$[b1,\ x,\ y,\ z,\ t,\ v,\ u,\ w] \tag{5}$$

List (Symbol)

and make it an ordered set;

```
V := OVAR(ls)
```

$$\text{OrderedVariableList}([b1,\ x,\ y,\ z,\ t,\ v,\ u,\ w]) \tag{6}$$

Type

then define the exponent monoid.

```
E := IndexedExponents V
```

$$\text{IndexedExponents}(\text{OrderedVariableList}([b1,\ x,\ y,\ z,\ t,\ v,\ u,\ w])) \tag{7}$$

Type

Define the polynomial ring.

```
P := NSMP(R, V)
```

$$\text{NewSparseMultivariatePolynomial}(Integer, \text{OrderedVariableList}([b1,\ x,\ y,\ z,\ t,\ v,\ u,\ w])) \tag{8}$$

Type

Let the variables be polynomial.

```
b1: P := 'b1
```

$$b1 \tag{9}$$

NewSparseMultivariatePolynomial(Integer , OrderedVariableList ([b1, x, y, z, t, v, u, w]))

```
x: P := 'x
```

$$x \tag{10}$$

NewSparseMultivariatePolynomial(Integer , OrderedVariableList ([b1, x, y, z, t, v, u, w]))

```
y: P := 'y
```

$$y \tag{11}$$

NewSparseMultivariatePolynomial(Integer , OrderedVariableList ([b1, x, y, z, t, v, u, w]))

```
z: P := 'z
```

$$z \tag{12}$$

NewSparseMultivariatePolynomial(Integer , OrderedVariableList ([b1, x, y, z, t, v, u, w]))

```
t: P := 't
```

$$t \tag{13}$$

NewSparseMultivariatePolynomial(Integer , OrderedVariableList ([b1, x, y, z, t, v, u, w]))

```
u: P := 'u
```

$$u \tag{14}$$

NewSparseMultivariatePolynomial(Integer, OrderedVariableList([b1, x, y, z, t, v, u, w]))

```
v: P := 'v
```

$$v \tag{15}$$

NewSparseMultivariatePolynomial(Integer, OrderedVariableList([b1, x, y, z, t, v, u, w]))

```
w: P := 'w
```

$$w \tag{16}$$

NewSparseMultivariatePolynomial(Integer, OrderedVariableList([b1, x, y, z, t, v, u, w]))

Now call the **RegularTriangularSet** domain constructor.

```
T := REGSET(R,E,V,P)
```

$$\begin{aligned}&\text{RegularTriangularSet}(Integer, \text{IndexedExponents}(\text{OrderedVariableList}([b1,\\ &x, y, z, t, v, u, w])), \text{OrderedVariableList}([b1, x, y, z, t, v, u,\\ &w]), \text{NewSparseMultivariatePolynomial}(Integer, \text{OrderedVariableList}([b1, x, y, z, t, v, u, w])))\end{aligned} \tag{17}$$

Type

Define a polynomial system (the Butcher example).

```
p0 := b1 + y + z - t - w
```

$$b1 + y + z - t - w \tag{18}$$

NewSparseMultivariatePolynomial(Integer, OrderedVariableList([b1, x, y, z, t, v, u, w]))

```
p1 := 2*z*u + 2*y*v + 2*t*w - 2*w^2 - w - 1
```

$$2\,v\,y + 2\,u\,z + 2\,w\,t - 2\,w^2 - w - 1 \tag{19}$$

NewSparseMultivariatePolynomial(Integer, OrderedVariableList([b1, x, y, z, t, v, u, w]))

```
p2 := 3*z*u^2 + 3*y*v^2 - 3*t*w^2 + 3*w^3 + 3*w^2 - t + 4*w
```

$$3\,v^2\,y + 3\,u^2\,z + \left(-3\,w^2 - 1\right)\,t + 3\,w^3 + 3\,w^2 + 4\,w \tag{20}$$

NewSparseMultivariatePolynomial(Integer, OrderedVariableList([b1, x, y, z, t, v, u, w]))

```
p3 := 6*x*z*v - 6*t*w^2 + 6*w^3 - 3*t*w + 6*w^2 - t + 4*w
```

$$6\,v\,z\,x + \left(-6\,w^2 - 3\,w - 1\right)\,t + 6\,w^3 + 6\,w^2 + 4\,w \tag{21}$$

NewSparseMultivariatePolynomial(Integer, OrderedVariableList([b1, x, y, z, t, v, u, w]))

```
p4 := 4*z*u^3+ 4*y*v^3+ 4*t*w^3- 4*w^4 - 6*w^3+ 4*t*w- 10*w^2- w- 1
```

$$4\,v^3\,y + 4\,u^3\,z + \left(4\,w^3 + 4\,w\right)\,t - 4\,w^4 - 6\,w^3 - 10\,w^2 - w - 1 \tag{22}$$

NewSparseMultivariatePolynomial(Integer , OrderedVariableList ([b1, x, y, z, t, v, u, w]))

```
p5 := 8*x*z*u*v +8*t*w^3 -8*w^4 +4*t*w^2 -12*w^3 +4*t*w -14*w^2 -3*w -1
```

$$8\,u\,v\,z\,x + \left(8\,w^3 + 4\,w^2 + 4\,w\right)\,t - 8\,w^4 - 12\,w^3 - 14\,w^2 - 3\,w - 1 \tag{23}$$

NewSparseMultivariatePolynomial(Integer , OrderedVariableList ([b1, x, y, z, t, v, u, w]))

```
p6 := 12*x*z*v^2+12*t*w^3 -12*w^4 +12*t*w^2 -18*w^3 +8*t*w -14*w^2 -w -1
```

$$12\,v^2\,z\,x + \left(12\,w^3 + 12\,w^2 + 8\,w\right)\,t - 12\,w^4 - 18\,w^3 - 14\,w^2 - w - 1 \tag{24}$$

NewSparseMultivariatePolynomial(Integer , OrderedVariableList ([b1, x, y, z, t, v, u, w]))

```
p7 := -24*t*w^3 + 24*w^4 - 24*t*w^2 + 36*w^3 - 8*t*w + 26*w^2 + 7*w + 1
```

$$\left(-24\,w^3 - 24\,w^2 - 8\,w\right)\,t + 24\,w^4 + 36\,w^3 + 26\,w^2 + 7\,w + 1 \tag{25}$$

NewSparseMultivariatePolynomial(Integer , OrderedVariableList ([b1, x, y, z, t, v, u, w]))

```
lp := [p0, p1, p2, p3, p4, p5, p6, p7]
```

$$\begin{aligned}
&\left[b1 + y + z - t - w,\ 2\,v\,y + 2\,u\,z + 2\,w\,t - 2\,w^2 - w - 1,\right.\\
&3\,v^2\,y + 3\,u^2\,z + \left(-3\,w^2 - 1\right)\,t + 3\,w^3 + 3\,w^2 + 4\,w,\\
&6\,v\,z\,x + \left(-6\,w^2 - 3\,w - 1\right)\,t + 6\,w^3 + 6\,w^2 + 4\,w,\\
&4\,v^3\,y + 4\,u^3\,z + \left(4\,w^3 + 4\,w\right)\,t - 4\,w^4 - 6\,w^3 - 10\,w^2 - w - 1,\\
&8\,u\,v\,z\,x + \left(8\,w^3 + 4\,w^2 + 4\,w\right)\,t - 8\,w^4 - 12\,w^3 - 14\,w^2 - 3\,w - 1,\\
&12\,v^2\,z\,x + \left(12\,w^3 + 12\,w^2 + 8\,w\right)\,t - 12\,w^4 - 18\,w^3 - 14\,w^2 - w - 1,\\
&\left.\left(-24\,w^3 - 24\,w^2 - 8\,w\right)\,t + 24\,w^4 + 36\,w^3 + 26\,w^2 + 7\,w + 1\right]
\end{aligned} \tag{26}$$

List (NewSparseMultivariatePolynomial(Integer , OrderedVariableList ([b1, x, y, z, t, v, u, w])))

First of all, let us solve this system in the sense of Lazard by means of the **REGSET** constructor:

```
lts := zeroSetSplit(lp,false)$T
```

$$\begin{aligned}
&\left[\left\{w + 1,\ u,\ v,\ t + 1,\ b1 + y + z + 2\right\},\ \left\{w + 1,\ v,\ t + 1,\ z,\ b1 + y + 2\right\},\right.\\
&\left\{w + 1,\ t + 1,\ z,\ y,\ b1 + 2\right\},\ \left\{w + 1,\ v - u,\ t + 1,\ y + z,\ x,\ b1 + 2\right\},\ \left\{w + 1,\right.\\
&u,\ t + 1,\ y,\ x,\ b1 + z + 2\right\},\ \left\{144\,w^5 + 216\,w^4 + 96\,w^3 + 6\,w^2 - 11\,w - 1,\right.\\
&\left(12\,w^2 + 9\,w + 1\right)\,u - 72\,w^5 - 108\,w^4 - 42\,w^3 - 9\,w^2 - 3\,w,\\
&\left(12\,w^2 + 9\,w + 1\right)\,v + 36\,w^4 + 54\,w^3 + 18\,w^2,\\
&\left(24\,w^3 + 24\,w^2 + 8\,w\right)\,t - 24\,w^4 - 36\,w^3 - 26\,w^2 - 7\,w - 1,\\
&\left(12\,u\,v - 12\,u^2\right)\,z + \left(12\,w\,v + 12\,w^2 + 4\right)\,t + \left(3\,w - 5\right)\,v + 36\,w^4 + 42\,w^3 + 6\,w^2 - 16\,w,\\
&2\,v\,y + 2\,u\,z + 2\,w\,t - 2\,w^2 - w - 1,\\
&\left.\left.6\,v\,z\,x + \left(-6\,w^2 - 3\,w - 1\right)\,t + 6\,w^3 + 6\,w^2 + 4\,w,\ b1 + y + z - t - w\right\}\right]
\end{aligned} \tag{27}$$

List (RegularTriangularSet (Integer , IndexedExponents(OrderedVariableList ([b1, x, y, z, t, v, u, w])), OrderedVariableList ([b1, x, y, z, t, v, u, w]), NewSparseMultivariatePolynomial(Integer , OrderedVariableList ([b1, x, y, z, t, v, u, w])))))

We can get the dimensions of each component of a decomposition as follows.

```
[coHeight(ts) for ts in lts]
```

$$[3,\ 3,\ 3,\ 2,\ 2,\ 0] \tag{28}$$

List (NonNegativeInteger)

The first five sets have a simple shape. However, the last one, which has dimension zero, can be simplified by using Lazard triangular sets.

Thus we call the **SquareFreeRegularTriangularSet** domain constructor,

```
ST := SREGSET(R,E,V,P)
```

$$\begin{aligned}&\text{SquareFreeRegularTriangularSet}(Integer, \text{IndexedExponents}(\text{OrderedVariableList}([b1,\\ &\quad x,\ y,\ z,\ t,\ v,\ u,\ w])), \text{OrderedVariableList}([b1,\ x,\ y,\ z,\ t,\ v,\ u,\\ &\quad w]), \text{NewSparseMultivariatePolynomial}(Integer, \text{OrderedVariableList}([b1,\ x,\ y,\ z,\ t,\ v,\ u,\ w])))\end{aligned} \tag{29}$$

Type

and set the **LAZM3PK** package constructor to our situation.

```
pack := LAZM3PK(R,E,V,P,T,ST)
```

$$\begin{aligned}&\text{LazardSetSolvingPackage}(Integer, \text{IndexedExponents}(\text{OrderedVariableList}([b1,\\ &\quad x,\ y,\ z,\ t,\ v,\ u,\ w])), \text{OrderedVariableList}([b1,\ x,\ y,\ z,\ t,\ v,\ u,\\ &\quad w]), \text{NewSparseMultivariatePolynomial}(Integer, \text{OrderedVariableList}([b1,\ x,\ y,\\ &\quad z,\ t,\ v,\ u,\ w])), \text{RegularTriangularSet}(Integer, \text{IndexedExponents}(\text{OrderedVariableList}([b1,\\ &\quad x,\ y,\ z,\ t,\ v,\ u,\ w])), \text{OrderedVariableList}([b1,\ x,\ y,\ z,\ t,\ v,\ u,\\ &\quad w]), \text{NewSparseMultivariatePolynomial}(Integer, \text{OrderedVariableList}([b1,\ x,\ y,\ z,\ t,\ v,\ u,\\ &\quad w]))), \text{SquareFreeRegularTriangularSet}(Integer, \text{IndexedExponents}(\text{OrderedVariableList}([b1,\\ &\quad x,\ y,\ z,\ t,\ v,\ u,\ w])), \text{OrderedVariableList}([b1,\ x,\ y,\ z,\ t,\ v,\ u,\\ &\quad w]), \text{NewSparseMultivariatePolynomial}(Integer, \text{OrderedVariableList}([b1,\ x,\ y,\ z,\ t,\ v,\ u,\ w]))))\end{aligned} \tag{30}$$

Type

We are ready to solve the system by means of Lazard triangular sets:

```
zeroSetSplit(lp,false)$pack
```

$$\begin{aligned}&\left[\{w+1,\ t+1,\ z,\ y,\ b1+2\},\ \{w+1,\ v,\ t+1,\ z,\ b1+y+2\},\ \{w+1,\ u,\right.\\&v,\ t+1,\ b1+y+z+2\},\ \{w+1,\ v-u,\ t+1,\ y+z,\ x,\ b1+2\},\ \{w+1,\\&u,\ t+1,\ y,\ x,\ b1+z+2\},\ \left\{144\,w^5+216\,w^4+96\,w^3+6\,w^2-11\,w-1,\right.\\&u-24\,w^4-36\,w^3-14\,w^2+w+1,\ 3\,v-48\,w^4-60\,w^3-10\,w^2+8\,w+2,\\&t-24\,w^4-36\,w^3-14\,w^2-w+1,\ 486\,z-2772\,w^4-4662\,w^3-2055\,w^2+30\,w+127,\\&2916\,y-22752\,w^4-30312\,w^3-8220\,w^2+2064\,w+1561,\\&356\,x-3696\,w^4-4536\,w^3-968\,w^2+822\,w+371,\\&\left.\left.2916\,b1-30600\,w^4-46692\,w^3-20274\,w^2-8076\,w+593\right\}\right]\end{aligned} \tag{31}$$

List(SquareFreeRegularTriangularSet(Integer, IndexedExponents(OrderedVariableList([b1, x, y, z, t, v, u, w])), OrderedVariableList([b1, x, y, z, t, v, u, w]), NewSparseMultivariatePolynomial(Integer, OrderedVariableList([b1, x, y, z, t, v, u, w]))))

We see the sixth triangular set is *nicer* now: each one of its polynomials has a constant initial.

We follow with the Vermeer example. The ordering is the usual one for this system.

Define the polynomial system.

```
f0 := (w - v) ^ 2 + (u - t) ^ 2 - 1
```

$$t^2-2\,u\,t+v^2-2\,w\,v+u^2+w^2-1 \tag{32}$$

NewSparseMultivariatePolynomial(Integer, OrderedVariableList([b1, x, y, z, t, v, u, w]))

```
f1 := t ^ 2 - v ^ 3
```

$$t^2-v^3 \tag{33}$$

NewSparseMultivariatePolynomial(Integer, OrderedVariableList([b1, x, y, z, t, v, u, w]))

```
f2 := 2 * t * (w - v) + 3 * v ^ 2 * (u - t)
```

$$\left(-3\,v^2-2\,v+2\,w\right)\,t+3\,u\,v^2 \tag{34}$$

NewSparseMultivariatePolynomial(Integer, OrderedVariableList([b1, x, y, z, t, v, u, w]))

```
f3 := (3 * z * v ^ 2 - 1) * (2 * z * t - 1)
```

$$6\,v^2\,t\,z^2+\left(-2\,t-3\,v^2\right)\,z+1 \tag{35}$$

NewSparseMultivariatePolynomial(Integer, OrderedVariableList([b1, x, y, z, t, v, u, w]))

```
lf := [f0, f1, f2, f3]
```

$$\left[t^2-2\,u\,t+v^2-2\,w\,v+u^2+w^2-1,\ t^2-v^3,\ \left(-3\,v^2-2\,v+2\,w\right)\,t+3\,u\,v^2,\ 6\,v^2\,t\,z^2+\left(-2\,t-3\,v^2\right)\,z+1\right] \tag{36}$$

List(NewSparseMultivariatePolynomial(Integer, OrderedVariableList([b1, x, y, z, t, v, u, w])))

First of all, let us solve this system in the sense of Kalkbrener by means of the **REGSET** constructor:

```
zeroSetSplit(lf,true)$T
```

$$
\begin{aligned}
&\left[\left\{729\,u^6+\left(-1458\,w^3+729\,w^2-4158\,w-1685\right)\,u^4\right.\right. \qquad (37)\\
&+\left(729\,w^6-1458\,w^5-2619\,w^4-4892\,w^3-297\,w^2+5814\,w+427\right)\,u^2+729\,w^8\\
&+216\,w^7-2900\,w^6-2376\,w^5+3870\,w^4+4072\,w^3-1188\,w^2-1656\,w+529,\\
&\left(2187\,u^4+\left(-4374\,w^3-972\,w^2-12474\,w-2868\right)\,u^2+2187\,w^6-1944\,w^5-10125\,w^4-4800\,w^3+2501\,w^2+4968\,w-1587\right)\,v\\
&+\left(1944\,w^3-108\,w^2\right)\,u^2+972\,w^6+3024\,w^5-1080\,w^4+496\,w^3+1116\,w^2,\\
&\left.\left.\left(3\,v^2+2\,v-2\,w\right)\,t-3\,u\,v^2,\ \left(\left(4\,v-4\,w\right)\,t-6\,u\,v^2\right)\,z^2+\left(2\,t+3\,v^2\right)\,z-1\right\}\right]
\end{aligned}
$$

List (RegularTriangularSet (Integer , IndexedExponents(OrderedVariableList ([b1, x, y, z, t, v, u, w])), OrderedVariableList ([b1, x, y, z, t, v, u, w]), NewSparseMultivariatePolynomial(Integer , OrderedVariableList ([b1, x, y, z, t, v, u, w]))))

We have obtained one regular chain (i.e. regular triangular set) with dimension 1. This set is in fact a characterist set of the (radical of) of the ideal generated by the input system **lf**. Thus we have only the *generic points* of the variety associated with **lf** (for the elimination ordering given by **ls**).

So let us get now a full description of this variety. Hence, we solve this system in the sense of Lazard by means of the **REGSET** constructor:

```
zeroSetSplit(lf,false)$T
```

$$
\begin{aligned}
&\left[\left\{729\,u^6+\left(-1458\,w^3+729\,w^2-4158\,w-1685\right)\,u^4\right.\right. \qquad (38)\\
&+\left(729\,w^6-1458\,w^5-2619\,w^4-4892\,w^3-297\,w^2+5814\,w+427\right)\,u^2+729\,w^8\\
&+216\,w^7-2900\,w^6-2376\,w^5+3870\,w^4+4072\,w^3-1188\,w^2-1656\,w+529,\\
&\left(2187\,u^4+\left(-4374\,w^3-972\,w^2-12474\,w-2868\right)\,u^2+2187\,w^6-1944\,w^5-10125\,w^4-4800\,w^3+2501\,w^2+4968\,w-1587\right)\,v\\
&+\left(1944\,w^3-108\,w^2\right)\,u^2+972\,w^6+3024\,w^5-1080\,w^4+496\,w^3+1116\,w^2,\\
&\left.\left.\left(3\,v^2+2\,v-2\,w\right)\,t-3\,u\,v^2,\ \left(\left(4\,v-4\,w\right)\,t-6\,u\,v^2\right)\,z^2+\left(2\,t+3\,v^2\right)\,z-1\right\}\right.,\\
&\left\{27\,w^4+4\,w^3-54\,w^2-36\,w+23,\ u,\ \left(12\,w+2\right)\,v-9\,w^2-2\,w+9,\ 6\,t^2-2\,v-3\,w^2+2\,w+3,\right.\\
&\left.2\,t\,z-1\right\},\ \left\{59049\,w^6+91854\,w^5-45198\,w^4+145152\,w^3+63549\,w^2+60922\,w+21420,\right.\\
&\left(31484448266904\,w^5-18316865522574\,w^4+23676995746098\,w^3+6657857188965\,w^2+8904703998546\,w+3890631403260\right)\,u^2\\
&+94262810316408\,w^5-82887296576616\,w^4+89801831438784\,w^3\\
&+28141734167208\,w^2+38070359425432\,w+16003865949120,\\
&\left(243\,w^2+36\,w+85\right)\,v^2+\left(-81\,u^2-162\,w^3+36\,w^2+154\,w+72\right)\,v-72\,w^3+4\,w^2,\\
&\left(3\,v^2+2\,v-2\,w\right)\,t-3\,u\,v^2,\ \left(\left(4\,v-4\,w\right)\,t-6\,u\,v^2\right)\,z^2+\left(2\,t+3\,v^2\right)\,z-1\right\},\\
&\left.\left\{27\,w^4+4\,w^3-54\,w^2-36\,w+23,\ u,\ \left(12\,w+2\right)\,v-9\,w^2-2\,w+9,\ 6\,t^2-2\,v-3\,w^2+2\,w+3,\ 3\,v^2\,z-1\right\}\right]
\end{aligned}
$$

List (RegularTriangularSet (Integer , IndexedExponents(OrderedVariableList ([b1, x, y, z, t, v, u, w])), OrderedVariableList ([b1, x, y, z, t, v, u, w]), NewSparseMultivariatePolynomial(Integer , OrderedVariableList ([b1, x, y, z, t, v, u, w]))))

We retrieve our regular chain of dimension 1 and we get three regular chains of dimension 0 corresponding to the *degenerate cases*. We want now to simplify these zero-dimensional regular chains by using Lazard triangular sets. Moreover, this will allow us to prove that the above decomposition has no redundant component. **N.B.** Generally, decompositions computed by the **REGSET** constructor do not have redundant components. However, to be sure that no redundant component occurs one needs to use the **SREGSET** or **LAZM3PK** constructors.

So let us solve the input system in the sense of Lazard by means of the **LAZM3PK** constructor:

```
zeroSetSplit(lf,false)$pack
```

$$
\begin{aligned}
&\left[\left\{729\,u^6+\left(-1458\,w^3+729\,w^2-4158\,w-1685\right)\,u^4\right.\right. \qquad (39)\\
&+\left(729\,w^6-1458\,w^5-2619\,w^4-4892\,w^3-297\,w^2+5814\,w+427\right)\,u^2+729\,w^8\\
&+216\,w^7-2900\,w^6-2376\,w^5+3870\,w^4+4072\,w^3-1188\,w^2-1656\,w+529,\\
&\left(2187\,u^4+\left(-4374\,w^3-972\,w^2-12474\,w-2868\right)\,u^2+2187\,w^6-1944\,w^5-10125\,w^4-4800\,w^3+2501\,w^2+4968\,w-1587\right)\,v\\
&+\left(1944\,w^3-108\,w^2\right)\,u^2+972\,w^6+3024\,w^5-1080\,w^4+496\,w^3+1116\,w^2,\\
&\left(3\,v^2+2\,v-2\,w\right)\,t-3\,u\,v^2,\ \left(\left(4\,v-4\,w\right)\,t-6\,u\,v^2\right)\,z^2+\left(2\,t+3\,v^2\right)\,z-1\Big\},\\
&\left\{81\,w^2+18\,w+28,\ 729\,u^2-1890\,w-533,\ 81\,v^2+\left(-162\,w+27\right)\,v-72\,w-112,\right.\\
&11881\,t+\left(972\,w+2997\right)\,u\,v+\left(-11448\,w-11536\right)\,u,\ 641237934604288\,z^2\\
&+(((78614584763904\,w+26785578742272)\,u+236143618655616\,w+70221988585728)\,v+(358520253138432\,w+101922133759488)\\
&\left.+\left(32655103844499\,w-44224572465882\right)\,u\,v+\left(43213900115457\,w-32432039102070\right)\,u\right\},\\
&\left\{27\,w^4+4\,w^3-54\,w^2-36\,w+23,\ u,\ 218\,v-162\,w^3+3\,w^2+160\,w+153,\right.\\
&\left.109\,t^2-27\,w^3-54\,w^2+63\,w+80,\ 1744\,z+\left(-1458\,w^3+27\,w^2+1440\,w+505\right)\,t\right\},\\
&\left\{27\,w^4+4\,w^3-54\,w^2-36\,w+23,\ u,\ 218\,v-162\,w^3+3\,w^2+160\,w+153,\right.\\
&\left.109\,t^2-27\,w^3-54\,w^2+63\,w+80,\ 1308\,z+162\,w^3-3\,w^2-814\,w-153\right\},\\
&\left\{729\,w^4+972\,w^3-1026\,w^2+1684\,w+765,\ 81\,u^2+72\,w^2+16\,w-72,\right.\\
&702\,v-162\,w^3-225\,w^2+40\,w-99,\ 11336\,t+\left(324\,w^3-603\,w^2-1718\,w-1557\right)\,u,\ 595003968\,z^2\\
&+\left(\left(-963325386\,w^3-898607682\,w^2+1516286466\,w-3239166186\right)\,u-1579048992\,w^3-1796454288\,w^2+2428328160\,w-436849502\right.\\
&\left.\left.\left.+\left(9713133306\,w^3+9678670317\,w^2-16726834476\,w+28144233593\right)\,u\right\}\right]
\end{aligned}
$$

```
List(SquareFreeRegularTriangularSet(Integer, IndexedExponents(OrderedVariableList([b1, x, y, z, t, v, u, w])),
OrderedVariableList([b1, x, y, z, t, v, u, w]), NewSparseMultivariatePolynomial(Integer, OrderedVariableList([b1, x,
y, z, t, v, u, w]))))
```

Due to square-free factorization, we obtained now four zero-dimensional regular chains. Moreover, each of them is normalized (the initials are constant). Note that these zero-dimensional components may be investigated further with the **ZeroDimensionalSolvePackage** package constructor.

9.45 LexTriangularPackage

The **LexTriangularPackage** package constructor provides an implementation of the *lexTriangular* algorithm (D. Lazard "Solving Zero-dimensional Algebraic Systems", J. of Symbol. Comput., 1992). This algorithm decomposes a zero-dimensional variety into zero-sets of regular triangular sets. Thus the input system must have a finite number of complex solutions. Moreover, this system needs to be a lexicographical Groebner basis.

This package takes two arguments: the coefficient-ring **R** of the polynomials, which must be a **GcdDomain** and their set of variables given by **ls** a **List Symbol**. The type of the input polynomials must be **NewSparseMultivariatePolynomial(R,V)** where **V** is **OrderedVariableList(ls)**. The abbreviation for **LexTriangularPackage** is **LEXTRIPK**. The main operations are `lexTriangular` and `squareFreeLexTriangular`. The latter provides decompositions by means of square-free regular triangular sets, built with the **SREGSET** constructor, whereas the former uses the **REGSET** constructor. Note that these constructors also implement another algorithm for solving algebraic systems by means of regular triangular sets; in that case no computations of Groebner bases are needed and the input system may have any dimension (i.e. it may have an infinite number of solutions).

The implementation of the *lexTriangular* algorithm provided in the **LexTriangularPackage** constructor differs from that reported in "Computations of gcd over algebraic towers of simple extensions" by M. Moreno Maza and R. Rioboo (in proceedings of AAECC11, Paris, 1995). Indeed, the `squareFreeLexTriangular` operation removes all multiplicities of the solutions (i.e. the computed solutions are pairwise different) and the `lexTriangular` operation may keep some multiplicities; this latter operation runs generally faster than the former.

The interest of the *lexTriangular* algorithm is due to the following experimental remark. For some examples, a triangular decomposition of a zero-dimensional variety can be computed faster via a lexicographical Groebner basis computation than by using a direct method (like that of **SREGSET** and **REGSET**). This happens typically when the total degree of the system relies essentially on its smallest variable (like in the *Katsura* systems). When this is not the case, the direct method may give better timings (like in the *Rose* system).

Of course, the direct method can also be applied to a lexicographical Groebner basis. However, the *lexTriangular* algorithm takes advantage of the structure of this basis and avoids many unnecessary computations which are performed by the direct method.

For this purpose of solving algebraic systems with a finite number of solutions, see also the **ZeroDimensionalSolvePackage**. It allows one to use both strategies (the lexTriangular algorithm and the direct method) for computing either the complex or real roots of a system.

Note that the way of understanding triangular decompositions is detailed in the example of the **RegularTriangularSet** constructor.

Since the **LEXTRIPK** package constructor is limited to zero-dimensional systems, it provides a `zeroDimensional?` operation to check whether this requirement holds. There is also a `groebner` operation to compute the lexicographical Groebner basis of a set of polynomials with type **NewSparseMultivariatePolynomial(R,V)**. The elimination ordering is that given by **ls** (the greatest variable being the first element of **ls**). This basis is computed by the *FLGM* algorithm (Faugere et al. "Efficient Computation of Zero-Dimensional Groebner Bases by Change of Ordering" , J. of Symbol. Comput., 1993) implemented in the **LinGroebnerPackage** package constructor. Once a lexicographical Groebner basis is computed, then one can call the operations `lexTriangular` and `squareFreeLexTriangular`. Note that these operations admit an optional argument to produce normalized triangular sets. There is also a `zeroSetSplit` operation which does all the job from the input system; an error is produced if this system is not zero-dimensional.

Let us illustrate the facilities of the **LEXTRIPK** constructor by a famous example, the *cyclic-6 root* system. Define the coefficient ring.

```
R := Integer
```

$$Integer \tag{4}$$

Type

Define the list of variables,

```
ls : List Symbol := [a,b,c,d,e,f]
```

$$[a,\ b,\ c,\ d,\ e,\ f] \tag{5}$$

List (Symbol)

and make it an ordered set.

```
V := OVAR(ls)
```

$$\text{OrderedVariableList}([a,\ b,\ c,\ d,\ e,\ f]) \tag{6}$$

Type

Define the polynomial ring.

```
P := NSMP(R, V)
```

$$\text{NewSparseMultivariatePolynomial}(Integer, \text{OrderedVariableList}([a,\ b,\ c,\ d,\ e,\ f])) \tag{7}$$

Type

Define the polynomials.

```
p1: P :=  a*b*c*d*e*f - 1
```

$$f\,e\,d\,c\,b\,a - 1 \tag{8}$$

NewSparseMultivariatePolynomial(Integer, OrderedVariableList([a, b, c, d, e, f]))

```
p2: P := a*b*c*d*e +a*b*c*d*f +a*b*c*e*f +a*b*d*e*f +a*c*d*e*f +b*c*d*e*f
```

$$((((e+f)\,d+f\,e)\,c+f\,e\,d)\,b+f\,e\,d\,c)\,a+f\,e\,d\,c\,b \tag{9}$$

NewSparseMultivariatePolynomial(Integer, OrderedVariableList([a, b, c, d, e, f]))

```
p3: P :=  a*b*c*d + a*b*c*f + a*b*e*f + a*d*e*f + b*c*d*e + c*d*e*f
```

$$(((d+f)\,c+f\,e)\,b+f\,e\,d)\,a+e\,d\,c\,b+f\,e\,d\,c \tag{10}$$

NewSparseMultivariatePolynomial(Integer, OrderedVariableList([a, b, c, d, e, f]))

```
p4: P := a*b*c + a*b*f + a*e*f + b*c*d + c*d*e + d*e*f
```

$$((c+f)\,b+f\,e)\,a+d\,c\,b+e\,d\,c+f\,e\,d \tag{11}$$

NewSparseMultivariatePolynomial(Integer, OrderedVariableList([a, b, c, d, e, f]))

```
p5: P := a*b + a*f + b*c + c*d + d*e + e*f
```

$$(b+f)\,a+c\,b+d\,c+e\,d+f\,e \tag{12}$$

NewSparseMultivariatePolynomial(Integer, OrderedVariableList([a, b, c, d, e, f]))

```
p6: P := a + b + c + d + e + f
```

$$a+b+c+d+e+f \tag{13}$$

NewSparseMultivariatePolynomial(Integer, OrderedVariableList([a, b, c, d, e, f]))

```
lp := [p1, p2, p3, p4, p5, p6]
```

$$[f\,e\,d\,c\,b\,a-1,\ ((((e+f)\,d+f\,e)\,c+f\,e\,d)\,b+f\,e\,d\,c)\,a+f\,e\,d\,c\,b,\ (((d+f)\,c+f\,e)\,b+f\,e\,d)\,a+e\,d\,c\,b+f\,e\,d\,c,\ ((c+f)\,b+f\,e)\,a+d\,c\,b+e\,d\,c+f\,e\,d,\ (b+f)\,a+c\,b+d\,c+e\,d+f\,e,\ a+b+c+d+e+f] \tag{14}$$

List(NewSparseMultivariatePolynomial(Integer, OrderedVariableList([a, b, c, d, e, f])))

Now call **LEXTRIPK** .

```
lextripack :=  LEXTRIPK(R,ls)
```

$$\text{LexTriangularPackage}(Integer, [a,\ b,\ c,\ d,\ e,\ f]) \tag{15}$$

Type

Compute the lexicographical Groebner basis of the system. This may take between 5 minutes and one hour, depending on your machine.

```
lg := groebner(lp)$lextripack;
```

List(NewSparseMultivariatePolynomial(Integer, OrderedVariableList([a, b, c, d, e, f])))

Apply lexTriangular to compute a decomposition into regular triangular sets. This should not take more than 5 seconds.

```
lexTriangular(lg,false)$lextripack
```

$$\begin{aligned}&[\{f^6+1,\ e^6-3\,f\,e^5+3\,f^2\,e^4-4\,f^3\,e^3+3\,f^4\,e^2-3\,f^5\,e-1,\ 3\,d+f^2\,e^5-4\,f^3\,e^4+4\,f^4\,e^3-2\,f^5\,e^2-2\,e+2\,f,\\ &c+f,\ 3\,b+2\,f^2\,e^5-5\,f^3\,e^4+5\,f^4\,e^3-10\,f^5\,e^2-4\,e+7\,f,\ a-f^2\,e^5+3\,f^3\,e^4-3\,f^4\,e^3+4\,f^5\,e^2+3\,e-3\,f\},\\ &\{f^6-1,\ e-f,\ d-f,\ c^2+4\,f\,c+f^2,\ (c-f)\,b-f\,c-5\,f^2,\ a+b+c+3\,f\},\ \{f^6-1,\ e-f,\ d-f,\ c-f,\\ &b^2+4\,f\,b+f^2,\ a+b+4\,f\},\ \{f^6-1,\ e-f,\ d^2+4\,f\,d+f^2,\ (d-f)\,c-f\,d-5\,f^2,\ b-f,\ a+c+d+3\,f\},\\ &\{f^{36}-2554\,f^{30}-399709\,f^{24}-502276\,f^{18}-399709\,f^{12}-2554\,f^6+1,\ (161718564\,f^{12}-161718564)\,e^2\\ &+(-504205\,f^{31}+1287737951\,f^{25}+201539391380\,f^{19}+253982817368\,f^{13}+201940704665\,f^7+1574134601\,f)\,e\\ &-2818405\,f^{32}+7198203911\,f^{26}+1126548149060\,f^{20}+1416530563364\,f^{14}+1127377589345\,f^8\\ &+7988820725\,f^2,\ (693772639560\,f^6-693772639560)\,d-462515093040\,f^2\,e^5\\ &+1850060372160\,f^3\,e^4-1850060372160\,f^4\,e^3+(-24513299931120\,f^{11}-23588269745040\,f^5)\,e^2\\ &+(-890810428\,f^{30}+2275181044754\,f^{24}+355937263869776\,f^{18}+413736880104344\,f^{12}+342849304487996\,f^6+3704966481878)\,e\\ &-4163798003\,f^{31}+10634395752169\,f^{25}+1664161760192806\,f^{19}+2079424391370694\,f^{13}\\ &+1668153650635921\,f^7+10924274392693\,f,\ (12614047992\,f^6-12614047992)\,c-7246825\,f^{31}\\ &+18508536599\,f^{25}+2896249516034\,f^{19}+3581539649666\,f^{13}+2796477571739\,f^7-48094301893\,f,\\ &(693772639560\,f^6-693772639560)\,b-925030186080\,f^2\,e^5+2312575465200\,f^3\,e^4\\ &-2312575465200\,f^4\,e^3+(-40007555547960\,f^{11}-35382404617560\,f^5)\,e^2\\ &+(-3781280823\,f^{30}+9657492291789\,f^{24}+1511158913397906\,f^{18}+1837290892286154\,f^{12}+1487216006594361\,f^6+8077238712093\\ &-9736390478\,f^{31}+24866827916734\,f^{25}+3891495681905296\,f^{19}+4872556418871424\,f^{13}\\ &+3904047887269606\,f^7+27890075838538\,f,\ a+b+c+d+e+f\},\ \{f^6-1,\\ &e^2+4\,f\,e+f^2,\ (e-f)\,d-f\,e-5\,f^2,\ c-f,\ b-f,\ a+d+e+3\,f\}]\end{aligned} \tag{17}$$

List(RegularChain(Integer, [a, b, c, d, e, f]))

Note that the first set of the decomposition is normalized (all initials are integer numbers) but not the second one (normalized triangular sets are defined in the description of the **NormalizedTrian-**

gularSetCategory constructor). So apply now lexTriangular to produce normalized triangular sets.

```
lts := lexTriangular(lg,true)$lextripack
```

$$
\begin{aligned}
&[\{f^6+1,\ e^6-3\,f\,e^5+3\,f^2\,e^4-4\,f^3\,e^3+3\,f^4\,e^2-3\,f^5\,e-1,\ 3\,d+f^2\,e^5-4\,f^3\,e^4+4\,f^4\,e^3-2\,f^5\,e^2-2\,e+2\,f, \\
&c+f,\ 3\,b+2\,f^2\,e^5-5\,f^3\,e^4+5\,f^4\,e^3-10\,f^5\,e^2-4\,e+7\,f,\ a-f^2\,e^5+3\,f^3\,e^4-3\,f^4\,e^3+4\,f^5\,e^2+3\,e-3\,f\},\\
&\{f^6-1,\ e-f,\ d-f,\ c^2+4\,f\,c+f^2,\ b+c+4\,f,\ a-f\},\ \{f^6-1,\ e-f,\ d-f,\ c-f,\\
&b^2+4\,f\,b+f^2,\ a+b+4\,f\},\ \{f^6-1,\ e-f,\ d^2+4\,f\,d+f^2,\ c+d+4\,f,\ b-f,\ a-f\},\\
&\{f^{36}-2554\,f^{30}-399709\,f^{24}-502276\,f^{18}-399709\,f^{12}-2554\,f^6+1,\ 1387545279120\,e^2\\
&+\left(4321823003\,f^{31}-11037922310209\,f^{25}-1727506390124986\,f^{19}-2176188913464634\,f^{13}-1732620732685741\,f^7-13506088516033\right.\\
&+24177661775\,f^{32}-61749727185325\,f^{26}-9664082618092450\,f^{20}\\
&-12152237485813570\,f^{14}-9672870290826025\,f^8-68544102808525\,f^2,\ 1387545279120\,d\\
&+\left(-1128983050\,f^{30}+2883434331830\,f^{24}+451234998755840\,f^{18}+562426491685760\,f^{12}+447129055314890\,f^6-165557857270\right)\,e\\
&-1816935351\,f^{31}+4640452214013\,f^{25}+726247129626942\,f^{19}\\
&+912871801716798\,f^{13}+726583262666877\,f^7+4909358645961\,f,\\
&1387545279120\,c+778171189\,f^{31}-1987468196267\,f^{25}-310993556954378\,f^{19}\\
&-383262822316802\,f^{13}-300335488637543\,f^7+5289595037041\,f,\ 1387545279120\,b\\
&+\left(1128983050\,f^{30}-2883434331830\,f^{24}-451234998755840\,f^{18}-562426491685760\,f^{12}-447129055314890\,f^6+165557857270\right)\,e\\
&-3283058841\,f^{31}+8384938292463\,f^{25}+1312252817452422\,f^{19}\\
&+1646579934064638\,f^{13}+1306372958656407\,f^7+4694680112151\,f,\\
&1387545279120\,a+1387545279120\,e+4321823003\,f^{31}-11037922310209\,f^{25}\\
&-1727506390124986\,f^{19}-2176188913464634\,f^{13}-1732620732685741\,f^7-13506088516033\,f\},\\
&\{f^6-1,\ e^2+4\,f\,e+f^2,\ d+e+4\,f,\ c-f,\ b-f,\ a-f\}]
\end{aligned}
\tag{18}
$$

List(RegularChain(Integer, [a, b, c, d, e, f]))

We check that all initials are constant.

```
[[init(p) for p in (ts :: List(P))] for ts in lts]
```

$$
[[1,\ 3,\ 1,\ 3,\ 1,\ 1],\ [1,\ 1,\ 1,\ 1,\ 1,\ 1],\ [1,\ 1,\ 1,\ 1,\ 1,\ 1],\ [1,\ 1,\ 1,\ 1,\ 1,\ 1],\ [1387545279120,\ 1387545279120,\ 1387545279120,\ 1387545279120,\ 1387545279120,\ 1],\ [1,\ 1,\ 1,\ 1,\ 1,\ 1]] \tag{19}
$$

List(List(NewSparseMultivariatePolynomial(Integer, OrderedVariableList([a, b, c, d, e, f]))))

Note that each triangular set in **lts** is a lexicographical Groebner basis. Recall that a point belongs to the variety associated with **lp** if and only if it belongs to that associated with one triangular set **ts** in **lts**.

By running the squareFreeLexTriangular operation, we retrieve the above decomposition.

```
squareFreeLexTriangular(lg,true)$lextripack
```

$$\left[\left\{f^6+1,\ e^6-3\,f\,e^5+3\,f^2\,e^4-4\,f^3\,e^3+3\,f^4\,e^2-3\,f^5\,e-1,\ 3\,d+f^2\,e^5-4\,f^3\,e^4+4\,f^4\,e^3-2\,f^5\,e^2-2\,e+2\,f,\right.\right. \quad (20)$$
$$c+f,\ 3\,b+2\,f^2\,e^5-5\,f^3\,e^4+5\,f^4\,e^3-10\,f^5\,e^2-4\,e+7\,f,\ a-f^2\,e^5+3\,f^3\,e^4-3\,f^4\,e^3+4\,f^5\,e^2+3\,e-3\,f\Big\},$$
$$\left\{f^6-1,\ e-f,\ d-f,\ c^2+4\,f\,c+f^2,\ b+c+4\,f,\ a-f\right\},\ \left\{f^6-1,\ e-f,\ d-f,\ c-f,\right.$$
$$b^2+4\,f\,b+f^2,\ a+b+4\,f\Big\},\ \left\{f^6-1,\ e-f,\ d^2+4\,f\,d+f^2,\ c+d+4\,f,\ b-f,\ a-f\right\},$$
$$\left\{f^{36}-2554\,f^{30}-399709\,f^{24}-502276\,f^{18}-399709\,f^{12}-2554\,f^6+1,\ 1387545279120\,e^2\right.$$
$$+\left(4321823003\,f^{31}-11037922310209\,f^{25}-1727506390124986\,f^{19}-2176188913464634\,f^{13}-1732620732685741\,f^7-13506088516033\right.$$
$$+24177661775\,f^{32}-61749727185325\,f^{26}-9664082618092450\,f^{20}$$
$$-12152237485813570\,f^{14}-9672870290826025\,f^8-68544102808525\,f^2,\ 1387545279120\,d$$
$$+\left(-1128983050\,f^{30}+2883434331830\,f^{24}+451234998755840\,f^{18}+562426491685760\,f^{12}+447129055314890\,f^6-165557857270\right)\,e$$
$$-1816935351\,f^{31}+4640452214013\,f^{25}+726247129626942\,f^{19}$$
$$+912871801716798\,f^{13}+726583262666877\,f^7+4909358645961\,f,$$
$$1387545279120\,c+778171189\,f^{31}-1987468196267\,f^{25}-310993556954378\,f^{19}$$
$$-383262822316802\,f^{13}-300335488637543\,f^7+5289595037041\,f,\ 1387545279120\,b$$
$$+\left(1128983050\,f^{30}-2883434331830\,f^{24}-451234998755840\,f^{18}-562426491685760\,f^{12}-447129055314890\,f^6+165557857270\right)\,e$$
$$-3283058841\,f^{31}+8384938292463\,f^{25}+1312252817452422\,f^{19}$$
$$+1646579934064638\,f^{13}+1306372958656407\,f^7+4694680112151\,f,$$
$$1387545279120\,a+1387545279120\,e+4321823003\,f^{31}-11037922310209\,f^{25}$$
$$-1727506390124986\,f^{19}-2176188913464634\,f^{13}-1732620732685741\,f^7-13506088516033\,f\Big\},$$
$$\left.\left\{f^6-1,\ e^2+4\,f\,e+f^2,\ d+e+4\,f,\ c-f,\ b-f,\ a-f\right\}\right]$$

List (SquareFreeRegularTriangularSet (Integer , IndexedExponents(OrderedVariableList ([a, b, c, d, e, f])) , OrderedVariableList ([a, b, c, d, e, f]) , NewSparseMultivariatePolynomial(Integer , OrderedVariableList ([a, b, c, d, e , f]))))

Thus the solutions given by **lts** are pairwise different. We count them as follows.

```
reduce(+,[degree(ts) for ts in lts])
```

$$156 \quad (21)$$

PositiveInteger

We can investigate the triangular decomposition **lts** by using the **ZeroDimensionalSolvePackage**. This requires adding an extra variable (smaller than the others) as follows.

```
ls2 : List Symbol := concat(ls,new()$Symbol)
```

$$[a,\ b,\ c,\ d,\ e,\ f,\ \%A] \quad (22)$$

List (Symbol)

Then we call the package.

```
zdpack := ZDSOLVE(R,ls,ls2)
```

$$\text{ZeroDimensionalSolvePackage}(Integer, [a,\ b,\ c,\ d,\ e,\ f], [a,\ b,\ c,\ d,\ e,\ f,\ \%A]) \quad (23)$$

Type

We compute a univariate representation of the variety associated with the input system as follows.

```
concat [univariateSolve(ts)$zdpack for ts in lts];
```

List (Record(complexRoots: SparseUnivariatePolynomial(Integer), coordinates : List (Polynomial(Integer))))

Since the `univariateSolve` operation may split a regular set, it returns a list. This explains the use of `concat`.

Look at the last item of the result. It consists of two parts. For any complex root **?** of the univariate polynomial in the first part, we get a tuple of univariate polynomials (in **a**, ..., **f** respectively) by replacing **%A** by **?** in the second part. Each of these tuples **t** describes a point of the variety associated with **lp** by equaling to zero the polynomials in **t**.

Note that the way of reading these univariate representations is explained also in the example illustrating the **ZeroDimensionalSolvePackage** constructor.

Now, we compute the points of the variety with real coordinates.

```
concat [realSolve(ts)$zdpack for ts in lts];
```

List (List (RealClosure(Fraction (Integer))))

We obtain 24 points given by lists of elements in the **RealClosure** of **Fraction** of **R**. In each list, the first value corresponds to the indeterminate **f**, the second to **e** and so on. See **ZeroDimensionalSolvePackage** to learn more about the `realSolve` operation.

9.46 Library

The **Library** domain provides a simple way to store FriCAS values in a file. This domain is similar to **KeyedAccessFile** but fewer declarations are needed and items of different types can be saved together in the same file.

To create a library, you supply a file name.

```
stuff := library "/tmp/Neat.stuff"
```

"/tmp/Neat.stuff" (4)

Library

Now values can be saved by key in the file. The keys should be mnemonic, just as the field names are for records. They can be given either as strings or symbols.

```
stuff.int    := 32^2
```

1024 (5)

PositiveInteger

```
stuff."poly" := x^2 + 1
```

$x^2 + 1$ (6)

Polynomial(Integer)

```
stuff.str    := "Hello"
```

```
"Hello"                                                    (7)
                                                          String
```

You obtain the set of available keys using the keys operation.

```
keys stuff
```

```
["str", "poly", "int"]                                     (8)
                                                  List ( String )
```

You extract values by giving the desired key in this way.

```
stuff.poly
```

$$x^2 + 1 \tag{9}$$

Polynomial(Integer)

```
stuff("poly")
```

$$x^2 + 1 \tag{10}$$

Polynomial(Integer)

When the file is no longer needed, you should remove it from the file system.

```
)system rm -rf /tmp/Neat.stuff
```

For more information on related topics, see 'File' on page 412, 'TextFile' on page 591, and 'KeyedAccessFile' on page 457. Issue the system command)show Library to display the full list of operations defined by **Library**.

9.47 LieExponentials

This example is in just two variables

```
a: Symbol := 'a
```

$$a \tag{4}$$

Symbol

```
b: Symbol := 'b
```

$$b \tag{5}$$

Symbol

Declarations of domains

```
coef      := Fraction(Integer)
```

$$\text{Fraction}(Integer) \tag{6}$$

Type

```
group    := LieExponentials(Symbol, coef, 3)
```

$$\text{LieExponentials}(Symbol, \text{Fraction}(Integer), 3) \tag{7}$$

Type

```
lpoly    := LiePolynomial(Symbol, coef)
```

$$\text{LiePolynomial}(Symbol, \text{Fraction}(Integer)) \tag{8}$$

Type

```
poly     := XPBWPolynomial(Symbol, coef)
```

$$\text{XPBWPolynomial}(Symbol, \text{Fraction}(Integer)) \tag{9}$$

Type

Calculations

```
ea := exp(a::lpoly)$group
```

$$\mathsf{e}^{[a]} \tag{10}$$

LieExponentials(Symbol, Fraction(Integer), 3)

```
eb := exp(b::lpoly)$group
```

$$\mathsf{e}^{[b]} \tag{11}$$

LieExponentials(Symbol, Fraction(Integer), 3)

```
g: group := ea*eb
```

$$\mathsf{e}^{[b]}\,\mathsf{e}^{\frac{1}{2}\left[a\,b^2\right]}\,\mathsf{e}^{[a\,b]}\,\mathsf{e}^{\frac{1}{2}\left[a^2\,b\right]}\,\mathsf{e}^{[a]} \tag{12}$$

LieExponentials(Symbol, Fraction(Integer), 3)

```
g :: poly
```

$$\begin{aligned} &1 + [a] + [b] + \frac{1}{2}\,[a]\,[a] + [a\,b] + [b]\,[a] + \frac{1}{2}\,[b]\,[b] + \frac{1}{6}\,[a]\,[a]\,[a] + \frac{1}{2}\,\left[a^2\,b\right] \\ &+ [a\,b]\,[a] + \frac{1}{2}\,\left[a\,b^2\right] + \frac{1}{2}\,[b]\,[a]\,[a] + [b]\,[a\,b] + \frac{1}{2}\,[b]\,[b]\,[a] + \frac{1}{6}\,[b]\,[b]\,[b] \end{aligned} \tag{13}$$

XPBWPolynomial(Symbol, Fraction(Integer))

```
log(g)$group
```

$$[a] + [b] + \frac{1}{2}\,[a\,b] + \frac{1}{12}\,\left[a^2\,b\right] + \frac{1}{12}\,\left[a\,b^2\right] \tag{14}$$

LiePolynomial(Symbol, Fraction(Integer))

```
g1: group := inv(g)
```

$$\mathsf{e}^{-[b]}\,\mathsf{e}^{-[a]} \tag{15}$$

LieExponentials(Symbol, Fraction(Integer), 3)

```
g*g1
```

$$1 \tag{16}$$

LieExponentials(Symbol, Fraction(Integer), 3)

9.48 LiePolynomial

Declaration of domains

```
RN    := Fraction Integer
```

$$\mathrm{Fraction}(Integer) \tag{4}$$

Type

```
Lpoly := LiePolynomial(Symbol,RN)
```

$$\mathrm{LiePolynomial}(Symbol, \mathrm{Fraction}(Integer)) \tag{5}$$

Type

```
Dpoly := XDPOLY(Symbol,RN)
```

$$\mathrm{XDistributedPolynomial}(Symbol, \mathrm{Fraction}(Integer)) \tag{6}$$

Type

```
Lword := LyndonWord Symbol
```

$$\mathrm{LyndonWord}(Symbol) \tag{7}$$

Type

Initialisation

```
a:Symbol := 'a
```

$$a \tag{8}$$

Symbol

```
b:Symbol := 'b
```

$$b \tag{9}$$

Symbol

```
c:Symbol := 'c
```

$$c \tag{10}$$

Symbol

```
aa: Lpoly := a
```

$$[a] \tag{11}$$

LiePolynomial(Symbol, Fraction(Integer))

```
bb: Lpoly := b
```

$$[b] \tag{12}$$

LiePolynomial(Symbol, Fraction(Integer))

```
cc: Lpoly := c
```

$$[c] \tag{13}$$

LiePolynomial(Symbol, Fraction(Integer))

```
p : Lpoly := [aa,bb]
```

$$[a\,b] \tag{14}$$

LiePolynomial(Symbol, Fraction(Integer))

```
q : Lpoly := [p,bb]
```

$$\left[a\,b^2\right] \tag{15}$$

LiePolynomial(Symbol, Fraction(Integer))

All the Lyndon words of order 4

```
liste : List Lword := LyndonWordsList([a,b], 4)
```

$$\left[[a],\,[b],\,[a\,b],\,\left[a^2\,b\right],\,\left[a\,b^2\right],\,\left[a^3\,b\right],\,\left[a^2\,b^2\right],\,\left[a\,b^3\right]\right] \tag{16}$$

List(LyndonWord(Symbol))

```
r: Lpoly := p + q + 3*LiePoly(liste.4)$Lpoly
```

$$[a\,b] + 3\,\left[a^2\,b\right] + \left[a\,b^2\right] \tag{17}$$

LiePolynomial(Symbol, Fraction(Integer))

```
s:Lpoly := [p,r]
```

$$-3\,\left[a^2\,b\,a\,b\right] + \left[a\,b\,a\,b^2\right] \tag{18}$$

LiePolynomial(Symbol, Fraction(Integer))

```
t:Lpoly  := s  + 2*LiePoly(liste.3) - 5*LiePoly(liste.5)
```

$$2\,[a\,b] - 5\,\left[a\,b^2\right] - 3\,\left[a^2\,b\,a\,b\right] + \left[a\,b\,a\,b^2\right] \tag{19}$$

LiePolynomial(Symbol, Fraction(Integer))

```
degree t
```

$$5 \tag{20}$$

PositiveInteger

```
mirror t
```

$$-2\,[a\,b] - 5\,\left[a\,b^2\right] - 3\,\left[a^2\,b\,a\,b\right] + \left[a\,b\,a\,b^2\right] \tag{21}$$

LiePolynomial(Symbol, Fraction(Integer))

Jacobi Relation

```
Jacobi(p: Lpoly, q: Lpoly, r: Lpoly): Lpoly == [[p,q]$Lpoly, r] + [[q,r]$Lpoly, p] + _
    [[r,p]$Lpoly, q]
```

```
   Function declaration Jacobi : (LiePolynomial(Symbol,Fraction(Integer)),
      LiePolynomial(Symbol,Fraction(Integer)), LiePolynomial(Symbol,Fraction(
      Integer))) -> LiePolynomial(Symbol,Fraction(Integer)) has been added to
      workspace.
```

Tests

```
test: Lpoly := Jacobi(a,b,b)
```

```
   Compiling function Jacobi with type (LiePolynomial(Symbol,Fraction(Integer)),
      LiePolynomial(Symbol,Fraction(Integer)), LiePolynomial(Symbol,Fraction(
      Integer))) -> LiePolynomial(Symbol,Fraction(Integer))
```

$$0 \tag{23}$$

LiePolynomial(Symbol, Fraction(Integer))

```
test: Lpoly := Jacobi(p,q,r)
```

$$0 \tag{24}$$

LiePolynomial(Symbol, Fraction(Integer))

```
test: Lpoly := Jacobi(r,s,t)
```

$$0 \tag{25}$$

LiePolynomial(Symbol, Fraction(Integer))

Evaluation

```
eval(p, a, p)$Lpoly
```

$$[a\,b^2] \tag{26}$$

LiePolynomial(Symbol, Fraction(Integer))

```
eval(p, [a,b], [2*bb, 3*aa])$Lpoly
```

$$-6\,[a\,b] \tag{27}$$

LiePolynomial(Symbol, Fraction(Integer))

```
r: Lpoly := [p,c]
```

$$[a\,b\,c] + [a\,c\,b] \tag{28}$$

LiePolynomial(Symbol, Fraction(Integer))

```
r1: Lpoly := eval(r, [a,b,c], [bb, cc, aa])$Lpoly
```

$$-[a\,b\,c] \tag{29}$$

LiePolynomial(Symbol, Fraction(Integer))

```
r2: Lpoly := eval(r, [a,b,c], [cc, aa, bb])$Lpoly
```

$$-[a\,c\,b] \tag{30}$$

LiePolynomial(Symbol, Fraction(Integer))

```
r + r1 + r2
```

$$0 \tag{31}$$

LiePolynomial(Symbol, Fraction(Integer))

9.49 LinearOrdinaryDifferentialOperator

LinearOrdinaryDifferentialOperator(A, diff) is the domain of linear ordinary differential operators with coefficients in a ring `A` with a given derivation. Issue the system command `)show LinearOrdinaryDifferentialOperator` to display the full list of operations defined by **LinearOrdinaryDifferentialOperator**.

9.49.1 Differential Operators with Series Coefficients

Problem: Find the first few coefficients of `exp(x)/x^i` of `Dop(φ)` where

```
Dop := D^3 + G/x^2 * D + H/x^3 - 1
phi := sum(s[i]*exp(x)/x^i, i = 0..)
```

Solution: Define the differential.

```
Dx: LODO(EXPR INT, f +-> D(f, x))
```

```
Dx := D()
```

$$D \tag{2}$$

LinearOrdinaryDifferentialOperator (Expression(Integer), theMap(anonymousFunction))

Now define the differential operator `Dop`.

```
Dop:= Dx^3 + G/x^2*Dx + H/x^3 - 1
```

$$D^3 + \frac{G}{x^2}\,D + \frac{-x^3 + H}{x^3} \tag{3}$$

LinearOrdinaryDifferentialOperator (Expression(Integer), theMap(anonymousFunction))

The first n coefficients can be obtained in the following way.

```
firstCoefficients(n) ==
  phi := reduce(+, [subscript(s,[i])*exp(x)/x^i for i in 0..n])
  phi1 := retract((Dop(phi) * x^(n+3) / exp x))@Polynomial(Integer)
  phi2 := phi1 :: UnivariatePolynomial(x, Polynomial Integer)
  sys := [coefficient(phi2, (n+3-i)::NNI) for i in 2..n+1]
  solve(sys, [subscript(s,[i]) for i in 1..n])
```

Evaluate this function for several values of `n`.

```
firstCoefficients(3)
```

```
   Compiling function firstCoefficients with type PositiveInteger -> List(List(
      Equation(Fraction(Polynomial(Integer)))))
   Compiling function G27 with type Integer -> Boolean
```

$$\left[\left[s_1 = \frac{s_0\,G}{3},\ s_2 = \frac{3\,s_0\,H + s_0\,G^2 + 6\,s_0\,G}{18},\ s_3 = \frac{(9\,s_0\,G + 54\,s_0)\,H + s_0\,G^3 + 18\,s_0\,G^2 + 72\,s_0\,G}{162}\right]\right] \tag{5}$$

List(List(Equation(Fraction(Polynomial(Integer))))))

```
firstCoefficients(4)
```

$$\left[\left[s_1 = \frac{s_0\,G}{3},\ s_2 = \frac{3\,s_0\,H + s_0\,G^2 + 6\,s_0\,G}{18},\ s_3 = \frac{(9\,s_0\,G + 54\,s_0)\,H + s_0\,G^3 + 18\,s_0\,G^2 + 72\,s_0\,G}{162},\right.\right.$$
$$\left.\left. s_4 = \frac{27\,s_0\,H^2 + \left(18\,s_0\,G^2 + 378\,s_0\,G + 1296\,s_0\right)\,H + s_0\,G^4 + 36\,s_0\,G^3 + 396\,s_0\,G^2 + 1296\,s_0\,G}{1944}\right]\right] \tag{6}$$

List (List (Equation(Fraction (Polynomial(Integer)))))

```
firstCoefficients(7)
```

$$\left[\left[s_1 = \frac{s_0\,G}{3},\ s_2 = \frac{3\,s_0\,H + s_0\,G^2 + 6\,s_0\,G}{18},\ s_3 = \frac{\left(9\,s_0\,G + 54\,s_0\right)\,H + s_0\,G^3 + 18\,s_0\,G^2 + 72\,s_0\,G}{162},\right.\right. \tag{7}$$

$$s_4 = \frac{27\,s_0\,H^2 + \left(18\,s_0\,G^2 + 378\,s_0\,G + 1296\,s_0\right)\,H + s_0\,G^4 + 36\,s_0\,G^3 + 396\,s_0\,G^2 + 1296\,s_0\,G}{1944},$$

$$s_5 = \frac{\left(135\,s_0\,G + 2268\,s_0\right)\,H^2 + \left(30\,s_0\,G^3 + 1350\,s_0\,G^2 + 16416\,s_0\,G + 38880\,s_0\right)\,H + s_0\,G^5 + 60\,s_0\,G^4 + 1188\,s_0\,G^3 + 9504\,s_0\,G^2 + 2}{29160}$$

$$s_6 = \frac{405\,s_0\,H^3 + \left(405\,s_0\,G^2 + 18468\,s_0\,G + 174960\,s_0\right)\,H^2 + \left(45\,s_0\,G^4 + 3510\,s_0\,G^3 + 88776\,s_0\,G^2 + 777600\,s_0\,G + 1166400\,s_0\right)\,H +}{524880}$$

$$s_7 = \left(2835\,s_0\,G + 91854\,s_0\right)\,H^3 + \left(945\,s_0\,G^3 + 81648\,s_0\,G^2 + 2082996\,s_0\,G + 14171760\,s_0\right)\,H^2 + \left(63\,s_0\,G^5 + 7560\,s_0\,G^4 + 317520\,s_0\right.$$

List (List (Equation(Fraction (Polynomial(Integer)))))

9.50 LinearOrdinaryDifferentialOperator1

LinearOrdinaryDifferentialOperator1(A) is the domain of linear ordinary differential operators with coefficients in the differential ring `A`. Issue the system command `)show LinearOrdinaryDifferentialOperator1` to display the full list of operations defined by **LinearOrdinaryDifferentialOperator1**.

9.50.1 Differential Operators with Rational Function Coefficients

This example shows differential operators with rational function coefficients. In this case operator multiplication is non-commutative and, since the coefficients form a field, an operator division algorithm exists.

We begin by defining `RFZ` to be the rational functions in `x` with integer coefficients and `Dx` to be the differential operator for `d/dx`.

```
RFZ := Fraction UnivariatePolynomial('x, Integer)
```

$$\text{Fraction}(\text{UnivariatePolynomial}(x, Integer)) \tag{1}$$

Type

```
x : RFZ := 'x
```

$$x \tag{2}$$

Fraction (UnivariatePolynomial (x, Integer))

```
Dx : LODO1 RFZ := D()
```

$$D \tag{3}$$

LinearOrdinaryDifferentialOperator1 (Fraction (UnivariatePolynomial (x, Integer)))

Operators are created using the usual arithmetic operations.

```
b : LODO1 RFZ := 3*x^2*Dx^2 + 2*Dx + 1/x
```

$$3\,x^2\,D^2 + 2\,D + \frac{1}{x} \tag{4}$$

LinearOrdinaryDifferentialOperator1 (Fraction (UnivariatePolynomial (x, Integer)))

```
a : LODO1 RFZ := b*(5*x*Dx + 7)
```

$$15\,x^3\,D^3 + \left(51\,x^2 + 10\,x\right)\,D^2 + 29\,D + \frac{7}{x} \tag{5}$$

LinearOrdinaryDifferentialOperator1 (Fraction (UnivariatePolynomial (x, Integer)))

Operator multiplication corresponds to functional composition.

```
p := x^2 + 1/x^2
```

$$\frac{x^4 + 1}{x^2} \tag{6}$$

Fraction (UnivariatePolynomial (x, Integer))

Since operator coefficients depend on `x`, the multiplication is not commutative.

```
(a*b - b*a) p
```

$$\frac{-75\,x^4 + 540\,x - 75}{x^4} \tag{7}$$

Fraction (UnivariatePolynomial (x, Integer))

When the coefficients of operator polynomials come from a field, as in this case, it is possible to define operator division. Division on the left and division on the right yield different results when the multiplication is non-commutative.

The results of leftDivide and rightDivide are quotient-remainder pairs satisfying:
`leftDivide(a,b)= [q, r]` such that `a = b*q + r`
`rightDivide(a,b)= [q, r]` such that `a = q*b + r`
In both cases, the degree of the remainder, `r`, is less than the degree of `b`.

```
ld := leftDivide(a,b)
```

$$[quotient = 5\,x\,D + 7,\ remainder = 0] \tag{8}$$

Record(quotient: LinearOrdinaryDifferentialOperator1 (Fraction (UnivariatePolynomial (x, Integer))), remainder: LinearOrdinaryDifferentialOperator1 (Fraction (UnivariatePolynomial (x, Integer))))

```
a = b * ld.quotient + ld.remainder
```

$$15\,x^3\,D^3 + \left(51\,x^2 + 10\,x\right)\,D^2 + 29\,D + \frac{7}{x} = 15\,x^3\,D^3 + \left(51\,x^2 + 10\,x\right)\,D^2 + 29\,D + \frac{7}{x} \tag{9}$$

Equation(LinearOrdinaryDifferentialOperator1 (Fraction(UnivariatePolynomial(x, Integer))))

The operations of left and right division are so-called because the quotient is obtained by dividing a on that side by b.

```
rd := rightDivide(a,b)
```

$$\left[quotient = 5\,x\,D + 7,\ remainder = 10\,D + \frac{5}{x}\right] \tag{10}$$

Record(quotient: LinearOrdinaryDifferentialOperator1 (Fraction(UnivariatePolynomial(x, Integer))), remainder: LinearOrdinaryDifferentialOperator1 (Fraction(UnivariatePolynomial(x, Integer))))

```
a = rd.quotient * b + rd.remainder
```

$$15\,x^3\,D^3 + \left(51\,x^2 + 10\,x\right)\,D^2 + 29\,D + \frac{7}{x} = 15\,x^3\,D^3 + \left(51\,x^2 + 10\,x\right)\,D^2 + 29\,D + \frac{7}{x} \tag{11}$$

Equation(LinearOrdinaryDifferentialOperator1 (Fraction(UnivariatePolynomial(x, Integer))))

Operations rightQuotient and rightRemainder are available if only one of the quotient or remainder is of interest to you. This is the quotient from right division.

```
rightQuotient(a,b)
```

$$5\,x\,D + 7 \tag{12}$$

LinearOrdinaryDifferentialOperator1 (Fraction(UnivariatePolynomial(x, Integer)))

This is the remainder from right division. The corresponding "left" functions leftQuotient and leftRemainder are also available.

```
rightRemainder(a,b)
```

$$10\,D + \frac{5}{x} \tag{13}$$

LinearOrdinaryDifferentialOperator1 (Fraction(UnivariatePolynomial(x, Integer)))

For exact division, the operations leftExactQuotient and rightExactQuotient are supplied. These return the quotient but only if the remainder is zero. The call `rightExactQuotient(a,b)` would yield an error.

```
leftExactQuotient(a,b)
```

$$5\,x\,D + 7 \tag{14}$$

Union(LinearOrdinaryDifferentialOperator1 (Fraction(UnivariatePolynomial(x, Integer))), ...)

The division operations allow the computation of left and right greatest common divisors (leftGcd and rightGcd) via remainder sequences, and consequently the computation of left and right least common multiples (rightLcm and leftLcm).

```
e := leftGcd(a,b)
```

$$3\,x^2\,D^2 + 2\,D + \frac{1}{x} \tag{15}$$

LinearOrdinaryDifferentialOperator1 (Fraction (UnivariatePolynomial (x, Integer)))

Note that a greatest common divisor doesn't necessarily divide `a` and `b` on both sides. Here the left greatest common divisor does not divide `a` on the right.

```
leftRemainder(a, e)
```

$$0 \tag{16}$$

LinearOrdinaryDifferentialOperator1 (Fraction (UnivariatePolynomial (x, Integer)))

```
rightRemainder(a, e)
```

$$10\,D + \frac{5}{x} \tag{17}$$

LinearOrdinaryDifferentialOperator1 (Fraction (UnivariatePolynomial (x, Integer)))

Similarly, a least common multiple is not necessarily divisible from both sides.

```
f := rightLcm(a,b)
```

$$15\,x^3\,D^3 + \left(51\,x^2 + 10\,x\right)\,D^2 + 29\,D + \frac{7}{x} \tag{18}$$

LinearOrdinaryDifferentialOperator1 (Fraction (UnivariatePolynomial (x, Integer)))

```
rightRemainder(f, b)
```

$$10\,D + \frac{5}{x} \tag{19}$$

LinearOrdinaryDifferentialOperator1 (Fraction (UnivariatePolynomial (x, Integer)))

```
leftRemainder(f, b)
```

$$0 \tag{20}$$

LinearOrdinaryDifferentialOperator1 (Fraction (UnivariatePolynomial (x, Integer)))

9.51 LinearOrdinaryDifferentialOperator2

LinearOrdinaryDifferentialOperator2(A, M) is the domain of linear ordinary differential operators with coefficients in the differential ring `A` and operating on `M`, an `A`-module. This includes the cases of operators which are polynomials in `D` acting upon scalar or vector expressions of a single variable. The coefficients of the operator polynomials can be integers, rational functions, matrices or elements of other domains. Issue the system command `)show LinearOrdinaryDifferentialOperator2` to display the full list of operations defined by **LinearOrdinaryDifferentialOperator2**.

9.51.1 Differential Operators with Constant Coefficients

This example shows differential operators with rational number coefficients operating on univariate polynomials.

We begin by making type assignments so we can conveniently refer to univariate polynomials in `x` over the rationals.

```
Q  := Fraction Integer
```

$$\text{Fraction}(Integer) \tag{1}$$

Type

```
PQ := UnivariatePolynomial('x, Q)
```

$$\text{UnivariatePolynomial}(x, \text{Fraction}(Integer)) \tag{2}$$

Type

```
x: PQ := 'x
```

$$x \tag{3}$$

UnivariatePolynomial(x, Fraction(Integer))

Now we assign `Dx` to be the differential operator `D` corresponding to `d/dx`.

```
Dx: LODO2(Q, PQ) := D()
```

$$D \tag{4}$$

LinearOrdinaryDifferentialOperator2(Fraction(Integer), UnivariatePolynomial(x, Fraction(Integer)))

New operators are created as polynomials in `D()`.

```
a := Dx  + 1
```

$$D + 1 \tag{5}$$

LinearOrdinaryDifferentialOperator2(Fraction(Integer), UnivariatePolynomial(x, Fraction(Integer)))

```
b := a + 1/2*Dx^2 - 1/2
```

$$\frac{1}{2}D^2 + D + \frac{1}{2} \tag{6}$$

LinearOrdinaryDifferentialOperator2(Fraction(Integer), UnivariatePolynomial(x, Fraction(Integer)))

To apply the operator `a` to the value `p` the usual function call syntax is used.

```
p := 4*x^2 + 2/3
```

$$4\,x^2 + \frac{2}{3} \tag{7}$$

UnivariatePolynomial(x, Fraction(Integer))

```
a p
```

$$4\,x^2 + 8\,x + \frac{2}{3} \tag{8}$$

UnivariatePolynomial(x, Fraction(Integer))

Operator multiplication is defined by the identity `(a*b) p = a(b(p))`

```
(a * b) p = a b p
```

$$2\,x^2 + 12\,x + \frac{37}{3} = 2\,x^2 + 12\,x + \frac{37}{3} \tag{9}$$

Equation(UnivariatePolynomial(x, Fraction(Integer)))

Exponentiation follows from multiplication.

```
c := (1/9)*b*(a + b)^2
```

$$\frac{1}{72}\,D^6 + \frac{5}{36}\,D^5 + \frac{13}{24}\,D^4 + \frac{19}{18}\,D^3 + \frac{79}{72}\,D^2 + \frac{7}{12}\,D + \frac{1}{8} \tag{10}$$

LinearOrdinaryDifferentialOperator2(Fraction(Integer), UnivariatePolynomial(x, Fraction(Integer)))

Finally, note that operator expressions may be applied directly.

```
(a^2 - 3/4*b + c) (p + 1)
```

$$3\,x^2 + \frac{44}{3}\,x + \frac{541}{36} \tag{11}$$

UnivariatePolynomial(x, Fraction(Integer))

9.51.2 Differential Operators with Matrix Coefficients Operating on Vectors

This is another example of linear ordinary differential operators with non-commutative multiplication. Unlike the rational function case, the differential ring of square matrices (of a given dimension) with univariate polynomial entries does not form a field. Thus the number of operations available is more limited.

In this section, the operators have three by three matrix coefficients with polynomial entries.

```
PZ   := UnivariatePolynomial(x,Integer)
```

$$\mathrm{UnivariatePolynomial}(x, Integer) \tag{1}$$

Type

```
x:PZ := 'x
```

$$x \tag{2}$$

UnivariatePolynomial(x, Integer)

```
Mat  := SquareMatrix(3,PZ)
```

$$\text{SquareMatrix}(3, \text{UnivariatePolynomial}(x, Integer)) \tag{3}$$

Type

The operators act on the vectors considered as a Mat-module.

```
Vect := DPMM(3, PZ, Mat, PZ);
```

Type

```
Modo := LODO2(Mat, Vect);
```

Type

The matrix m is used as a coefficient and the vectors p and q are operated upon.

```
m:Mat := matrix [[x^2,1,0],[1,x^4,0],[0,0,4*x^2]]
```

$$\begin{bmatrix} x^2 & 1 & 0 \\ 1 & x^4 & 0 \\ 0 & 0 & 4\,x^2 \end{bmatrix} \tag{6}$$

SquareMatrix(3, UnivariatePolynomial(x, Integer))

```
p:Vect := directProduct [3*x^2+1,2*x,7*x^3+2*x]
```

$$\left[3\,x^2+1,\ 2\,x,\ 7\,x^3+2\,x\right] \tag{7}$$

DirectProductMatrixModule(3, UnivariatePolynomial(x, Integer), SquareMatrix(3, UnivariatePolynomial(x, Integer)), UnivariatePolynomial(x, Integer))

```
q: Vect := m * p
```

$$\left[3\,x^4+x^2+2\,x,\ 2\,x^5+3\,x^2+1,\ 28\,x^5+8\,x^3\right] \tag{8}$$

DirectProductMatrixModule(3, UnivariatePolynomial(x, Integer), SquareMatrix(3, UnivariatePolynomial(x, Integer)), UnivariatePolynomial(x, Integer))

Now form a few operators.

```
Dx : Modo := D()
```

$$D \tag{9}$$

LinearOrdinaryDifferentialOperator2 (SquareMatrix(3, UnivariatePolynomial(x, Integer)), DirectProductMatrixModule(3, UnivariatePolynomial(x, Integer), SquareMatrix(3, UnivariatePolynomial(x, Integer)), UnivariatePolynomial(x, Integer)))

```
a : Modo := Dx  + m
```

$$D+\begin{bmatrix} x^2 & 1 & 0 \\ 1 & x^4 & 0 \\ 0 & 0 & 4\,x^2 \end{bmatrix} \tag{10}$$

LinearOrdinaryDifferentialOperator2 (SquareMatrix(3, UnivariatePolynomial(x, Integer)), DirectProductMatrixModule(3, UnivariatePolynomial(x, Integer), SquareMatrix(3, UnivariatePolynomial(x, Integer)), UnivariatePolynomial(x, Integer)))

```
b : Modo := m*Dx  + 1
```

$$\begin{bmatrix} x^2 & 1 & 0 \\ 1 & x^4 & 0 \\ 0 & 0 & 4\,x^2 \end{bmatrix} D + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{11}$$

LinearOrdinaryDifferentialOperator2 (SquareMatrix(3, UnivariatePolynomial(x, Integer)), DirectProductMatrixModule(3, UnivariatePolynomial(x, Integer), SquareMatrix(3, UnivariatePolynomial(x, Integer)), UnivariatePolynomial(x, Integer)))

```
c := a*b
```

$$\begin{bmatrix} x^2 & 1 & 0 \\ 1 & x^4 & 0 \\ 0 & 0 & 4\,x^2 \end{bmatrix} D^2 + \begin{bmatrix} x^4+2\,x+2 & x^4+x^2 & 0 \\ x^4+x^2 & x^8+4\,x^3+2 & 0 \\ 0 & 0 & 16\,x^4+8\,x+1 \end{bmatrix} D + \begin{bmatrix} x^2 & 1 & 0 \\ 1 & x^4 & 0 \\ 0 & 0 & 4\,x^2 \end{bmatrix} \tag{12}$$

LinearOrdinaryDifferentialOperator2 (SquareMatrix(3, UnivariatePolynomial(x, Integer)), DirectProductMatrixModule(3, UnivariatePolynomial(x, Integer), SquareMatrix(3, UnivariatePolynomial(x, Integer)), UnivariatePolynomial(x, Integer)))

These operators can be applied to vector values.

```
a p
```

$$\left[3\,x^4+x^2+8\,x,\ 2\,x^5+3\,x^2+3,\ 28\,x^5+8\,x^3+21\,x^2+2\right] \tag{13}$$

DirectProductMatrixModule(3, UnivariatePolynomial(x, Integer), SquareMatrix(3, UnivariatePolynomial(x, Integer)), UnivariatePolynomial(x, Integer))

```
b p
```

$$\left[6\,x^3+3\,x^2+3,\ 2\,x^4+8\,x,\ 84\,x^4+7\,x^3+8\,x^2+2\,x\right] \tag{14}$$

DirectProductMatrixModule(3, UnivariatePolynomial(x, Integer), SquareMatrix(3, UnivariatePolynomial(x, Integer)), UnivariatePolynomial(x, Integer))

```
(a + b + c) (p + q)
```

$$\begin{aligned} &\left[10\,x^8+12\,x^7+16\,x^6+30\,x^5+85\,x^4+94\,x^3+40\,x^2+40\,x+17,\right. \\ &\ 10\,x^{12}+10\,x^9+12\,x^8+92\,x^7+6\,x^6+32\,x^5+72\,x^4+28\,x^3+49\,x^2+32\,x+19, \\ &\left.\ 2240\,x^8+224\,x^7+1280\,x^6+3508\,x^5+492\,x^4+751\,x^3+98\,x^2+18\,x+4\right] \end{aligned} \tag{15}$$

DirectProductMatrixModule(3, UnivariatePolynomial(x, Integer), SquareMatrix(3, UnivariatePolynomial(x, Integer)), UnivariatePolynomial(x, Integer))

9.52 List

A *list* is a finite collection of elements in a specified order that can contain duplicates. A list is a convenient structure to work with because it is easy to add or remove elements and the length need not be constant. There are many different kinds of lists in FriCAS, but the default types (and those used most often) are created by the **List** constructor. For example, there are objects of type **List Integer**, **List Float** and **List Polynomial Fraction Integer**. Indeed, you can even have **List List List Boolean** (that is, lists of lists of Boolean values). You can have lists of any type of FriCAS object.

9.52.1 Creating Lists

The easiest way to create a list with, for example, the elements `2, 4, 5, 6` is to enclose the elements with square brackets and separate the elements with commas. The spaces after the commas are optional, but they do improve the readability.

```
[2, 4, 5, 6]
```

$$[2,\ 4,\ 5,\ 6] \tag{1}$$

List (PositiveInteger)

To create a list with the single element `1`, you can use either `[1]` or the operation list.

```
[1]
```

$$[1] \tag{2}$$

List (PositiveInteger)

```
list(1)
```

$$[1] \tag{3}$$

List (PositiveInteger)

Once created, two lists `k` and `m` can be concatenated by issuing `append(k,m)`. append does *not* physically join the lists, but rather produces a new list with the elements coming from the two arguments.

```
append([1,2,3],[5,6,7])
```

$$[1,\ 2,\ 3,\ 5,\ 6,\ 7] \tag{4}$$

List (PositiveInteger)

Use cons to append an element onto the front of a list.

```
cons(10,[9,8,7])
```

$$[10,\ 9,\ 8,\ 7] \tag{5}$$

List (PositiveInteger)

9.52.2 Accessing List Elements

To determine whether a list has any elements, use the operation empty?.

```
empty? [x+1]
```

false (1)

Boolean

Alternatively, equality with explicit empty list [] can be tested.

```
(rest([1]) = [])@Boolean
```

true (2)

Boolean

We'll use this in some of the following examples.

```
k := [4,3,7,3,8,5,9,2]
```

[4, 3, 7, 3, 8, 5, 9, 2] (3)

List (PositiveInteger)

Each of the next four expressions extracts the first element of k.

```
first k
```

4 (4)

PositiveInteger

```
k.first
```

4 (5)

PositiveInteger

```
k.1
```

4 (6)

PositiveInteger

```
k(1)
```

4 (7)

PositiveInteger

The last two forms generalize to `k.i` and `k(i)`, respectively, where $1 \leq i \leq n$ and `n` equals the length of `k`. This length is calculated by `#`.

```
n := #k
```

$$8 \tag{8}$$

PositiveInteger

Performing an operation such as `k.i` is sometimes referred to as *indexing into k* or *elting into k*. The latter phrase comes about because the name of the operation that extracts elements is called elt. That is, `k.3` is just alternative syntax for `elt(k,3)`. It is important to remember that list indices begin with 1. If we issue `k := [1,3,2,9,5]` then `k.4` returns `9`. It is an error to use an index that is not in the range from `1` to the length of the list.

The last element of a list is extracted by any of the following three expressions.

```
last k
```

$$2 \tag{9}$$

PositiveInteger

```
k.last
```

$$2 \tag{10}$$

PositiveInteger

This form computes the index of the last element and then extracts the element from the list.

```
k.(#k)
```

$$2 \tag{11}$$

PositiveInteger

9.52.3 Changing List Elements

We'll use this in some of the following examples.

```
k := [4,3,7,3,8,5,9,2]
```

$$[4,\ 3,\ 7,\ 3,\ 8,\ 5,\ 9,\ 2] \tag{1}$$

List (PositiveInteger)

List elements are reset by using the `k.i` form on the left-hand side of an assignment. This expression resets the first element of `k` to `999`.

```
k.1 := 999
```

999 (2)

PositiveInteger

As with indexing into a list, it is an error to use an index that is not within the proper bounds. Here you see that `k` was modified.

```
k
```

[999, 3, 7, 3, 8, 5, 9, 2] (3)

List (PositiveInteger)

The operation that performs the assignment of an element to a particular position in a list is called setelt!. This operation is *destructive* in that it changes the list. In the above example, the assignment returned the value `999` and `k` was modified. For this reason, lists are called *mutable* objects: it is possible to change part of a list (mutate it) rather than always returning a new list reflecting the intended modifications. Moreover, since lists can share structure, changes to one list can sometimes affect others.

```
k := [1,2]
```

[1, 2] (4)

List (PositiveInteger)

```
m := cons(0,k)
```

[0, 1, 2] (5)

List (Integer)

Change the second element of `m`.

```
m.2 := 99
```

99 (6)

PositiveInteger

See, `m` was altered.

```
m
```

[0, 99, 2] (7)

List (Integer)

But what about `k`? It changed too!

```
k
```

[99, 2] (8)

List (PositiveInteger)

9.52.4 Other Functions

An operation that is used frequently in list processing is that which returns all elements in a list after the first element.

```
k := [1,2,3]
```

$$[1,\ 2,\ 3] \tag{1}$$

List (PositiveInteger)

Use the rest operation to do this.

```
rest k
```

$$[2,\ 3] \tag{2}$$

List (PositiveInteger)

To remove duplicate elements in a list `k`, use removeDuplicates.

```
removeDuplicates [4,3,4,3,5,3,4]
```

$$[4,\ 3,\ 5] \tag{3}$$

List (PositiveInteger)

To get a list with elements in the order opposite to those in a list `k`, use reverse.

```
reverse [1,2,3,4,5,6]
```

$$[6,\ 5,\ 4,\ 3,\ 2,\ 1] \tag{4}$$

List (PositiveInteger)

To test whether an element is in a list, use member?: `member?(a,k)` returns `true` or `false` depending on whether `a` is in `k` or not.

```
member?(1/2,[3/4,5/6,1/2])
```

$$\texttt{true} \tag{5}$$

Boolean

```
member?(1/12,[3/4,5/6,1/2])
```

$$\texttt{false} \tag{6}$$

Boolean

As an exercise, the reader should determine how to get a list containing all but the last of the elements in a given non-empty list `k`.[4]

[4] `reverse(rest(reverse(k)))` works.

9.52.5 Dot, Dot

Certain lists are used so often that FriCAS provides an easy way of constructing them. If `n` and `m` are integers, then `expand [n..m]` creates a list containing `n, n+1, ... m`. If `n > m` then the list is empty. It is actually permissible to leave off the `m` in the dot-dot construction (see below).

The dot-dot notation can be used more than once in a list construction and with specific elements being given. Items separated by dots are called *segments.*

```
[1..3,10,20..23]
```

$$[1..3,\ 10..10,\ 20..23] \tag{1}$$

List (Segment(PositiveInteger))

Segments can be expanded into the range of items between the endpoints by using expand.

```
expand [1..3,10,20..23]
```

$$[1,\ 2,\ 3,\ 10,\ 20,\ 21,\ 22,\ 23] \tag{2}$$

List (Integer)

What happens if we leave off a number on the right-hand side of `..`?

```
expand [1..]
```

$$[1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ \ldots] \tag{3}$$

Stream(Integer)

What is created in this case is a **Stream** which is a generalization of a list. See '`Stream`' on page 577 for more information.

9.53 LyndonWord

Initialisations

```
a:Symbol :='a
```

$$a \tag{4}$$

Symbol

```
b:Symbol :='b
```

$$b \tag{5}$$

Symbol

```
c:Symbol :='c
```

$$c \tag{6}$$

Symbol

```
lword:= LyndonWord(Symbol)
```

$$\text{LyndonWord}(Symbol) \tag{7}$$

Type

```
magma := FreeMagma(Symbol)
```

$$\text{FreeMagma}(Symbol) \tag{8}$$

Type

```
word  := FreeMonoid(Symbol)
```

$$\text{FreeMonoid}(Symbol) \tag{9}$$

Type

All Lyndon words with a, b, c to order 3

```
LyndonWordsList1([a,b,c],3)$lword
```

$$[[[a], [b], [c]], [[a\,b], [a\,c], [b\,c]], [[a^2\,b], [a^2\,c], [a\,b^2], [a\,b\,c], [a\,c\,b], [a\,c^2], [b^2\,c], [b\,c^2]]] \tag{10}$$

OneDimensionalArray(List(LyndonWord(Symbol)))

All Lyndon words with a, b, c to order 3 in a flat list

```
LyndonWordsList([a,b,c],3)$lword
```

$$[[a], [b], [c], [a\,b], [a\,c], [b\,c], [a^2\,b], [a^2\,c], [a\,b^2], [a\,b\,c], [a\,c\,b], [a\,c^2], [b^2\,c], [b\,c^2]] \tag{11}$$

List (LyndonWord(Symbol))

All Lyndon words with a, b to order 5

```
lw := LyndonWordsList([a,b],5)$lword
```

$$[[a], [b], [a\,b], [a^2\,b], [a\,b^2], [a^3\,b], [a^2\,b^2], [a\,b^3], [a^4\,b], [a^3\,b^2], [a^2\,b\,a\,b], [a^2\,b^3], [a\,b\,a\,b^2], [a\,b^4]] \tag{12}$$

List (LyndonWord(Symbol))

```
w1 : word := lw.4 :: word
```

$$a^2\,b \tag{13}$$

FreeMonoid(Symbol)

```
w2 : word := lw.5 :: word
```

$$a\,b^2 \tag{14}$$

FreeMonoid(Symbol)

Let's try factoring

```
factor(a::word)$lword
```

$$[[a]] \tag{15}$$

List (LyndonWord(Symbol))

```
factor(w1*w2)$lword
```

$$\left[\left[a^2\,b\,a\,b^2\right]\right] \tag{16}$$

List (LyndonWord(Symbol))

```
factor(w2*w2)$lword
```

$$\left[\left[a\,b^2\right],\ \left[a\,b^2\right]\right] \tag{17}$$

List (LyndonWord(Symbol))

```
factor(w2*w1)$lword
```

$$\left[\left[a\,b^2\right],\ \left[a^2\,b\right]\right] \tag{18}$$

List (LyndonWord(Symbol))

Checks and coercions

```
lyndon?(w1)$lword
```

```
true                                                    (19)
```

Boolean

```
lyndon?(w1*w2)$lword
```

```
true                                                    (20)
```

Boolean

```
lyndon?(w2*w1)$lword
```

```
false                                                   (21)
```

Boolean

```
lyndonIfCan(w1)$lword
```

$$\left[a^2\, b\right] \tag{22}$$

Union(LyndonWord(Symbol), ...)

```
lyndonIfCan(w2*w1)$lword
```

$$\text{"failed"} \tag{23}$$

Union(" failed ", ...)

```
lyndon(w1)$lword
```

$$\left[a^2\, b\right] \tag{24}$$

LyndonWord(Symbol)

```
lyndon(w1*w2)$lword
```

$$\left[a^2\, b\, a\, b^2\right] \tag{25}$$

LyndonWord(Symbol)

9.54 MakeFunction

It is sometimes useful to be able to define a function given by the result of a calculation. Suppose that you have obtained the following expression after several computations and that you now want to tabulate the numerical values of `f` for `x` between `-1` and `+1` with increment `0.1`.

```
expr := (x - exp x + 1)^2 * (sin(x^2) * x + 1)^3
```

$$\begin{aligned}&\left(x^3\,(e^x)^2+\left(-2\,x^4-2\,x^3\right)\,e^x+x^5+2\,x^4+x^3\right)\,\sin\left(x^2\right)^3\\&+\left(3\,x^2\,(e^x)^2+\left(-6\,x^3-6\,x^2\right)\,e^x+3\,x^4+6\,x^3+3\,x^2\right)\,\sin\left(x^2\right)^2\\&+\left(3\,x\,(e^x)^2+\left(-6\,x^2-6\,x\right)\,e^x+3\,x^3+6\,x^2+3\,x\right)\,\sin\left(x^2\right)+(e^x)^2+(-2\,x-2)\,e^x+x^2+2\,x+1\end{aligned} \tag{4}$$

Expression(Integer)

You could, of course, use the function eval within a loop and evaluate `expr` twenty-one times, but this would be quite slow. A better way is to create a numerical function `f` such that `f(x)` is defined by the expression `expr` above, but without retyping `expr`! The package **MakeFunction** provides the operation function which does exactly this. Issue this to create the function `f(x)` given by `expr`.

```
function(expr, f, x)
```

$$f \tag{5}$$

Symbol

To tabulate `expr`, we can now quickly evaluate `f` 21 times.

```
tbl := [f(0.1 * i - 1) for i in 0..20];
```

```
   Compiling function f with type Float -> Float
```

List (Float)

Use the list `[x1,...,xn]` as the third argument to function to create a multivariate function `f(x1,...,xn)`.

```
e := (x - y + 1)^2 * (x^2 * y + 1)^2
```

$$x^4\,y^4+\left(-2\,x^5-2\,x^4+2\,x^2\right)\,y^3+\left(x^6+2\,x^5+x^4-4\,x^3-4\,x^2+1\right)\,y^2 \\ +\left(2\,x^4+4\,x^3+2\,x^2-2\,x-2\right)\,y+x^2+2\,x+1 \tag{7}$$

Polynomial(Integer)

```
function(e, g, [x, y])
```

$$g \tag{8}$$

Symbol

In the case of just two variables, they can be given as arguments without making them into a list.

```
function(e, h, x, y)
```

$$h \tag{9}$$

Symbol

Note that the functions created by function are not limited to floating point numbers, but can be applied to any type for which they are defined.

```
m1 := squareMatrix [[1, 2], [3, 4]]
```

$$\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} \tag{10}$$

SquareMatrix(2, Integer)

```
m2 := squareMatrix [[1, 0], [-1, 1]]
```

$$\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix} \tag{11}$$

SquareMatrix(2, Integer)

```
h(m1, m2)
```

```
   Compiling function h with type (SquareMatrix(2,Integer), SquareMatrix(2,Integer
      )) -> SquareMatrix(2,Integer)
```

$$\begin{bmatrix}-7836 & 8960\\ -17132 & 19588\end{bmatrix} \tag{12}$$

SquareMatrix(2, Integer)

For more information, see Section 6.14 on page 169. Issue the system command `)show MakeFunction` to display the full list of operations defined by **MakeFunction**.

9.55 MappingPackage1

Functions are objects of type **Mapping**. In this section we demonstrate some library operations from the packages **MappingPackage1**, **MappingPackage2**, and **MappingPackage3** that manipulate and create functions. Some terminology: a *nullary* function takes no arguments, a *unary* function takes one argument, and a *binary* function takes two arguments.

We begin by creating an example function that raises a rational number to an integer exponent.

```
power(q: FRAC INT, n: INT): FRAC INT == q^n
```

```
   Function declaration power : (Fraction(Integer), Integer) -> Fraction(Integer)
      has been added to workspace.
```

```
power(2,3)
```

```
   Compiling function power with type (Fraction(Integer), Integer) -> Fraction(
      Integer)
```

$$8 \tag{5}$$

Fraction(Integer)

The twist operation transposes the arguments of a binary function. Here `rewop(a, b)` is `power(b, a)`.

```
rewop := twist power
```

$$\text{theMap(twist)} \tag{6}$$

((Integer, Fraction(Integer)) → Fraction(Integer))

This is 2^3.

```
rewop(3, 2)
```

$$8 \tag{7}$$

Fraction(Integer)

Now we define **square** in terms of **power**.

```
square: FRAC INT -> FRAC INT
```

The curryRight operation creates a unary function from a binary one by providing a constant argument on the right.

```
square:= curryRight(power, 2)
```

$$\text{theMap(curryRight)} \tag{9}$$

(Fraction(Integer) → Fraction(Integer))

Likewise, the curryLeft operation provides a constant argument on the left.

```
square 4
```

$$16 \tag{10}$$

Fraction(Integer)

The constantRight operation creates (in a trivial way) a binary function from a unary one: `constantRight(f)` is the function `g` such that `g(a,b)= f(a)`.

```
squirrel:= constantRight(square)$MAPPKG3(FRAC INT,FRAC INT,FRAC INT)
```

$$\text{theMap(constantRight)} \tag{11}$$

((Fraction(Integer), Fraction(Integer)) → Fraction(Integer))

Likewise, `constantLeft(f)` is the function `g` such that `g(a,b)= f(b)`.

```
squirrel(1/2, 1/3)
```

$$\frac{1}{4} \tag{12}$$

Fraction(Integer)

The curry operation makes a unary function nullary.

```
sixteen := curry(square, 4/1)
```

$$\text{theMap(curry)} \tag{13}$$

(() → Fraction(Integer))

```
sixteen()
```

$$16 \tag{14}$$

Fraction(Integer)

The * operation constructs composed functions.

```
square2:=square*square
```

$$\text{theMap}(*) \tag{15}$$

(Fraction(Integer) → Fraction(Integer))

```
square2  3
```

$$81 \tag{16}$$

Fraction (Integer)

Use the ^ operation to create functions that are n-fold iterations of other functions.

```
sc(x: FRAC INT): FRAC INT == x + 1
```

```
   Function declaration sc : Fraction(Integer) -> Fraction(Integer) has been added
      to workspace.
```

This is a list of **Mapping** objects.

```
incfns := [sc^i for i in 0..10]
```

```
   Compiling function sc with type Fraction(Integer) -> Fraction(Integer)
```

$$[\text{theMap}(^), \text{theMap}(^), \text{theMap}(^), \text{theMap}(^), \text{theMap}(^), \text{theMap}(^), \text{theMap}(^), \text{theMap}(^), \text{theMap}(^), \text{theMap}(^), \text{theMap}(^)] \tag{18}$$

List ((Fraction (Integer) → Fraction (Integer)))

This is a list of applications of those functions.

```
[f 4 for f in incfns]
```

$$[4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14] \tag{19}$$

List (Fraction (Integer))

Use the recur operation for recursion: `g := recur f` means `g(n,x) == f(n,f(n-1,...f(1,x)))`.

```
times(n:NNI, i:INT):INT == n*i
```

```
   Function declaration times : (NonNegativeInteger, Integer) -> Integer has been
      added to workspace.
```

```
r := recur(times)
```

```
   Compiling function times with type (NonNegativeInteger, Integer) -> Integer
```

$$\text{theMap}(\text{recur}) \tag{21}$$

((NonNegativeInteger, Integer) → Integer)

This is a factorial function.

```
fact := curryRight(r, 1)
```

$$\text{theMap}(\text{curryRight}) \tag{22}$$

(NonNegativeInteger → Integer)

```
fact 4
```

$$24 \tag{23}$$

PositiveInteger

Constructed functions can be used within other functions.

```
mto2ton(m, n) ==
  raiser := square^n
  raiser m
```

This is 3^{2^3}.

```
mto2ton(3, 3)
```

```
   Compiling function mto2ton with type (PositiveInteger, PositiveInteger) ->
      Fraction(Integer)
```

$$6561 \tag{25}$$

Fraction (Integer)

Here **shiftfib** is a unary function that modifies its argument.

```
shiftfib(r: List INT) : INT ==
  t := r.1
  r.1 := r.2
  r.2 := r.2 + t
  t
```

```
   Function declaration shiftfib : List(Integer) -> Integer has been added to
      workspace.
```

By currying over the argument we get a function with private state.

```
fibinit: List INT := [0, 1]
```

$$[0,\ 1] \tag{27}$$

List (Integer)

```
fibs := curry(shiftfib, fibinit)
```

```
   Compiling function shiftfib with type List(Integer) -> Integer
```

$$\text{theMap(curry)} \tag{28}$$

(() → Integer)

```
[fibs() for i in 0..30]
```

$$[0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ 233,\ 377,\ 610,\ 987,\ 1597,\ 2584,\ 4181,\ 6765,\ 10946,\ 17711,\ 28657,\ 46368,\ 75025,\ 121393,\ 196418,\ 317811,\ 514229,\ 832040] \tag{29}$$

List (Integer)

9.56 Matrix

The **Matrix** domain provides arithmetic operations on matrices and standard functions from linear algebra. This domain is similar to the **TwoDimensionalArray** domain, except that the entries for **Matrix** must belong to a **Ring**.

9.56.1 Creating Matrices

There are many ways to create a matrix from a collection of values or from existing matrices.

If the matrix has almost all items equal to the same value, use new to create a matrix filled with that value and then reset the entries that are different.

```
m : Matrix(Integer) := new(3,3,0)
```

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1}$$

Matrix(Integer)

To change the entry in the second row, third column to 5, use setelt!.

```
setelt!(m, 2, 3, 5)
```

$$5 \tag{2}$$

PositiveInteger

An alternative syntax is to use assignment.

```
m(1,2) := 10
```

$$10 \tag{3}$$

PositiveInteger

The matrix was *destructively modified*.

```
m
```

$$\begin{bmatrix} 0 & 10 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} \tag{4}$$

Matrix(Integer)

If you already have the matrix entries as a list of lists, use matrix.

```
matrix [[1,2,3,4],[0,9,8,7]]
```

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 9 & 8 & 7 \end{bmatrix} \tag{5}$$

Matrix(NonNegativeInteger)

If the matrix is diagonal, use diagonalMatrix.

```
dm := diagonalMatrix [1,x^2,x^3,x^4,x^5]
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & x^2 & 0 & 0 & 0 \\ 0 & 0 & x^3 & 0 & 0 \\ 0 & 0 & 0 & x^4 & 0 \\ 0 & 0 & 0 & 0 & x^5 \end{bmatrix} \tag{6}$$

Matrix(Polynomial(Integer))

Use setRow! and setColumn! to change a row or column of a matrix.

```
setRow!(dm,5,vector [1,1,1,1,1])
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & x^2 & 0 & 0 & 0 \\ 0 & 0 & x^3 & 0 & 0 \\ 0 & 0 & 0 & x^4 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \tag{7}$$

Matrix(Polynomial(Integer))

```
setColumn!(dm,2,vector [y,y,y,y,y])
```

$$\begin{bmatrix} 1 & y & 0 & 0 & 0 \\ 0 & y & 0 & 0 & 0 \\ 0 & y & x^3 & 0 & 0 \\ 0 & y & 0 & x^4 & 0 \\ 1 & y & 1 & 1 & 1 \end{bmatrix} \tag{8}$$

Matrix(Polynomial(Integer))

Use copy to make a copy of a matrix.

```
cdm := copy(dm)
```

$$\begin{bmatrix} 1 & y & 0 & 0 & 0 \\ 0 & y & 0 & 0 & 0 \\ 0 & y & x^3 & 0 & 0 \\ 0 & y & 0 & x^4 & 0 \\ 1 & y & 1 & 1 & 1 \end{bmatrix} \tag{9}$$

Matrix(Polynomial(Integer))

This is useful if you intend to modify a matrix destructively but want a copy of the original.

```
setelt!(dm, 4, 1, 1 - x^7)
```

$$-x^7+1 \tag{10}$$

Polynomial(Integer)

```
[dm,cdm]
```

$$\left[\begin{bmatrix} 1 & y & 0 & 0 & 0 \\ 0 & y & 0 & 0 & 0 \\ 0 & y & x^3 & 0 & 0 \\ -x^7+1 & y & 0 & x^4 & 0 \\ 1 & y & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & y & 0 & 0 & 0 \\ 0 & y & 0 & 0 & 0 \\ 0 & y & x^3 & 0 & 0 \\ 0 & y & 0 & x^4 & 0 \\ 1 & y & 1 & 1 & 1 \end{bmatrix}\right] \tag{11}$$

List (Matrix(Polynomial(Integer)))

Use subMatrix to extract part of an existing matrix. The syntax is subMatrix(*m, firstrow, lastrow, firstcol, lastcol*).

```
subMatrix(dm,2,3,2,4)
```

$$\begin{bmatrix} y & 0 & 0 \\ y & x^3 & 0 \end{bmatrix} \tag{12}$$

Matrix(Polynomial(Integer))

To change a submatrix, use setsubMatrix!.

```
d := diagonalMatrix [1.2,-1.3,1.4,-1.5]
```

$$\begin{bmatrix} 1.2 & 0.0 & 0.0 & 0.0 \\ 0.0 & -1.3 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.4 & 0.0 \\ 0.0 & 0.0 & 0.0 & -1.5 \end{bmatrix} \tag{13}$$

Matrix(Float)

If e is too big to fit where you specify, an error message is displayed. Use subMatrix to extract part of e, if necessary.

```
e := matrix [[6.7,9.11],[-31.33,67.19]]
```

$$\begin{bmatrix} 6.7 & 9.11 \\ -31.33 & 67.19 \end{bmatrix} \tag{14}$$

Matrix(Float)

This changes the submatrix of d whose upper left corner is at the first row and second column and whose size is that of e.

```
setsubMatrix!(d,1,2,e)
```

$$\begin{bmatrix} 1.2 & 6.7 & 9.11 & 0.0 \\ 0.0 & -31.33 & 67.19 & 0.0 \\ 0.0 & 0.0 & 1.4 & 0.0 \\ 0.0 & 0.0 & 0.0 & -1.5 \end{bmatrix} \tag{15}$$

Matrix(Float)

```
d
```

$$\begin{bmatrix} 1.2 & 6.7 & 9.11 & 0.0 \\ 0.0 & -31.33 & 67.19 & 0.0 \\ 0.0 & 0.0 & 1.4 & 0.0 \\ 0.0 & 0.0 & 0.0 & -1.5 \end{bmatrix} \tag{16}$$

Matrix(Float)

Matrices can be joined either horizontally or vertically to make new matrices.

```
a := matrix [[1/2,1/3,1/4],[1/5,1/6,1/7]]
```

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{bmatrix} \tag{17}$$

Matrix(Fraction(Integer))

```
b := matrix [[3/5,3/7,3/11],[3/13,3/17,3/19]]
```

$$\begin{bmatrix} \frac{3}{5} & \frac{3}{7} & \frac{3}{11} \\ \frac{3}{13} & \frac{3}{17} & \frac{3}{19} \end{bmatrix} \tag{18}$$

Matrix(Fraction(Integer))

Use horizConcat to append them side to side. The two matrices must have the same number of rows.

```
horizConcat(a,b)
```

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{3}{5} & \frac{3}{7} & \frac{3}{11} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{3}{13} & \frac{3}{17} & \frac{3}{19} \end{bmatrix} \tag{19}$$

Matrix(Fraction(Integer))

Use vertConcat to stack one upon the other. The two matrices must have the same number of columns.

```
vab := vertConcat(a,b)
```

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \\ \frac{3}{5} & \frac{3}{7} & \frac{3}{11} \\ \frac{3}{13} & \frac{3}{17} & \frac{3}{19} \end{bmatrix} \tag{20}$$

Matrix(Fraction(Integer))

The operation transpose is used to create a new matrix by reflection across the main diagonal.

```
transpose vab
```

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{5} & \frac{3}{5} & \frac{3}{13} \\ \frac{1}{3} & \frac{1}{6} & \frac{3}{7} & \frac{3}{17} \\ \frac{1}{4} & \frac{1}{7} & \frac{3}{11} & \frac{3}{19} \end{bmatrix} \tag{21}$$

Matrix(Fraction(Integer))

9.56.2 Operations on Matrices

FriCAS provides both left and right scalar multiplication.

```
m := matrix [[1,2],[3,4]]
```

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \tag{1}$$

Matrix(Integer)

```
4 * m * (-5)
```

$$\begin{bmatrix} -20 & -40 \\ -60 & -80 \end{bmatrix} \tag{2}$$

Matrix(Integer)

You can add, subtract, and multiply matrices provided, of course, that the matrices have compatible dimensions. If not, an error message is displayed.

```
n := matrix([[1,0,-2],[-3,5,1]])
```

$$\begin{bmatrix} 1 & 0 & -2 \\ -3 & 5 & 1 \end{bmatrix} \tag{3}$$

Matrix(Integer)

The following product is defined but `n * m` is not.

```
m * n
```

$$\begin{bmatrix} -5 & 10 & 0 \\ -9 & 20 & -2 \end{bmatrix} \tag{4}$$

Matrix(Integer)

The operations nrows and ncols return the number of rows and columns of a matrix. You can extract a row or a column of a matrix using the operations row and column. The object returned is a **Vector**. Here is the third column of the matrix `n`.

```
vec := column(n,3)
```

$$[-2,\ 1] \tag{5}$$

Vector(Integer)

You can multiply a matrix on the left by a "row vector" and on the right by a "column vector."

```
vec * m
```

$$[1,\ 0] \tag{6}$$

Vector(Integer)

Of course, the dimensions of the vector and the matrix must be compatible or an error message is returned.

```
m * vec
```

$$[0,\ -2] \tag{7}$$

Vector(Integer)

The operation inverse computes the inverse of a matrix if the matrix is invertible, and returns "failed" if not. This Hilbert matrix is invertible.

```
hilb := matrix([[1/(i + j) for i in 1..3] for j in 1..3])
```

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \end{bmatrix} \tag{8}$$

Matrix(Fraction (Integer))

```
inverse(hilb)
```

$$\begin{bmatrix} 72 & -240 & 180 \\ -240 & 900 & -720 \\ 180 & -720 & 600 \end{bmatrix} \tag{9}$$

Union(Matrix(Fraction(Integer)), ...)

This matrix is not invertible.

```
mm := matrix([[1,2,3,4], [5,6,7,8], [9,10,11,12], [13,14,15,16]])
```

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \tag{10}$$

Matrix(Integer)

```
inverse(mm)
```

$$\text{"failed"} \tag{11}$$

Union(" failed ", ...)

The operation determinant computes the determinant of a matrix provided that the entries of the matrix belong to a **CommutativeRing**. The above matrix mm is not invertible and, hence, must have determinant 0.

```
determinant(mm)
```

$$0 \tag{12}$$

NonNegativeInteger

The operation trace computes the trace of a *square* matrix.

```
trace(mm)
```

$$34 \tag{13}$$

PositiveInteger

The operation rank computes the *rank* of a matrix: the maximal number of linearly independent rows or columns.

```
rank(mm)
```

$$2 \tag{14}$$

PositiveInteger

The operation nullity computes the *nullity* of a matrix: the dimension of its null space.

```
nullity(mm)
```

$$2 \tag{15}$$

PositiveInteger

The operation nullSpace returns a list containing a basis for the null space of a matrix. Note that the nullity is the number of elements in a basis for the null space.

```
nullSpace(mm)
```

$$[[1,\ -2,\ 1,\ 0]\,,\ [2,\ -3,\ 0,\ 1]] \tag{16}$$

List(Vector(Integer))

The operation rowEchelon returns the row echelon form of a matrix. It is easy to see that the rank of this matrix is two and that its nullity is also two.

```
rowEchelon(mm)
```

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 4 & 8 & 12 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{17}$$

Matrix(Integer)

For more information on related topics, see Section 1.7 on page 43, Section 8.4 on page 267, Section 9.27.4 on page 423, '`Permanent`' on page 531, '`Vector`' on page 603, '`OneDimensionalArray`' on page 517, and '`TwoDimensionalArray`' on page 592. Issue the system command `)show Matrix` to display the full list of operations defined by **Matrix**.

9.57 Multiset

The domain **Multiset(R)** is similar to **Set(R)** except that multiplicities (counts of duplications) are maintained and displayed. Use the operation multiset to create multisets from lists. All the standard

operations from sets are available for multisets. An element with multiplicity greater than one has the multiplicity displayed first, then a colon, and then the element.

Create a multiset of integers.

```
s := multiset [1,2,3,4,5,4,3,2,3,4,5,6,7,4,10]
```

$$\{1,\ 2\ :\ 2,\ 3\ :\ 3,\ 4\ :\ 4,\ 2\ :\ 5,\ 6,\ 7,\ 10\} \tag{4}$$

Multiset(PositiveInteger)

The operation insert! adds an element to a multiset.

```
insert!(3,s)
```

$$\{1,\ 2\ :\ 2,\ 4\ :\ 3,\ 4\ :\ 4,\ 2\ :\ 5,\ 6,\ 7,\ 10\} \tag{5}$$

Multiset(PositiveInteger)

Use remove! to remove an element. If a third argument is present, it specifies how many instances to remove. Otherwise all instances of the element are removed. Display the resulting multiset.

```
remove!(3,s,1); s
```

$$\{1,\ 2\ :\ 2,\ 3\ :\ 3,\ 4\ :\ 4,\ 2\ :\ 5,\ 6,\ 7,\ 10\} \tag{6}$$

Multiset(PositiveInteger)

```
remove!(5,s); s
```

$$\{1,\ 2\ :\ 2,\ 3\ :\ 3,\ 4\ :\ 4,\ 6,\ 7,\ 10\} \tag{7}$$

Multiset(PositiveInteger)

The operation count returns the number of copies of a given value.

```
count(5,s)
```

$$0 \tag{8}$$

NonNegativeInteger

A second multiset.

```
t := multiset [2,2,2,-9]
```

$$\{3\ :\ 2,\ -9\} \tag{9}$$

Multiset(Integer)

The union of two multisets is additive.

```
U := union(s,t)
```

$$\{10,\ 7,\ 6,\ 4\ :\ 4,\ 3\ :\ 3,\ 5\ :\ 2,\ 1,\ -9\} \tag{10}$$

Multiset (Integer)

The intersect operation gives the elements that are in common, with additive multiplicity.

```
I := intersect(s,t)
```

$$\{5 : 2\} \tag{11}$$

Multiset (Integer)

The difference of s and t consists of the elements that s has but t does not. Elements are regarded as indistinguishable, so that if s and t have any element in common, the difference does not contain that element.

```
difference(s,t)
```

$$\{10,\ 7,\ 6,\ 4 : 4,\ 3 : 3,\ 1\} \tag{12}$$

Multiset (Integer)

The symmetricDifference is the union of `difference(s, t)` and `difference(t, s)`.

```
S := symmetricDifference(s,t)
```

$$\{1,\ 3 : 3,\ 4 : 4,\ 6,\ 7,\ 10,\ -9\} \tag{13}$$

Multiset (Integer)

Check that the union of the symmetricDifference and the intersect equals the union of the elements.

```
(U = union(S,I))@Boolean
```

$$\texttt{true} \tag{14}$$

Boolean

Check some inclusion relations.

```
t1 := multiset [1,2,2,3]; [t1 < t, t1 < s, t < s, t1 <= s]
```

$$[\texttt{false},\ \texttt{true},\ \texttt{false},\ \texttt{true}] \tag{15}$$

List (Boolean)

9.58 MultivariatePolynomial

The domain constructor **MultivariatePolynomial** is similar to **Polynomial** except that it specifies the variables to be used. Most functions available for **Polynomial** are available for **MultivariatePolynomial**. The abbreviation for **MultivariatePolynomial** is **MPOLY**. The type expressions

MultivariatePolynomial([x,y],Integer) and **MPOLY([x,y],INT)**

refer to the domain of multivariate polynomials in the variables `x` and `y` where the coefficients are restricted to be integers. The first variable specified is the main variable and the display of the polynomial reflects this. This polynomial appears with terms in descending powers of the variable `x`.

```
m : MPOLY([x,y],INT) := (x^2 - x*y^3 +3*y)^2
```

$$x^4 - 2\,y^3\,x^3 + \left(y^6 + 6\,y\right)\,x^2 - 6\,y^4\,x + 9\,y^2 \tag{4}$$

MultivariatePolynomial ([x, y], Integer)

It is easy to see a different variable ordering by doing a conversion.

```
m :: MPOLY([y,x],INT)
```

$$x^2\,y^6 - 6\,x\,y^4 - 2\,x^3\,y^3 + 9\,y^2 + 6\,x^2\,y + x^4 \tag{5}$$

MultivariatePolynomial ([y, x], Integer)

You can use other, unspecified variables, by using **Polynomial** in the coefficient type of **MPOLY**.

```
p : MPOLY([x,y],POLY INT)
```

```
p := (a^2*x - b*y^2 + 1)^2
```

$$a^4\,x^2 + \left(-2\,a^2\,b\,y^2 + 2\,a^2\right)\,x + b^2\,y^4 - 2\,b\,y^2 + 1 \tag{7}$$

MultivariatePolynomial ([x, y], Polynomial(Integer))

Conversions can be used to re-express such polynomials in terms of the other variables. For example, you can first push all the variables into a polynomial with integer coefficients.

```
p :: POLY INT
```

$$b^2\,y^4 + \left(-2\,a^2\,b\,x - 2\,b\right)\,y^2 + a^4\,x^2 + 2\,a^2\,x + 1 \tag{8}$$

Polynomial(Integer)

Now pull out the variables of interest.

```
% :: MPOLY([a,b],POLY INT)
```

$$x^2\,a^4 + \left(-2\,x\,y^2\,b + 2\,x\right)\,a^2 + y^4\,b^2 - 2\,y^2\,b + 1 \tag{9}$$

MultivariatePolynomial ([a, b], Polynomial(Integer))

Restriction:

FriCAS does not allow you to create types where **MultivariatePolynomial** is contained in the coefficient type of **Polynomial**. Therefore, `MPOLY([x,y],POLY INT)` is legal but `POLY MPOLY([x,y],INT)` is not.

Multivariate polynomials may be combined with univariate polynomials to create types with special structures.

```
q : UP(x, FRAC MPOLY([y,z],INT))
```

This is a polynomial in x whose coefficients are quotients of polynomials in y and z.

```
q := (x^2 - x*(z+1)/y +2)^2
```

$$x^4+\frac{-2\,z-2}{y}\,x^3+\frac{4\,y^2+z^2+2\,z+1}{y^2}\,x^2+\frac{-4\,z-4}{y}\,x+4 \tag{11}$$

UnivariatePolynomial(x, Fraction(MultivariatePolynomial([y, z], Integer)))

Use conversions for structural rearrangements. z does not appear in a denominator and so it can be made the main variable.

```
q :: UP(z, FRAC MPOLY([x,y],INT))
```

$$\frac{x^2}{y^2}\,z^2+\frac{-2\,y\,x^3+2\,x^2-4\,y\,x}{y^2}\,z+\frac{y^2\,x^4-2\,y\,x^3+\left(4\,y^2+1\right)\,x^2-4\,y\,x+4\,y^2}{y^2} \tag{12}$$

UnivariatePolynomial(z, Fraction(MultivariatePolynomial([x, y], Integer)))

Or you can make a multivariate polynomial in x and z whose coefficients are fractions in polynomials in y.

```
q :: MPOLY([x,z], FRAC UP(y,INT))
```

$$x^4+\left(-\frac{2}{y}\,z-\frac{2}{y}\right)\,x^3+\left(\frac{1}{y^2}\,z^2+\frac{2}{y^2}\,z+\frac{4\,y^2+1}{y^2}\right)\,x^2+\left(-\frac{4}{y}\,z-\frac{4}{y}\right)\,x+4 \tag{13}$$

MultivariatePolynomial([x, z], Fraction(UnivariatePolynomial(y, Integer)))

A conversion like `q :: MPOLY([x,y], FRAC UP(z,INT))` is not possible in this example because y appears in the denominator of a fraction. As you can see, FriCAS provides extraordinary flexibility in the manipulation and display of expressions via its conversion facility.

For more information on related topics, see '`Polynomial`' on page 531, '`UnivariatePolynomial`' on page 596, and '`DistributedMultivariatePolynomial`' on page 394. Issue the system command `)show MultivariatePolynomial` to display the full list of operations defined by **MultivariatePolynomial**.

9.59 None

The **None** domain is not very useful for interactive work but it is provided nevertheless for completeness of the FriCAS type system. Probably the only place you will ever see it is if you enter an empty list with no type information.

```
[]
```

$$[] \tag{4}$$

List(None)

Such an empty list can be converted into an empty list of any other type.

```
[] :: List Float
```

$$[] \tag{5}$$

List (Float)

If you wish to produce an empty list of a particular type directly, such as **List NonNegativeInteger**, do it this way.

```
[]$List(NonNegativeInteger)
```

$$[] \tag{6}$$

List (NonNegativeInteger)

9.60 Octonion

The Octonions, also called the Cayley-Dickson algebra, defined over a commutative ring are an eight-dimensional non-associative algebra. Their construction from quaternions is similar to the construction of quaternions from complex numbers (see '`Quaternion`' on page 539). As **Octonion** creates an eight-dimensional algebra, you have to give eight components to construct an octonion.

```
oci1 := octon(1,2,3,4,5,6,7,8)
```

$$1 + 2\,i + 3\,j + 4\,k + 5\,E + 6\,I + 7\,J + 8\,K \tag{4}$$

Octonion(Integer)

```
oci2 := octon(7,2,3,-4,5,6,-7,0)
```

$$7 + 2\,i + 3\,j - 4\,k + 5\,E + 6\,I - 7\,J \tag{5}$$

Octonion(Integer)

Or you can use two quaternions to create an octonion.

```
oci3 := octon(quatern(-7,-12,3,-10), quatern(5,6,9,0))
```

$$-7 - 12\,i + 3\,j - 10\,k + 5\,E + 6\,I + 9\,J \tag{6}$$

Octonion(Integer)

You can easily demonstrate the non-associativity of multiplication.

```
(oci1 * oci2) * oci3 - oci1 * (oci2 * oci3)
```

$$2696\,i - 2928\,j - 4072\,k + 16\,E - 1192\,I + 832\,J + 2616\,K \tag{7}$$

Octonion(Integer)

As with the quaternions, we have a real part, the imaginary parts `i`, `j`, `k`, and four additional imaginary parts `E`, `I`, `J` and `K`. These parts correspond to the canonical basis `(1,i,j,k,E,I,J,K)`. For each

basis element there is a component operation to extract the coefficient of the basis element for a given octonion.

```
[real oci1, imagi oci1, imagj oci1, imagk oci1, imagE oci1, imagI oci1, imagJ oci1, _
    imagK oci1]
```

$$[1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8] \tag{8}$$

List (PositiveInteger)

A basis with respect to the quaternions is given by `(1,E)`. However, you might ask, what then are the commuting rules? To answer this, we create some generic elements. We do this in FriCAS by simply changing the ground ring from **Integer** to **Polynomial Integer**.

```
q : Quaternion Polynomial Integer := quatern(q1, qi, qj, qk)
```

$$q1 + qi\,i + qj\,j + qk\,k \tag{9}$$

Quaternion(Polynomial(Integer))

```
E : Octonion Polynomial Integer:= octon(0,0,0,0,1,0,0,0)
```

$$E \tag{10}$$

Octonion(Polynomial(Integer))

Note that quaternions are automatically converted to octonions in the obvious way.

```
q * E
```

$$q1\,E + qi\,I + qj\,J + qk\,K \tag{11}$$

Octonion(Polynomial(Integer))

```
E * q
```

$$q1\,E - qi\,I - qj\,J - qk\,K \tag{12}$$

Octonion(Polynomial(Integer))

```
q * 1$(Octonion Polynomial Integer)
```

$$q1 + qi\,i + qj\,j + qk\,k \tag{13}$$

Octonion(Polynomial(Integer))

```
1$(Octonion Polynomial Integer) * q
```

$$q1 + qi\,i + qj\,j + qk\,k \tag{14}$$

Octonion(Polynomial(Integer))

Finally, we check that the norm, defined as the sum of the squares of the coefficients, is a multiplicative map.

```
o : Octonion Polynomial Integer := octon(o1, oi, oj, ok, oE, oI, oJ, oK)
```

$$o1 + oi\,i + oj\,j + ok\,k + oE\,E + oI\,I + oJ\,J + oK\,K \tag{15}$$

Octonion(Polynomial(Integer))

```
norm o
```

$$ok^2 + oj^2 + oi^2 + oK^2 + oJ^2 + oI^2 + oE^2 + o1^2 \tag{16}$$

Polynomial(Integer)

```
p : Octonion Polynomial Integer := octon(p1, pi, pj, pk, pE, pI, pJ, pK)
```

$$p1 + pi\,i + pj\,j + pk\,k + pE\,E + pI\,I + pJ\,J + pK\,K \tag{17}$$

Octonion(Polynomial(Integer))

Since the result is 0, the norm is multiplicative.

```
norm(o*p)-norm(o)*norm(p)
```

$$0 \tag{18}$$

Polynomial(Integer)

Issue the system command `)show Octonion` to display the full list of operations defined by **Octonion**.

9.61 OneDimensionalArray

The **OneDimensionalArray** domain is used for storing data in a one-dimensional indexed data structure. Such an array is a homogeneous data structure in that all the entries of the array must belong to the same FriCAS domain. Each array has a fixed length specified by the user and arrays are not extensible. The indexing of one-dimensional arrays is one-based. This means that the "first" element of an array is given the index 1. See also '`Vector`' on page 603 and '`FlexibleArray`' on page 416. To create a one-dimensional array, apply the operation oneDimensionalArray to a list.

```
oneDimensionalArray [i^2 for i in 1..10]
```

$$[1,\ 4,\ 9,\ 16,\ 25,\ 36,\ 49,\ 64,\ 81,\ 100] \tag{4}$$

OneDimensionalArray(PositiveInteger)

Another approach is to first create a, a one-dimensional array of 10 0's. **OneDimensionalArray** has the convenient abbreviation **ARRAY1**.

```
a : ARRAY1 INT := new(10,0)
```

$$[0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0] \tag{5}$$

OneDimensionalArray(Integer)

Set each ith element to i, then display the result.

```
for i in 1..10 repeat a.i := i; a
```

$$[1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10] \tag{6}$$

OneDimensionalArray(Integer)

Square each element by mapping the function $i \mapsto i^2$ onto each element.

```
map!(i +-> i ^ 2,a); a
```

$$[1,\ 4,\ 9,\ 16,\ 25,\ 36,\ 49,\ 64,\ 81,\ 100] \tag{7}$$

OneDimensionalArray(Integer)

Reverse the elements in place.

```
reverse! a
```

$$[100,\ 81,\ 64,\ 49,\ 36,\ 25,\ 16,\ 9,\ 4,\ 1] \tag{8}$$

OneDimensionalArray(Integer)

Swap the 4th and 5th element.

```
swap!(a,4,5); a
```

$$[100,\ 81,\ 64,\ 36,\ 49,\ 25,\ 16,\ 9,\ 4,\ 1] \tag{9}$$

OneDimensionalArray(Integer)

Sort the elements in place.

```
sort! a
```

$$[1,\ 4,\ 9,\ 16,\ 25,\ 36,\ 49,\ 64,\ 81,\ 100] \tag{10}$$

OneDimensionalArray(Integer)

Create a new one-dimensional array `b` containing the last 5 elements of `a`.

```
b := a(6..10)
```

$$[36,\ 49,\ 64,\ 81,\ 100] \tag{11}$$

OneDimensionalArray(Integer)

Replace the first 5 elements of `a` with those of `b`.

```
copyInto!(a,b,1)
```

$$[36,\ 49,\ 64,\ 81,\ 100,\ 36,\ 49,\ 64,\ 81,\ 100] \tag{12}$$

OneDimensionalArray(Integer)

9.62 Operator

Given any ring R, the ring of the **Integer**-linear operators over R is called **Operator(R)**. To create an operator over R, first create a basic operator using the operation operator, and then convert it to **Operator(R)** for the R you want. We choose R to be the two by two matrices over the integers.

```
R := SQMATRIX(2, INT)
```

$$\text{SquareMatrix}(2, Integer) \tag{4}$$

Type

Create the operator tilde on R.

```
t := operator("tilde") :: OP(R)
```

$$tilde \tag{5}$$

Operator(SquareMatrix(2, Integer))

Since **Operator** is unexposed we must either package-call operations from it, or expose it explicitly. For convenience we will do the latter. Expose **Operator**.

```
)set expose add constructor Operator
```

To attach an evaluation function (from R to R) to an operator over R, use evaluate(op, f) where op is an operator over R and f is a function $R \to R$. This needs to be done only once when the operator is defined. Note that f must be **Integer**-linear (that is, f(ax+y) = a f(x) + f(y) for any integer a, and any x and y in R). We now attach the transpose map to the above operator t.

```
evaluate(t, m +-> transpose m)
```

$$tilde \tag{6}$$

Operator(SquareMatrix(2, Integer))

Operators can be manipulated formally as in any ring: + is the pointwise addition and * is composition. Any element x of R can be converted to an operator op_x over R, and the evaluation function of op_x is left-multiplication by x. Multiplying on the left by this matrix swaps the two rows.

```
s := matrix([[0, 1], [1, 0]])$R::OP(R)
```

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{7}$$

Operator(SquareMatrix(2, Integer))

Can you guess what is the action of the following operator?

```
rho := t * s
```

$$tilde \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{8}$$

Operator(SquareMatrix(2, Integer))

Hint: applying `rho` four times gives the identity, so `rho^4-1` should return 0 when applied to any two by two matrix.

```
z := rho^4 - 1
```

$$-1+tilde\begin{bmatrix}0&1\\1&0\end{bmatrix}tilde\begin{bmatrix}0&1\\1&0\end{bmatrix}tilde\begin{bmatrix}0&1\\1&0\end{bmatrix}tilde\begin{bmatrix}0&1\\1&0\end{bmatrix} \tag{9}$$

Operator(SquareMatrix(2, Integer))

Now check with this matrix.

```
m:R := matrix [[1, 2], [3, 4]]
```

$$\begin{bmatrix}1&2\\3&4\end{bmatrix} \tag{10}$$

SquareMatrix(2, Integer)

```
z m
```

$$\begin{bmatrix}0&0\\0&0\end{bmatrix} \tag{11}$$

SquareMatrix(2, Integer)

As you have probably guessed by now, `rho` acts on matrices by rotating the elements clockwise.

```
rho m
```

$$\begin{bmatrix}3&1\\4&2\end{bmatrix} \tag{12}$$

SquareMatrix(2, Integer)

```
rho rho m
```

$$\begin{bmatrix}4&3\\2&1\end{bmatrix} \tag{13}$$

SquareMatrix(2, Integer)

```
(rho^3) m
```

$$\begin{bmatrix}2&4\\1&3\end{bmatrix} \tag{14}$$

SquareMatrix(2, Integer)

Do the swapping of rows and transposition commute? We can check by computing their bracket.

```
b := t * s - s * t
```

$$-\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} tilde + tilde \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \tag{15}$$

Operator(SquareMatrix(2, Integer))

Now apply it to m.

```
b m
```

$$\begin{bmatrix}1 & -3\\ 3 & -1\end{bmatrix} \tag{16}$$

SquareMatrix(2, Integer)

Next we demonstrate how to define a differential operator on a polynomial ring. This is the recursive definition of the n-th Legendre polynomial.

```
L n ==
  n = 0 => 1
  n = 1 => x
  (2*n-1)/n * x * L(n-1) - (n-1)/n * L(n-2)
```

Create the differential operator $\frac{d}{dx}$ on polynomials in x over the rational numbers.

```
dx := operator("D") :: OP(POLY FRAC INT)
```

$$D \tag{18}$$

Operator(Polynomial(Fraction(Integer)))

Now attach the map to it.

```
evaluate(dx, p +-> D(p, 'x))
```

$$D \tag{19}$$

Operator(Polynomial(Fraction(Integer)))

This is the differential equation satisfied by the n-th Legendre polynomial.

```
E n == (1 - x^2) * dx^2 - 2 * x * dx + n*(n+1)
```

Now we verify this for n = 15. Here is the polynomial.

```
L 15
```

```
   Compiling function L with type Integer -> Polynomial(Fraction(Integer))
   Compiling function L as a recurrence relation.
```

$$\frac{9694845}{2048}x^{15} - \frac{35102025}{2048}x^{13} + \frac{50702925}{2048}x^{11} - \frac{37182145}{2048}x^{9} + \frac{14549535}{2048}x^{7} - \frac{2909907}{2048}x^{5} + \frac{255255}{2048}x^{3} - \frac{6435}{2048}x \tag{21}$$

Polynomial(Fraction(Integer))

Here is the operator.

```
E 15
```

```
   Compiling function E with type PositiveInteger -> Operator(Polynomial(Fraction(
      Integer)))
```

$$240 - 2\,x\,D + \left(-x^2 + 1\right)\,D^2 \tag{22}$$

Operator(Polynomial(Fraction(Integer)))

Here is the evaluation.

```
(E 15)(L 15)
```

$$0 \tag{23}$$

Polynomial(Fraction(Integer))

9.63 OrderedVariableList

The domain **OrderedVariableList** provides symbols which are restricted to a particular list and have a definite ordering. Those two features are specified by a **List Symbol** object that is the argument to the domain. This is a sample ordering of three symbols.

```
ls:List Symbol:=['x,'a,'z]
```

$$[x,\ a,\ z] \tag{4}$$

List(Symbol)

Let's build the domain

```
Z:=OVAR ls
```

$$\text{OrderedVariableList}([x,\ a,\ z]) \tag{5}$$

Type

How many variables does it have?

```
size()$Z
```

$$3 \tag{6}$$

NonNegativeInteger

They are (in the imposed order)

```
lv:=[index(i::PI)$Z for i in 1..size()$Z]
```

```
   Compiling function G9 with type Integer -> Boolean
   Compiling function G11 with type NonNegativeInteger -> Boolean
```

$$[x,\ a,\ z] \tag{7}$$

List(OrderedVariableList([x, a, z]))

Check that the ordering is right

```
sorted?(>,lv)
```

$$\texttt{true} \tag{8}$$

Boolean

9.64 OrderlyDifferentialPolynomial

Many systems of differential equations may be transformed to equivalent systems of ordinary differential equations where the equations are expressed polynomially in terms of the unknown functions. In FriCAS, the domain constructors **OrderlyDifferentialPolynomial** (abbreviated **ODPOL**) and **SequentialDifferentialPolynomial** (abbreviation **SDPOL**) implement two domains of ordinary differential polynomials over any differential ring. In the simplest case, this differential ring is usually either the ring of integers, or the field of rational numbers. However, FriCAS can handle ordinary differential polynomials over a field of rational functions in a single indeterminate.

The two domains **ODPOL** and **SDPOL** are almost identical, the only difference being the choice of a different ranking, which is an ordering of the derivatives of the indeterminates. The first domain uses an orderly ranking, that is, derivatives of higher order are ranked higher, and derivatives of the same order are ranked alphabetically. The second domain uses a sequential ranking, where derivatives are ordered first alphabetically by the differential indeterminates, and then by order. A more general domain constructor, **DifferentialSparseMultivariatePolynomial** (abbreviation **DSMP**) allows both a user-provided list of differential indeterminates as well as a user-defined ranking. We shall illustrate **ODPOL(FRAC INT)**, which constructs a domain of ordinary differential polynomials in an arbitrary number of differential indeterminates with rational numbers as coefficients.

```
dpol:= ODPOL(FRAC INT)
```

$$\text{OrderlyDifferentialPolynomial}(\text{Fraction}(\mathit{Integer})) \tag{4}$$

Type

A differential indeterminate `w` may be viewed as an infinite sequence of algebraic indeterminates, which are the derivatives of `w`. To facilitate referencing these, FriCAS provides the operation makeVariable to convert an element of type **Symbol** to a map from the natural numbers to the differential polynomial ring.

```
w := makeVariable('w)$dpol
```

$$\text{theMap}(\text{makeVariable}) \tag{5}$$

(NonNegativeInteger → OrderlyDifferentialPolynomial (Fraction (Integer)))

```
z := makeVariable('z)$dpol
```

$$\text{theMap}(\text{makeVariable}) \tag{6}$$

(NonNegativeInteger → OrderlyDifferentialPolynomial (Fraction (Integer)))

The fifth derivative of `w` can be obtained by applying the map `w` to the number `5`. Note that the order of differentiation is given as a subscript (except when the order is 0).

```
w.5
```

$$w_5 \tag{7}$$

OrderlyDifferentialPolynomial (Fraction (Integer))

```
w 0
```

$$w \tag{8}$$

OrderlyDifferentialPolynomial (Fraction (Integer))

The first five derivatives of z can be generated by a list.

```
[z.i for i in 1..5]
```

$$[z_1,\ z_2,\ z_3,\ z_4,\ z_5] \tag{9}$$

List (OrderlyDifferentialPolynomial (Fraction (Integer)))

The usual arithmetic can be used to form a differential polynomial from the derivatives.

```
f:= w.4 - w.1 * w.1 * z.3
```

$$w_4 - (w_1)^2\, z_3 \tag{10}$$

OrderlyDifferentialPolynomial (Fraction (Integer))

```
g:=(z.1)^3 * (z.2)^2 - w.2
```

$$(z_1)^3\,(z_2)^2 - w_2 \tag{11}$$

OrderlyDifferentialPolynomial (Fraction (Integer))

The operation D computes the derivative of any differential polynomial.

```
D(f)
```

$$w_5 - (w_1)^2\, z_4 - 2\, w_1\, w_2\, z_3 \tag{12}$$

OrderlyDifferentialPolynomial (Fraction (Integer))

The same operation can compute higher derivatives, like the fourth derivative.

```
D(f,4)
```

$$\begin{aligned} &w_8 - (w_1)^2\, z_7 - 8\, w_1\, w_2\, z_6 + \left(-12\, w_1\, w_3 - 12\,(w_2)^2\right)\, z_5 - 2\, w_1\, z_3\, w_5 \\ &+ \left(-8\, w_1\, w_4 - 24\, w_2\, w_3\right)\, z_4 - 8\, w_2\, z_3\, w_4 - 6\,(w_3)^2\, z_3 \end{aligned} \tag{13}$$

OrderlyDifferentialPolynomial (Fraction (Integer))

The operation makeVariable creates a map to facilitate referencing the derivatives of f, similar to the map w.

```
df:=makeVariable(f)$dpol
```

$$\text{theMap}(\text{makeVariable}) \tag{14}$$

(NonNegativeInteger → OrderlyDifferentialPolynomial (Fraction (Integer)))

The fourth derivative of f may be referenced easily.

```
df.4
```

$$\begin{aligned} & w_8 - (w_1)^2\, z_7 - 8\, w_1\, w_2\, z_6 + \left(-12\, w_1\, w_3 - 12\, (w_2)^2\right)\, z_5 - 2\, w_1\, z_3\, w_5 \\ & + \left(-8\, w_1\, w_4 - 24\, w_2\, w_3\right)\, z_4 - 8\, w_2\, z_3\, w_4 - 6\, (w_3)^2\, z_3 \end{aligned} \tag{15}$$

OrderlyDifferentialPolynomial (Fraction (Integer))

The operation order returns the order of a differential polynomial, or the order in a specified differential indeterminate.

```
order(g)
```

$$2 \tag{16}$$

PositiveInteger

```
order(g, 'w)
```

$$2 \tag{17}$$

PositiveInteger

The operation differentialVariables returns a list of differential indeterminates occurring in a differential polynomial.

```
differentialVariables(g)
```

$$[z,\ w] \tag{18}$$

List (Symbol)

The operation degree returns the degree, or the degree in the differential indeterminate specified.

```
degree(g)
```

$$(z_2)^2\, (z_1)^3 \tag{19}$$

IndexedExponents(OrderlyDifferentialVariable (Symbol))

```
degree(g, 'w)
```

$$1 \tag{20}$$

PositiveInteger

The operation weights returns a list of weights of differential monomials appearing in a differential polynomial, or a list of weights in a specified differential indeterminate.

```
weights(g)
```

$$[7,\ 2] \tag{21}$$

List (NonNegativeInteger)

```
weights(g,'w)
```

$$[2] \tag{22}$$

List (NonNegativeInteger)

The operation weight returns the maximum weight of all differential monomials appearing in the differential polynomial.

```
weight(g)
```

$$7 \tag{23}$$

PositiveInteger

A differential polynomial is *isobaric* if the weights of all differential monomials appearing in it are equal.

```
isobaric?(g)
```

$$\texttt{false} \tag{24}$$

Boolean

To substitute *differentially*, use eval. Note that we must coerce `'w` to **Symbol**, since in **ODPOL**, differential indeterminates belong to the domain **Symbol**. Compare this result to the next, which substitutes *algebraically* (no substitution is done since `w.0` does not appear in `g`).

```
eval(g,['w::Symbol],[f])
```

$$-w_6 + (w_1)^2\, z_5 + 4\, w_1\, w_2\, z_4 + \left(2\, w_1\, w_3 + 2\,(w_2)^2\right)\, z_3 + (z_1)^3\,(z_2)^2 \tag{25}$$

OrderlyDifferentialPolynomial (Fraction (Integer))

```
eval(g,variables(w.0),[f])
```

$$(z_1)^3\,(z_2)^2 - w_2 \tag{26}$$

OrderlyDifferentialPolynomial (Fraction (Integer))

Since **OrderlyDifferentialPolynomial** belongs to **PolynomialCategory**, all the operations defined in the latter category, or in packages for the latter category, are available.

```
monomials(g)
```

$$\left[(z_1)^3\,(z_2)^2,\ -w_2\right] \tag{27}$$

List (OrderlyDifferentialPolynomial (Fraction (Integer)))

```
variables(g)
```

$$[z_2,\ w_2,\ z_1] \tag{28}$$

List (OrderlyDifferentialVariable (Symbol))

```
gcd(f,g)
```

$$1 \tag{29}$$

OrderlyDifferentialPolynomial (Fraction (Integer))

```
groebner([f,g])
```

$$\left[w_4 - (w_1)^2\, z_3,\ (z_1)^3\, (z_2)^2 - w_2\right] \tag{30}$$

List (OrderlyDifferentialPolynomial (Fraction (Integer)))

The next three operations are essential for elimination procedures in differential polynomial rings. The operation leader returns the leader of a differential polynomial, which is the highest ranked derivative of the differential indeterminates that occurs.

```
lg:=leader(g)
```

$$z_2 \tag{31}$$

OrderlyDifferentialVariable (Symbol)

The operation separant returns the separant of a differential polynomial, which is the partial derivative with respect to the leader.

```
sg:=separant(g)
```

$$2\,(z_1)^3\, z_2 \tag{32}$$

OrderlyDifferentialPolynomial (Fraction (Integer))

The operation initial returns the initial, which is the leading coefficient when the given differential polynomial is expressed as a polynomial in the leader.

```
ig:=initial(g)
```

$$(z_1)^3 \tag{33}$$

OrderlyDifferentialPolynomial (Fraction (Integer))

Using these three operations, it is possible to reduce `f` modulo the differential ideal generated by `g`. The general scheme is to first reduce the order, then reduce the degree in the leader. First, eliminate `z.3` using the derivative of `g`.

```
g1 := D g
```

$$2\,(z_1)^3\, z_2\, z_3 - w_3 + 3\,(z_1)^2\, (z_2)^3 \tag{34}$$

OrderlyDifferentialPolynomial (Fraction (Integer))

Find its leader.

```
lg1:= leader g1
```

$$z_3 \tag{35}$$

OrderlyDifferentialVariable (Symbol)

Differentiate `f` partially with respect to this leader.

```
pdf:=D(f, lg1)
```

$$-(w_1)^2 \tag{36}$$

OrderlyDifferentialPolynomial (Fraction (Integer))

Compute the partial remainder of `f` with respect to `g`.

```
prf:=sg * f- pdf * g1
```

$$2(z_1)^3 z_2 w_4 - (w_1)^2 w_3 + 3(w_1)^2 (z_1)^2 (z_2)^3 \tag{37}$$

OrderlyDifferentialPolynomial (Fraction (Integer))

Note that high powers of `lg` still appear in `prf`. Compute the leading coefficient of `prf` as a polynomial in the leader of `g`.

```
lcf:=leadingCoefficient univariate(prf, lg)
```

$$3(w_1)^2 (z_1)^2 \tag{38}$$

OrderlyDifferentialPolynomial (Fraction (Integer))

Finally, continue eliminating the high powers of `lg` appearing in `prf` to obtain the (pseudo) remainder of `f` modulo `g` and its derivatives.

```
ig * prf - lcf * g * lg
```

$$2(z_1)^6 z_2 w_4 - (w_1)^2 (z_1)^3 w_3 + 3(w_1)^2 (z_1)^2 w_2 z_2 \tag{39}$$

OrderlyDifferentialPolynomial (Fraction (Integer))

Issue the system command `)show OrderlyDifferentialPolyomial` to display the full list of operations defined by **OrderlyDifferentialPolyomial**. Issue the system command `)show SequentialDifferentialPolynomial` to display the full list of operations defined by **SequentialDifferentialPolynomial**.

9.65 PartialFraction

A *partial fraction* is a decomposition of a quotient into a sum of quotients where the denominators of the summands are powers of primes.[5] For example, the rational number `1/6` is decomposed into `1/2 -1/3`.

[5]Most people first encounter partial fractions when they are learning integral calculus. For a technical discussion of partial fractions, see, for example, Lang's *Algebra*.

You can compute partial fractions of quotients of objects from domains belonging to the category **EuclideanDomain**. For example, **Integer**, **Complex Integer**, and **UnivariatePolynomial(x, Fraction Integer)** all belong to **EuclideanDomain**. In the examples following, we demonstrate how to decompose quotients of each of these kinds of object into partial fractions. Issue the system command `)show PartialFraction` to display the full list of operations defined by **PartialFraction**.

It is necessary that we know how to factor the denominator when we want to compute a partial fraction. Although the interpreter can often do this automatically, it may be necessary for you to include a call to factor. In these examples, it is not necessary to factor the denominators explicitly. The main operation for computing partial fractions is called partialFraction and we use this to compute a decomposition of `1 / 10!`. The first argument to partialFraction is the numerator of the quotient and the second argument is the factored denominator.

```
partialFraction(1,factorial 10)
```

$$-\frac{97}{2^8}+\frac{58}{3^4}+\frac{13}{5^2}-\frac{6}{7} \tag{4}$$

PartialFraction (Integer)

Since the denominators are powers of primes, it may be possible to expand the numerators further with respect to those primes. Use the operation padicFraction to do this.

```
f := padicFraction(%)
```

$$-\frac{1}{2^2}-\frac{1}{2^3}-\frac{1}{2^8}+\frac{2}{3}+\frac{1}{3^3}+\frac{1}{3^4}+\frac{2}{5}+\frac{3}{5^2}-\frac{6}{7} \tag{5}$$

PartialFraction (Integer)

The operation compactFraction returns an expanded fraction into the usual form. The compacted version is used internally for computational efficiency.

```
compactFraction(f)
```

$$-\frac{97}{2^8}+\frac{58}{3^4}+\frac{13}{5^2}-\frac{6}{7} \tag{6}$$

PartialFraction (Integer)

You can add, subtract, multiply and divide partial fractions. In addition, you can extract the parts of the decomposition. numberOfFractionalTerms computes the number of terms in the fractional part. This does not include the whole part of the fraction, which you get by calling wholePart. In this example, the whole part is just 0.

```
numberOfFractionalTerms(f)
```

$$9 \tag{7}$$

PositiveInteger

The operation fractionalTerms returns the individual terms in the decomposition. Notice that the object returned is a list of **Record(num : R, den : Factored(R))**, you can extract the numerator and denominator from this object.

```
fractionalTerms(f).3
```

$$[num = -1,\ d_fact = 2,\ d_exp = 8] \tag{8}$$

Record(num: Integer, d_fact: Integer, d_exp: NonNegativeInteger)

Given two gaussian integers (see '**Complex**' on page 372), you can decompose their quotient into a partial fraction.

```
partialFraction(1,- 13 + 14 * %i)
```

$$-\frac{1}{1+2\,i}+\frac{4}{3+8\,i} \tag{9}$$

PartialFraction (Complex(Integer))

To convert back to a quotient, simply use a conversion.

```
% :: Fraction Complex Integer
```

$$-\frac{i}{14+13\,i} \tag{10}$$

Fraction (Complex(Integer))

To conclude this section, we compute the decomposition of

$$\frac{1}{(x+1)(x+2)^2(x+3)^3(x+4)^4}$$

The polynomials in this object have type **UnivariatePolynomial(x, Fraction Integer)**. We use the primeFactor operation (see '**Factored**' on page 405) to create the denominator in factored form directly.

```
u : FR UP(x, FRAC INT) := reduce(*,[primeFactor(x+i,i) for i in 1..4])
```

$$(x+1)\ (x+2)^2\ (x+3)^3\ (x+4)^4 \tag{11}$$

Factored(UnivariatePolynomial (x, Fraction (Integer)))

These are the compact and expanded partial fractions for the quotient.

```
partialFraction(1,u)
```

$$\frac{\frac{1}{648}}{x+1}+\frac{\frac{1}{4}\,x+\frac{7}{16}}{(x+2)^2}+\frac{-\frac{17}{8}\,x^2-12\,x-\frac{139}{8}}{(x+3)^3}+\frac{\frac{607}{324}\,x^3+\frac{10115}{432}\,x^2+\frac{391}{4}\,x+\frac{44179}{324}}{(x+4)^4} \tag{12}$$

PartialFraction (UnivariatePolynomial (x, Fraction (Integer)))

```
padicFraction %
```

$$\frac{\frac{607}{324}}{x+4}+\frac{\frac{403}{432}}{(x+4)^2}+\frac{\frac{13}{36}}{(x+4)^3}+\frac{\frac{1}{12}}{(x+4)^4}-\frac{\frac{17}{8}}{x+3}+\frac{\frac{3}{4}}{(x+3)^2}-\frac{\frac{1}{2}}{(x+3)^3}+\frac{\frac{1}{4}}{x+2}-\frac{\frac{1}{16}}{(x+2)^2}+\frac{\frac{1}{648}}{x+1} \tag{13}$$

PartialFraction (UnivariatePolynomial (x, Fraction (Integer)))

Also see '`FullPartialFractionExpansion`' on page 429 for examples of factor-free conversion of quotients to full partial fractions.

9.66 Permanent

The package **Permanent** provides the function permanent for square matrices. The permanent of a square matrix can be computed in the same way as the determinant by expansion of minors except that for the permanent the sign for each element is `1`, rather than being `1` if the row plus column indices is positive and `-1` otherwise. This function is much more difficult to compute efficiently than the determinant. An example of the use of permanent is the calculation of the n^{th} derangement number, defined to be the number of different possibilities for `n` couples to dance but never with their own spouse. Consider an `n` by `n` matrix with entries `0` on the diagonal and `1` elsewhere. Think of the rows as one-half of each couple (for example, the males) and the columns as the other half. The permanent of such a matrix gives the desired derangement number.

```
kn n ==
  r : MATRIX INT := new(n,n,1)
  for i in 1..n repeat
      r(i, i) := 0
  r
```

Here are some derangement numbers, which you see grow quite fast.

```
permanent(kn(5) :: SQMATRIX(5,INT))
```

```
   Compiling function kn with type PositiveInteger -> Matrix(Integer)
```

$$44 \tag{5}$$

PositiveInteger

```
[permanent(kn(n) :: SQMATRIX(n,INT)) for n in 1..13]
```

```
   Cannot compile conversion for types involving local variables. In particular,
      could not compile the expression involving :: SQMATRIX(n,INT)
   FriCAS will attempt to step through and interpret the code.
```

$$[0,\ 1,\ 2,\ 9,\ 44,\ 265,\ 1854,\ 14833,\ 133496,\ 1334961,\ 14684570,\ 176214841,\ 2290792932] \tag{6}$$

List (NonNegativeInteger)

9.67 Polynomial

The domain constructor **Polynomial** (abbreviation: **POLY**) provides polynomials with an arbitrary number of unspecified variables.

It is used to create the default polynomial domains in FriCAS. Here the coefficients are integers.

```
x + 1
```

$$x + 1 \tag{4}$$

Polynomial(Integer)

Here the coefficients have type **Float**.

```
z - 2.3
```

$$z - 2.3 \tag{5}$$

Polynomial(Float)

And here we have a polynomial in two variables with coefficients which have type **Fraction Integer**.

```
y^2 - z + 3/4
```

$$-z + y^2 + \frac{3}{4} \tag{6}$$

Polynomial(Fraction (Integer))

The representation of objects of domains created by **Polynomial** is that of recursive univariate polynomials.[6] This recursive structure is sometimes obvious from the display of a polynomial.

```
y ^2 + x*y + y
```

$$y^2 + (x + 1)\, y \tag{7}$$

Polynomial(Integer)

In this example, you see that the polynomial is stored as a polynomial in `y` with coefficients that are polynomials in `x` with integer coefficients. In fact, you really don't need to worry about the representation unless you are working on an advanced application where it is critical. The polynomial types created from **DistributedMultivariatePolynomial** and **NewDistributedMultivariatePolynomial** (discussed in '`DistributedMultivariatePolynomial`' on page 394) are stored and displayed in a non-recursive manner. You see a "flat" display of the above polynomial by converting to one of those types.

```
% :: DMP([y,x],INT)
```

$$y^2 + y\,x + y \tag{8}$$

DistributedMultivariatePolynomial ([y, x], Integer)

We will demonstrate many of the polynomial facilities by using two polynomials with integer coefficients. By default, the interpreter expands polynomial expressions, even if they are written in a factored format.

```
p := (y-1)^2 * x * z
```

$$\left(x\,y^2 - 2\,x\,y + x\right)\, z \tag{9}$$

[6] The term `univariate` means "one variable." `multivariate` means "possibly more than one variable."

Polynomial(Integer)

See ‘Factored’ on page 405 to see how to create objects in factored form directly.

```
q := (y-1) * x * (z+5)
```

$$(x\,y - x)\,z + 5\,x\,y - 5\,x \tag{10}$$

Polynomial(Integer)

The fully factored form can be recovered by using factor.

```
factor(q)
```

$$x\,(y-1)\,(z+5) \tag{11}$$

Factored(Polynomial(Integer))

This is the same name used for the operation to factor integers. Such reuse of names is called *overloading* and makes it much easier to think of solving problems in general ways. FriCAS facilities for factoring polynomials created with **Polynomial** are currently restricted to the integer and rational number coefficient cases. There are more complete facilities for factoring univariate polynomials: see Section 8.2 on page 260.

The standard arithmetic operations are available for polynomials.

```
p - q^2
```

$$\left(-x^2\,y^2 + 2\,x^2\,y - x^2\right)\,z^2 + \left(\left(-10\,x^2 + x\right)\,y^2 + \left(20\,x^2 - 2\,x\right)\,y - 10\,x^2 + x\right)\,z - 25\,x^2\,y^2 + 50\,x^2\,y - 25\,x^2 \tag{12}$$

Polynomial(Integer)

The operation gcd is used to compute the greatest common divisor of two polynomials.

```
gcd(p,q)
```

$$x\,y - x \tag{13}$$

Polynomial(Integer)

In the case of p and q, the gcd is obvious from their definitions. We factor the gcd to show this relationship better.

```
factor %
```

$$x\,(y-1) \tag{14}$$

Factored(Polynomial(Integer))

The least common multiple is computed by using lcm.

```
lcm(p,q)
```

$$\left(x\,y^2 - 2\,x\,y + x\right)\,z^2 + \left(5\,x\,y^2 - 10\,x\,y + 5\,x\right)\,z \tag{15}$$

Polynomial(Integer)

Use content to compute the greatest common divisor of the coefficients of the polynomial.

```
content p
```

$$1 \tag{16}$$

PositiveInteger

Many of the operations on polynomials require you to specify a variable. For example, resultant requires you to give the variable in which the polynomials should be expressed. This computes the resultant of the values of p and q, considering them as polynomials in the variable z. They do not share a root when thought of as polynomials in z.

```
resultant(p,q,z)
```

$$5\,x^2\,y^3 - 15\,x^2\,y^2 + 15\,x^2\,y - 5\,x^2 \tag{17}$$

Polynomial(Integer)

This value is 0 because as polynomials in x the polynomials have a common root.

```
resultant(p,q,x)
```

$$0 \tag{18}$$

Polynomial(Integer)

The data type used for the variables created by **Polynomial** is **Symbol**. As mentioned above, the representation used by **Polynomial** is recursive and so there is a main variable for nonconstant polynomials. The operation mainVariable returns this variable. The return type is actually a union of **Symbol** and "failed".

```
mainVariable p
```

$$z \tag{19}$$

Union(Symbol, ...)

The latter branch of the union is used if the polynomial has no variables, that is, is a constant.

```
mainVariable(1 :: POLY INT)
```

$$\text{"failed"} \tag{20}$$

Union(" failed ", ...)

You can also use the predicate ground? to test whether a polynomial is in fact a member of its ground ring.

```
ground? p
```

$$\text{false} \tag{21}$$

Boolean

```
ground?(1 :: POLY INT)
```

$$\texttt{true} \tag{22}$$

Boolean

The complete list of variables actually used in a particular polynomial is returned by variables. For constant polynomials, this list is empty.

```
variables p
```

$$[z,\ y,\ x] \tag{23}$$

List (Symbol)

The degree operation returns the degree of a polynomial in a specific variable.

```
degree(p,x)
```

$$1 \tag{24}$$

PositiveInteger

```
degree(p,y)
```

$$2 \tag{25}$$

PositiveInteger

```
degree(p,z)
```

$$1 \tag{26}$$

PositiveInteger

If you give a list of variables for the second argument, a list of the degrees in those variables is returned.

```
degree(p,[x,y,z])
```

$$[1,\ 2,\ 1] \tag{27}$$

List (NonNegativeInteger)

The minimum degree of a variable in a polynomial is computed using minimumDegree.

```
minimumDegree(p,z)
```

$$1 \tag{28}$$

PositiveInteger

The total degree of a polynomial is returned by totalDegree.

```
totalDegree p
```

$$4 \tag{29}$$

PositiveInteger

It is often convenient to think of a polynomial as a leading monomial plus the remaining terms.

```
leadingMonomial p
```

$$x\,y^2\,z \tag{30}$$

Polynomial(Integer)

The reductum operation returns a polynomial consisting of the sum of the monomials after the first.

```
reductum p
```

$$(-2\,x\,y + x)\,z \tag{31}$$

Polynomial(Integer)

These have the obvious relationship that the original polynomial is equal to the leading monomial plus the reductum.

```
p - leadingMonomial p - reductum p
```

$$0 \tag{32}$$

Polynomial(Integer)

The value returned by leadingMonomial includes the coefficient of that term. This is extracted by using leadingCoefficient on the original polynomial.

```
leadingCoefficient p
```

$$1 \tag{33}$$

PositiveInteger

The operation eval is used to substitute a value for a variable in a polynomial.

```
p
```

$$\left(x\,y^2 - 2\,x\,y + x\right)\,z \tag{34}$$

Polynomial(Integer)

This value may be another variable, a constant or a polynomial.

```
eval(p,x,w)
```

$$\left(w\,y^2 - 2\,w\,y + w\right)\,z \tag{35}$$

Polynomial(Integer)

```
eval(p,x,1)
```

$$\left(y^2 - 2\,y + 1\right)\,z \tag{36}$$

Polynomial(Integer)

Actually, all the things being substituted are just polynomials, some more trivial than others.

```
eval(p,x,y^2 - 1)
```

$$\left(y^4 - 2\,y^3 + 2\,y - 1\right)\,z \tag{37}$$

Polynomial(Integer)

Derivatives are computed using the D operation.

```
D(p,x)
```

$$\left(y^2 - 2\,y + 1\right)\,z \tag{38}$$

Polynomial(Integer)

The first argument is the polynomial and the second is the variable.

```
D(p,y)
```

$$\left(2\,x\,y - 2\,x\right)\,z \tag{39}$$

Polynomial(Integer)

Even if the polynomial has only one variable, you must specify it.

```
D(p,z)
```

$$x\,y^2 - 2\,x\,y + x \tag{40}$$

Polynomial(Integer)

Integration of polynomials is similar and the integrate operation is used.

Integration requires that the coefficients support division. Consequently, FriCAS converts polynomials over the integers to polynomials over the rational numbers before integrating them.

```
integrate(p,y)
```

$$\left(\frac{1}{3}\,x\,y^3 - x\,y^2 + x\,y\right)\,z \tag{41}$$

Polynomial(Fraction (Integer))

It is not possible, in general, to divide two polynomials. In our example using polynomials over the integers, the operation monicDivide divides a polynomial by a monic polynomial (that is, a polynomial with leading coefficient equal to 1). The result is a record of the quotient and remainder of the division. You must specify the variable in which to express the polynomial.

```
qr := monicDivide(p,x+1,x)
```

$$[quotient = \left(y^2 - 2\,y + 1\right)\,z,\ remainder = \left(-y^2 + 2\,y - 1\right)\,z] \tag{42}$$

Record(quotient: Polynomial(Integer), remainder: Polynomial(Integer))

The selectors of the components of the record are `quotient` and `remainder`. Issue this to extract the remainder.

```
qr.remainder
```

$$\left(-y^2 + 2\,y - 1\right)\,z \tag{43}$$

Polynomial(Integer)

Now that we can extract the components, we can demonstrate the relationship among them and the arguments to our original expression `qr := monicDivide(p,x+1,x)`.

```
p - ((x+1) * qr.quotient + qr.remainder)
```

$$0 \tag{44}$$

Polynomial(Integer)

If the `/` operator is used with polynomials, a fraction object is created. In this example, the result is an object of type **Fraction Polynomial Integer**.

```
p/q
```

$$\frac{(y-1)\,z}{z+5} \tag{45}$$

Fraction(Polynomial(Integer))

If you use rational numbers as polynomial coefficients, the resulting object is of type **Polynomial Fraction Integer**.

```
(2/3) * x^2 - y + 4/5
```

$$-y + \frac{2}{3}\,x^2 + \frac{4}{5} \tag{46}$$

Polynomial(Fraction(Integer))

This can be converted to a fraction of polynomials and back again, if required.

```
% :: FRAC POLY INT
```

$$\frac{-15\,y + 10\,x^2 + 12}{15} \tag{47}$$

Fraction(Polynomial(Integer))

```
% :: POLY FRAC INT
```

$$-y+\frac{2}{3}x^2+\frac{4}{5} \tag{48}$$

Polynomial(Fraction(Integer))

To convert the coefficients to floating point, map the numeric operation on the coefficients of the polynomial.

```
map(numeric,%)
```

$$-1.0\,y+0.66666666666666666667\,x^2+0.8 \tag{49}$$

Polynomial(Float)

For more information on related topics, see 'UnivariatePolynomial' on page 596, 'MultivariatePolynomial' on page 512, and 'DistributedMultivariatePolynomial' on page 394. You can also issue the system command)show Polynomial to display the full list of operations defined by **Polynomial**.

9.68 Quaternion

The domain constructor **Quaternion** implements Hamilton quaternions over commutative rings. For information on related topics, see 'GeneralQuaternion' on page 432, 'Complex' on page 372 and 'Octonion' on page 515. You can also issue the system command)show Quaternion to display the full list of operations defined by **Quaternion**.

The basic operation for creating quaternions is quatern. This is a quaternion over the rational numbers.

```
q := quatern(2/11,-8,3/4,1)
```

$$\frac{2}{11}-8\,i+\frac{3}{4}\,j+k \tag{4}$$

Quaternion(Fraction(Integer))

The four arguments are the real part, the i imaginary part, the j imaginary part, and the k imaginary part, respectively.

```
[real q, imagI q, imagJ q, imagK q]
```

$$\left[\frac{2}{11},\ -8,\ \frac{3}{4},\ 1\right] \tag{5}$$

List(Fraction(Integer))

Because q is over the rationals (and nonzero), you can invert it.

```
inv q
```

$$\frac{352}{126993}+\frac{15488}{126993}\,i-\frac{484}{42331}\,j-\frac{1936}{126993}\,k \tag{6}$$

Quaternion(Fraction(Integer))

The usual arithmetic (ring) operations are available

```
q^6
```

$$-\frac{2029490709319345}{7256313856}-\frac{48251690851}{1288408}\,i+\frac{144755072553}{41229056}\,j+\frac{48251690851}{10307264}\,k \tag{7}$$

Quaternion(Fraction(Integer))

```
r := quatern(-2,3,23/9,-89); q + r
```

$$-\frac{20}{11}-5\,i+\frac{119}{36}\,j-88\,k \tag{8}$$

Quaternion(Fraction(Integer))

In general, multiplication is not commutative.

```
q * r - r * q
```

$$-\frac{2495}{18}\,i-1418\,j-\frac{817}{18}\,k \tag{9}$$

Quaternion(Fraction(Integer))

There are no predefined constants for the imaginary `i`, `j`, and `k` parts, but you can easily define them.

```
i := quatern(0,1,0,0); j := quatern(0,0,1,0); k := quatern(0,0,0,1)
```

$$k \tag{10}$$

Quaternion(Integer)

These satisfy the normal identities.

```
[i*i, j*j, k*k, i*j, j*k, k*i, q*i]
```

$$\left[-1,\ -1,\ -1,\ k,\ i,\ j,\ 8+\frac{2}{11}\,i+j-\frac{3}{4}\,k\right] \tag{11}$$

List(Quaternion(Fraction(Integer)))

The norm is the quaternion times its conjugate.

```
norm q
```

$$\frac{126993}{1936} \tag{12}$$

Fraction(Integer)

```
conjugate q
```

$$\frac{2}{11} + 8\,i - \frac{3}{4}\,j - k \tag{13}$$

Quaternion(Fraction(Integer))

```
q * %
```

$$\frac{126993}{1936} \tag{14}$$

Quaternion(Fraction(Integer))

9.69 RadixExpansion

It is possible to expand numbers in general bases.

Here we expand `111` in base `5`. This means $10^2 + 10^1 + 10^0 = 4 \cdot 5^2 + 2 \cdot 5^1 + 5^0$.

```
111::RadixExpansion(5)
```

$$421 \tag{4}$$

RadixExpansion(5)

You can expand fractions to form repeating expansions.

```
(5/24)::RadixExpansion(2)
```

$$0.001\overline{10} \tag{5}$$

RadixExpansion(2)

```
(5/24)::RadixExpansion(3)
```

$$0.0\overline{12} \tag{6}$$

RadixExpansion(3)

```
(5/24)::RadixExpansion(8)
```

$$0.1\overline{52} \tag{7}$$

RadixExpansion(8)

```
(5/24)::RadixExpansion(10)
```

$$0.208\overline{3} \tag{8}$$

RadixExpansion(10)

For bases from 11 to 36 the letters A through Z are used.

```
(5/24)::RadixExpansion(12)
```

$$0.26 \tag{9}$$

RadixExpansion(12)

```
(5/24)::RadixExpansion(16)
```

$$0.3\overline{5} \tag{10}$$

RadixExpansion(16)

```
(5/24)::RadixExpansion(36)
```

$$0.7\mathtt{I} \tag{11}$$

RadixExpansion(36)

For bases greater than 36, the ragits are separated by blanks.

```
(5/24)::RadixExpansion(38)
```

$$0\ .\ 7\ 34\ 31\ \overline{25\ 12} \tag{12}$$

RadixExpansion(38)

The **RadixExpansion** type provides operations to obtain the individual ragits. Here is a rational number in base 8.

```
a := (76543/210)::RadixExpansion(8)
```

$$554.3\overline{7307} \tag{13}$$

RadixExpansion(8)

The operation wholeRagits returns a list of the ragits for the integral part of the number.

```
w := wholeRagits a
```

$$[5,\ 5,\ 4] \tag{14}$$

List (Integer)

The operations prefixRagits and cycleRagits return lists of the initial and repeating ragits in the fractional part of the number.

```
f0 := prefixRagits a
```

$$[3] \tag{15}$$

List (Integer)

```
f1 := cycleRagits a
```

$$[7,\ 3,\ 0,\ 7] \tag{16}$$

List (Integer)

You can construct any radix expansion by giving the whole, prefix and cycle parts. The declaration is necessary to let FriCAS know the base of the ragits.

```
u:RadixExpansion(8):=wholeRadix(w)+fractRadix(f0,f1)
```

$$554.37\overline{307} \tag{17}$$

RadixExpansion(8)

If there is no repeating part, then the list [0] should be used.

```
v: RadixExpansion(12) := fractRadix([1,2,3,11], [0])
```

$$0.123\mathrm{B}\overline{0} \tag{18}$$

RadixExpansion(12)

If you are not interested in the repeating nature of the expansion, an infinite stream of ragits can be obtained using fractRagits.

```
fractRagits(u)
```

$$[3,\ 7,\ \overline{3,\ 0,\ 7,\ 7}] \tag{19}$$

Stream(Integer)

Of course, it's possible to recover the fraction representation:

```
a :: Fraction(Integer)
```

$$\frac{76543}{210} \tag{20}$$

Fraction (Integer)

Issue the system command `)show RadixExpansion` to display the full list of operations defined by **RadixExpansion**. More examples of expansions are available in '`DecimalExpansion`' on page 392, '`BinaryExpansion`' on page 348, and '`HexadecimalExpansion`' on page 440.

9.70 RealClosure

The Real Closure 1.0 package provided by Renaud Rioboo (Renaud.Rioboo@lip6.fr) consists of different packages, categories and domains :

- The package **RealPolynomialUtilitiesPackage** which needs a **Field** F and a **UnivariatePolynomialCategory** domain with coefficients in F. It computes some simple functions such as Sturm and Sylvester sequences (`sturmSequence`, `sylvesterSequence`).
- The category **RealRootCharacterizationCategory** provides abstract functions to work with "real roots" of univariate polynomials. These resemble variables with some functionality needed to compute important operations.

- The category **RealClosedField** provides common operations available over real closed fields. These include finding all the roots of a univariate polynomial, taking square (and higher) roots, ...
- The domain **RightOpenIntervalRootCharacterization** is the main code that provides the functionality of **RealRootCharacterizationCategory** for the case of archimedean fields. Abstract roots are encoded with a left closed right open interval containing the root together with a defining polynomial for the root.
- The **RealClosure** domain is the end-user code. It provides usual arithmetic with real algebraic numbers, along with the functionality of a real closed field. It also provides functions to approximate a real algebraic number by an element of the base field. This approximation may either be absolute (`approximate`) or relative (`relativeApprox`).

CAVEATS

Since real algebraic expressions are stored as depending on "real roots" which are managed like variables, there is an ordering on these. This ordering is dynamical in the sense that any new algebraic takes precedence over older ones. In particular every creation function raises a new "real root". This has the effect that when you type something like `sqrt(2)+ sqrt(2)` you have two new variables which happen to be equal. To avoid this name the expression such as in `s2 := sqrt(2); s2 + s2`

Also note that computing times depend strongly on the ordering you implicitly provide. Please provide algebraics in the order which seems most natural to you.

LIMITATIONS

This package uses algorithms which are published in [1] and [2] which are based on field arithmetics, in particular for polynomial gcd related algorithms. This can be quite slow for high degree polynomials and subresultants methods usually work best. Beta versions of the package try to use these techniques in a better way and work significantly faster. These are mostly based on unpublished algorithms and cannot be distributed. Please contact the author if you have a particular problem to solve or want to use these versions.

Be aware that approximations behave as post-processing and that all computations are done exactly. They can thus be quite time consuming when depending on several "real roots".

REFERENCES

[1] R. Rioboo : Real Algebraic Closure of an ordered Field : Implementation in Axiom. In proceedings of the ISSAC'92 Conference, Berkeley 1992 pp. 206-215.

[2] Z. Ligatsikas, R. Rioboo, M. F. Roy : Generic computation of the real closure of an ordered field. In Mathematics and Computers in Simulation Volume 42, Issue 4-6, November 1996.

EXAMPLES

We shall work with the real closure of the ordered field of rational numbers.

```
Ran := RECLOS(FRAC INT)
```

$$\text{RealClosure}(\text{Fraction}(\textit{Integer})) \tag{4}$$

Type

Some simple signs for square roots, these correspond to an extension of degree 16 of the rational numbers. Examples provided by J. Abbot.

```
fourSquares(a:Ran,b:Ran,c:Ran,d:Ran):Ran == sqrt(a)+sqrt(b) - sqrt(c)-sqrt(d)
```

```
Function declaration fourSquares : (RealClosure(Fraction(Integer)), RealClosure
   (Fraction(Integer)), RealClosure(Fraction(Integer)), RealClosure(Fraction(
   Integer))) -> RealClosure(Fraction(Integer)) has been added to workspace.
```

These produce values very close to zero.

```
squareDiff1 := fourSquares(73,548,60,586)
```

```
Compiling function fourSquares with type (RealClosure(Fraction(Integer)),
   RealClosure(Fraction(Integer)), RealClosure(Fraction(Integer)), RealClosure(
   Fraction(Integer))) -> RealClosure(Fraction(Integer))
```

$$-\sqrt{586}-\sqrt{60}+\sqrt{548}+\sqrt{73} \tag{6}$$

RealClosure(Fraction (Integer))

```
recip(squareDiff1)
```

$$\begin{aligned}&\left(\left(54602\sqrt{548}+149602\sqrt{73}\right)\sqrt{60}+49502\sqrt{73}\sqrt{548}+9900895\right)\sqrt{586}\\&+\left(154702\sqrt{73}\sqrt{548}+30941947\right)\sqrt{60}+10238421\sqrt{548}+28051871\sqrt{73}\end{aligned} \tag{7}$$

Union(RealClosure(Fraction (Integer)), ...)

```
sign(squareDiff1)
```

$$1 \tag{8}$$

PositiveInteger

```
squareDiff2 := fourSquares(165,778,86,990)
```

$$-\sqrt{990}-\sqrt{86}+\sqrt{778}+\sqrt{165} \tag{9}$$

RealClosure(Fraction (Integer))

```
recip(squareDiff2)
```

$$\begin{aligned}&\left(\left(556778\sqrt{778}+1209010\sqrt{165}\right)\sqrt{86}+401966\sqrt{165}\sqrt{778}+144019431\right)\sqrt{990}\\&+\left(1363822\sqrt{165}\sqrt{778}+488640503\right)\sqrt{86}+162460913\sqrt{778}+352774119\sqrt{165}\end{aligned} \tag{10}$$

Union(RealClosure(Fraction (Integer)), ...)

```
sign(squareDiff2)
```

$$1 \tag{11}$$

PositiveInteger

```
squareDiff3 := fourSquares(217,708,226,692)
```

$$-\sqrt{692}-\sqrt{226}+\sqrt{708}+\sqrt{217} \tag{12}$$

RealClosure(Fraction (Integer))

```
recip(squareDiff3)
```

$$\left(\left(-34102\sqrt{708}-61598\sqrt{217}\right)\sqrt{226}-34802\sqrt{217}\sqrt{708}-13641141\right)\sqrt{692} + \left(-60898\sqrt{217}\sqrt{708}-23869841\right)\sqrt{226}-13486123\sqrt{708}-24359809\sqrt{217} \tag{13}$$

Union(RealClosure(Fraction (Integer)), ...)

```
sign(squareDiff3)
```

$$-1 \tag{14}$$

Integer

```
squareDiff4 := fourSquares(155,836,162,820)
```

$$-\sqrt{820}-\sqrt{162}+\sqrt{836}+\sqrt{155} \tag{15}$$

RealClosure(Fraction (Integer))

```
recip(squareDiff4)
```

$$\left(\left(-37078\sqrt{836}-86110\sqrt{155}\right)\sqrt{162}-37906\sqrt{155}\sqrt{836}-13645107\right)\sqrt{820} + \left(-85282\sqrt{155}\sqrt{836}-30699151\right)\sqrt{162}-13513901\sqrt{836}-31384703\sqrt{155} \tag{16}$$

Union(RealClosure(Fraction (Integer)), ...)

```
sign(squareDiff4)
```

$$-1 \tag{17}$$

Integer

```
squareDiff5 := fourSquares(591,772,552,818)
```

$$-\sqrt{818}-\sqrt{552}+\sqrt{772}+\sqrt{591} \tag{18}$$

RealClosure(Fraction (Integer))

```
recip(squareDiff5)
```

$$\left(\left(70922\sqrt{772}+81058\sqrt{591}\right)\sqrt{552}+68542\sqrt{591}\sqrt{772}+46297673\right)\sqrt{818} + \left(83438\sqrt{591}\sqrt{772}+56359389\right)\sqrt{552}+47657051\sqrt{772}+54468081\sqrt{591} \tag{19}$$

Union(RealClosure(Fraction (Integer)), ...)

```
sign(squareDiff5)
```

$$1 \tag{20}$$

PositiveInteger

```
squareDiff6 := fourSquares(434,1053,412,1088)
```

$$-\sqrt{1088}-\sqrt{412}+\sqrt{1053}+\sqrt{434} \tag{21}$$

RealClosure(Fraction(Integer))

```
recip(squareDiff6)
```

$$\begin{aligned}&\left(\left(115442\sqrt{1053}+179818\sqrt{434}\right)\sqrt{412}+112478\sqrt{434}\sqrt{1053}+76037291\right)\sqrt{1088}\\&+\left(182782\sqrt{434}\sqrt{1053}+123564147\right)\sqrt{412}+77290639\sqrt{1053}+120391609\sqrt{434}\end{aligned} \tag{22}$$

Union(RealClosure(Fraction(Integer)), ...)

```
sign(squareDiff6)
```

$$1 \tag{23}$$

PositiveInteger

```
squareDiff7 := fourSquares(514,1049,446,1152)
```

$$-\sqrt{1152}-\sqrt{446}+\sqrt{1049}+\sqrt{514} \tag{24}$$

RealClosure(Fraction(Integer))

```
recip(squareDiff7)
```

$$\begin{aligned}&\left(\left(349522\sqrt{1049}+499322\sqrt{514}\right)\sqrt{446}+325582\sqrt{514}\sqrt{1049}+239072537\right)\sqrt{1152}\\&+\left(523262\sqrt{514}\sqrt{1049}+384227549\right)\sqrt{446}+250534873\sqrt{1049}+357910443\sqrt{514}\end{aligned} \tag{25}$$

Union(RealClosure(Fraction(Integer)), ...)

```
sign(squareDiff7)
```

$$1 \tag{26}$$

PositiveInteger

```
squareDiff8 := fourSquares(190,1751,208,1698)
```

$$-\sqrt{1698}-\sqrt{208}+\sqrt{1751}+\sqrt{190} \tag{27}$$

RealClosure(Fraction(Integer))

```
recip(squareDiff8)
```

$$\left(\left(-214702\sqrt{1751}-651782\sqrt{190}\right)\sqrt{208}-224642\sqrt{190}\sqrt{1751}-129571901\right)\sqrt{1698} \\ +\left(-641842\sqrt{190}\sqrt{1751}-370209881\right)\sqrt{208}-127595865\sqrt{1751}-387349387\sqrt{190} \tag{28}$$

Union(RealClosure(Fraction(Integer)), ...)

```
sign(squareDiff8)
```

$$-1 \tag{29}$$

Integer

This should give three digits of precision

```
relativeApprox(squareDiff8,10^(-3))::Float
```

$$-0.23405277715937700123E-10 \tag{30}$$

Float

The sum of these 4 roots is 0

```
l := allRootsOf((x^2-2)^2-2)$Ran
```

$$[\%A33,\ \%A34,\ \%A35,\ \%A36] \tag{31}$$

List(RealClosure(Fraction(Integer)))

Check that they are all roots of the same polynomial

```
removeDuplicates map(mainDefiningPolynomial,l)
```

$$[?^4-4?^2+2] \tag{32}$$

List(Union(SparseUnivariatePolynomial(RealClosure(Fraction(Integer))), " failed "))

We can see at a glance that they are separate roots

```
map(mainCharacterization,l)
```

$$[[-2,\ -1[,\ [-1,\ 0[,\ [0,\ 1[,\ [1,\ 2[] \tag{33}$$

List(Union(RightOpenIntervalRootCharacterization(RealClosure(Fraction(Integer)), SparseUnivariatePolynomial(RealClosure(Fraction(Integer)))), " failed "))

Check the sum and product

```
[reduce(+,l),reduce(*,l)-2]
```

$$[0,\ 0] \tag{34}$$

List(RealClosure(Fraction(Integer)))

A more complicated test that involve an extension of degree 256. This is a way of checking nested radical identities.

```
(s2, s5, s10) := (sqrt(2)$Ran, sqrt(5)$Ran, sqrt(10)$Ran)
```

$$\sqrt{10} \tag{35}$$

RealClosure(Fraction (Integer))

```
eq1:=sqrt(s10+3)*sqrt(s5+2) - sqrt(s10-3)*sqrt(s5-2) = sqrt(10*s2+10)
```

$$-\sqrt{\sqrt{10}-3}\sqrt{\sqrt{5}-2}+\sqrt{\sqrt{10}+3}\sqrt{\sqrt{5}+2}=\sqrt{10\sqrt{2}+10} \tag{36}$$

Equation(RealClosure(Fraction (Integer)))

```
eq1::Boolean
```

$$\texttt{true} \tag{37}$$

Boolean

```
eq2:=sqrt(s5+2)*sqrt(s2+1) - sqrt(s5-2)*sqrt(s2-1) = sqrt(2*s10+2)
```

$$-\sqrt{\sqrt{5}-2}\sqrt{\sqrt{2}-1}+\sqrt{\sqrt{5}+2}\sqrt{\sqrt{2}+1}=\sqrt{2\sqrt{10}+2} \tag{38}$$

Equation(RealClosure(Fraction (Integer)))

```
eq2::Boolean
```

$$\texttt{true} \tag{39}$$

Boolean

Some more examples from J. M. Arnaudies

```
s3 := sqrt(3)$Ran
```

$$\sqrt{3} \tag{40}$$

RealClosure(Fraction (Integer))

```
s7:= sqrt(7)$Ran
```

$$\sqrt{7} \tag{41}$$

RealClosure(Fraction (Integer))

```
e1 := sqrt(2*s7-3*s3,3)
```

$$\sqrt[3]{2\sqrt{7}-3\sqrt{3}} \tag{42}$$

RealClosure(Fraction (Integer))

```
e2 := sqrt(2*s7+3*s3,3)
```

$$\sqrt[3]{2\sqrt{7}+3\sqrt{3}} \tag{43}$$

RealClosure(Fraction(Integer))

This should be null

```
e2-e1-s3
```

$$0 \tag{44}$$

RealClosure(Fraction(Integer))

A quartic polynomial

```
pol : UP(x,Ran) := x^4+(7/3)*x^2+30*x-(100/3)
```

$$x^4+\frac{7}{3}x^2+30\,x-\frac{100}{3} \tag{45}$$

UnivariatePolynomial(x, RealClosure(Fraction(Integer)))

Add some cubic roots

```
r1 := sqrt(7633)$Ran
```

$$\sqrt{7633} \tag{46}$$

RealClosure(Fraction(Integer))

```
alpha := sqrt(5*r1-436,3)/3
```

$$\frac{1}{3}\sqrt[3]{5\sqrt{7633}-436} \tag{47}$$

RealClosure(Fraction(Integer))

```
beta := -sqrt(5*r1+436,3)/3
```

$$-\frac{1}{3}\sqrt[3]{5\sqrt{7633}+436} \tag{48}$$

RealClosure(Fraction(Integer))

this should be null

```
pol.(alpha+beta-1/3)
```

$$0 \tag{49}$$

RealClosure(Fraction(Integer))

A quintic polynomial

```
qol : UP(x,Ran) := x^5+10*x^3+20*x+22
```

$$x^5 + 10\,x^3 + 20\,x + 22 \tag{50}$$

UnivariatePolynomial(x, RealClosure(Fraction(Integer)))

Add some cubic roots

```
r2 := sqrt(153)$Ran
```

$$\sqrt{153} \tag{51}$$

RealClosure(Fraction(Integer))

```
alpha2 := sqrt(r2-11,5)
```

$$\sqrt[5]{\sqrt{153} - 11} \tag{52}$$

RealClosure(Fraction(Integer))

```
beta2 := -sqrt(r2+11,5)
```

$$-\sqrt[5]{\sqrt{153} + 11} \tag{53}$$

RealClosure(Fraction(Integer))

this should be null

```
qol(alpha2+beta2)
```

$$0 \tag{54}$$

RealClosure(Fraction(Integer))

Finally, some examples from the book Computer Algebra by Davenport, Siret and Tournier (page 77). The last one is due to Ramanujan.

```
dst1:=sqrt(9+4*s2)=1+2*s2
```

$$\sqrt{4\sqrt{2} + 9} = 2\sqrt{2} + 1 \tag{55}$$

Equation(RealClosure(Fraction(Integer)))

```
dst1::Boolean
```

$$\texttt{true} \tag{56}$$

Boolean

```
s6:Ran:=sqrt 6
```

$$\sqrt{6} \tag{57}$$

RealClosure(Fraction (Integer))

```
dst2:=sqrt(5+2*s6)+sqrt(5-2*s6) = 2*s3
```

$$\sqrt{-2\sqrt{6}+5}+\sqrt{2\sqrt{6}+5}=2\sqrt{3} \tag{58}$$

Equation(RealClosure(Fraction (Integer)))

```
dst2::Boolean
```

$$\texttt{true} \tag{59}$$

Boolean

```
s29:Ran:=sqrt 29
```

$$\sqrt{29} \tag{60}$$

RealClosure(Fraction (Integer))

```
dst4:=sqrt(16-2*s29+2*sqrt(55-10*s29)) = sqrt(22+2*s5)-sqrt(11+2*s29)+s5
```

$$\sqrt{2\sqrt{-10\sqrt{29}+55}-2\sqrt{29}+16}=-\sqrt{2\sqrt{29}+11}+\sqrt{2\sqrt{5}+22}+\sqrt{5} \tag{61}$$

Equation(RealClosure(Fraction (Integer)))

```
dst4::Boolean
```

$$\texttt{true} \tag{62}$$

Boolean

```
dst6:=sqrt((112+70*s2)+(46+34*s2)*s5) = (5+4*s2)+(3+s2)*s5
```

$$\sqrt{\left(34\sqrt{2}+46\right)\sqrt{5}+70\sqrt{2}+112}=\left(\sqrt{2}+3\right)\sqrt{5}+4\sqrt{2}+5 \tag{63}$$

Equation(RealClosure(Fraction (Integer)))

```
dst6::Boolean
```

$$\texttt{true} \tag{64}$$

Boolean

```
f3:Ran:=sqrt(3,5)
```

$$\sqrt[5]{3} \tag{65}$$

RealClosure(Fraction (Integer))

```
f25:Ran:=sqrt(1/25,5)
```

$$\sqrt[5]{\frac{1}{25}} \tag{66}$$

RealClosure(Fraction(Integer))

```
f32:Ran:=sqrt(32/5,5)
```

$$\sqrt[5]{\frac{32}{5}} \tag{67}$$

RealClosure(Fraction(Integer))

```
f27:Ran:=sqrt(27/5,5)
```

$$\sqrt[5]{\frac{27}{5}} \tag{68}$$

RealClosure(Fraction(Integer))

```
dst5:=sqrt((f32-f27,3)) = f25*(1+f3-f3^2)
```

$$\sqrt[3]{-\sqrt[5]{\frac{27}{5}}+\sqrt[5]{\frac{32}{5}}} = \left(-\sqrt[5]{3}^2+\sqrt[5]{3}+1\right)\sqrt[5]{\frac{1}{25}} \tag{69}$$

Equation(RealClosure(Fraction(Integer)))

```
dst5::Boolean
```

$$\texttt{true} \tag{70}$$

Boolean

9.71 RegularTriangularSet

The **RegularTriangularSet** domain constructor implements regular triangular sets. These particular triangular sets were introduced by M. Kalkbrener (1991) in his PhD Thesis under the name regular chains. Regular chains and their related concepts are presented in the paper "On the Theories of Triangular sets" By P. Aubry, D. Lazard and M. Moreno Maza (to appear in the Journal of Symbolic Computation). The **RegularTriangularSet** constructor also provides a new method (by the third author) for solving polynomial systems by means of regular chains. This method has two ways of solving. One has the same specifications as Kalkbrener's algorithm (1991) and the other is closer to Lazard's method (Discr. App. Math, 1991). Moreover, this new method removes redundant component from the decompositions when this is not *too expensive*. This is always the case with square-free regular chains. So if you want to obtain decompositions without redundant components just use the **SquareFreeRegularTriangularSet** domain constructor or the **LazardSetSolvingPackage** package constructor. See also the **LexTriangularPackage** and **ZeroDimensionalSolvePackage** for the case of algebraic systems with a finite number of (complex) solutions.

One of the main features of regular triangular sets is that they naturally define towers of simple extensions of a field. This allows one to perform with multivariate polynomials the same kind of operations as one can do in an **EuclideanDomain**.

The **RegularTriangularSet** constructor takes four arguments. The first one, **R**, is the coefficient ring of the polynomials; it must belong to the category **GcdDomain**. The second one, **E**, is the exponent monoid of the polynomials; it must belong to the category **OrderedAbelianMonoidSup**. The third one, **V**, is the ordered set of variables; it must belong to the category **OrderedSet**. The last one is the polynomial ring; it must belong to the category **RecursivePolynomialCategory(R,E,V)**. The abbreviation for **RegularTriangularSet** is **REGSET**. See also the constructor **RegularChain** which only takes two arguments, the coefficient ring and the ordered set of variables; in that case, polynomials are necessarily built with the **NewSparseMultivariatePolynomial** domain constructor.

We shall explain now how to use the constructor **REGSET** and how to read the decomposition of a polynomial system by means of regular sets.

Let us give some examples. We start with an easy one (Donati-Traverso) in order to understand the two ways of solving polynomial systems provided by the **REGSET** constructor. Define the coefficient ring.

```
R := Integer
```

$$Integer \tag{4}$$

Type

Define the list of variables,

```
ls : List Symbol := [x,y,z,t]
```

$$[x,\ y,\ z,\ t] \tag{5}$$

List (Symbol)

and make it an ordered set;

```
V := OVAR(ls)
```

$$\text{OrderedVariableList}([x,\ y,\ z,\ t]) \tag{6}$$

Type

then define the exponent monoid.

```
E := IndexedExponents V
```

$$\text{IndexedExponents}(\text{OrderedVariableList}([x,\ y,\ z,\ t])) \tag{7}$$

Type

Define the polynomial ring.

```
P := NSMP(R, V)
```

$$\text{NewSparseMultivariatePolynomial}(Integer, \text{OrderedVariableList}([x,\ y,\ z,\ t])) \tag{8}$$

Type

Let the variables be polynomial.

```
x: P := 'x
```

$$x \tag{9}$$

NewSparseMultivariatePolynomial(Integer, OrderedVariableList([x, y, z, t]))

```
y: P := 'y
```

$$y \tag{10}$$

NewSparseMultivariatePolynomial(Integer, OrderedVariableList([x, y, z, t]))

```
z: P := 'z
```

$$z \tag{11}$$

NewSparseMultivariatePolynomial(Integer, OrderedVariableList([x, y, z, t]))

```
t: P := 't
```

$$t \tag{12}$$

NewSparseMultivariatePolynomial(Integer, OrderedVariableList([x, y, z, t]))

Now call the **RegularTriangularSet** domain constructor.

```
T := REGSET(R,E,V,P)
```

$$\text{RegularTriangularSet}(Integer, \text{IndexedExponents}(\text{OrderedVariableList}([x, y, z, t])), \text{OrderedVariableList}([x, y, z, t]), \text{NewSparseMultivariatePolynomial}(Integer, \text{OrderedVariableList}([x, y, z, t]))) \tag{13}$$

Type

Define a polynomial system.

```
p1 := x ^ 31 - x ^ 6 - x - y
```

$$x^{31} - x^6 - x - y \tag{14}$$

NewSparseMultivariatePolynomial(Integer, OrderedVariableList([x, y, z, t]))

```
p2 := x ^ 8  - z
```

$$x^8 - z \tag{15}$$

NewSparseMultivariatePolynomial(Integer, OrderedVariableList([x, y, z, t]))

```
p3 := x ^ 10 - t
```

$$x^{10} - t \tag{16}$$

NewSparseMultivariatePolynomial(Integer, OrderedVariableList([x, y, z, t]))

```
lp := [p1, p2, p3]
```

$$\left[x^{31} - x^6 - x - y,\ x^8 - z,\ x^{10} - t\right] \tag{17}$$

List(NewSparseMultivariatePolynomial(Integer, OrderedVariableList([x, y, z, t])))

First of all, let us solve this system in the sense of Kalkbrener.

```
zeroSetSplit(lp)$T
```

$$\left[\left\{z^5 - t^4,\ t\,z\,y^2 + 2\,z^3\,y - t^8 + 2\,t^5 + t^3 - t^2,\ \left(t^4 - t\right)\,x - t\,y - z^2\right\}\right] \tag{18}$$

List(RegularTriangularSet(Integer, IndexedExponents(OrderedVariableList([x, y, z, t])), OrderedVariableList([x, y, z, t]), NewSparseMultivariatePolynomial(Integer, OrderedVariableList([x, y, z, t]))))

And now in the sense of Lazard (or Wu and other authors).

```
lts := zeroSetSplit(lp,false)$T
```

$$\begin{aligned}&\left[\left\{z^5 - t^4,\ t\,z\,y^2 + 2\,z^3\,y - t^8 + 2\,t^5 + t^3 - t^2,\ \left(t^4 - t\right)\,x - t\,y - z^2\right\},\right.\\&\left.\left\{t^3 - 1,\ z^5 - t,\ t\,z\,y^2 + 2\,z^3\,y + 1,\ z\,x^2 - t\right\},\ \{t,\ z,\ y,\ x\}\right]\end{aligned} \tag{19}$$

List(RegularTriangularSet(Integer, IndexedExponents(OrderedVariableList([x, y, z, t])), OrderedVariableList([x, y, z, t]), NewSparseMultivariatePolynomial(Integer, OrderedVariableList([x, y, z, t]))))

We can see that the first decomposition is a subset of the second. So how can both be correct ?

Recall first that polynomials from a domain of the category **RecursivePolynomialCategory** are regarded as univariate polynomials in their main variable. For instance the second polynomial in the first set of each decomposition has main variable **y** and its initial (i.e. its leading coefficient w.r.t. its main variable) is **t z**.

Now let us explain how to read the second decomposition. Note that the non-constant initials of the first set are $t^4 - t$ and tz. Then the solutions described by this first set are the common zeros of its polynomials that do not cancel the polynomials $t^4 - t$ and tyz. Now the solutions of the input system **lp** satisfying these equations are described by the second and the third sets of the decomposition. Thus, in some sense, they can be considered as degenerate solutions. The solutions given by the first set are called the generic points of the system; they give the general form of the solutions. The first decomposition only provides these generic points. This latter decomposition is useful when there are many degenerate solutions (which is sometimes hard to compute) and when one is only interested in general information, like the dimension of the input system.

We can get the dimensions of each component of a decomposition as follows.

```
[coHeight(ts) for ts in lts]
```

$$[1,\ 0,\ 0] \tag{20}$$

List (NonNegativeInteger)

Thus the first set has dimension one. Indeed **t** can take any value, except **0** or any third root of **1**, whereas **z** is completely determined from **t**, **y** is given by **z** and **t**, and finally **x** is given by the other three variables. In the second and the third sets of the second decomposition the four variables are completely determined and thus these sets have dimension zero.

We give now the precise specifications of each decomposition. This assumes some mathematical knowledge. However, for the non-expert user, the above explanations will be sufficient to understand the other features of the **RSEGSET** constructor.

The input system **lp** is decomposed in the sense of Kalkbrener as finitely many regular sets **T1,...,Ts** such that the radical ideal generated by **lp** is the intersection of the radicals of the saturated ideals of **T1,...,Ts**. In other words, the affine variety associated with **lp** is the union of the closures (w.r.t. Zariski topology) of the regular-zeros sets of **T1,...,Ts**.

N. B. The prime ideals associated with the radical of the saturated ideal of a regular triangular set have all the same dimension; moreover these prime ideals can be given by characteristic sets with the same main variables. Thus a decomposition in the sense of Kalkbrener is unmixed dimensional. Then it can be viewed as a *lazy* decomposition into prime ideals (some of these prime ideals being merged into unmixed dimensional ideals).

Now we explain the other way of solving by means of regular triangular sets. The input system **lp** is decomposed in the sense of Lazard as finitely many regular triangular sets **T1,...,Ts** such that the affine variety associated with **lp** is the union of the regular-zeros sets of **T1,...,Ts**. Thus a decomposition in the sense of Lazard is also a decomposition in the sense of Kalkbrener; the converse is false as we have seen before.

When the input system has a finite number of solutions, both ways of solving provide similar decompositions as we shall see with this second example (Caprasse).

Define a polynomial system.

```
f1 := y^2*z+2*x*y*t-2*x-z
```

$$(2\,t\,y-2)\,x+z\,y^2-z \tag{21}$$

NewSparseMultivariatePolynomial(Integer, OrderedVariableList ([x, y, z, t]))

```
f2 :=   -x^3*z+ 4*x*y^2*z+ 4*x^2*y*t+ 2*y^3*t+ 4*x^2- 10*y^2+ 4*x*z- 10*y*t+ 2
```

$$-z\,x^3+(4\,t\,y+4)\,x^2+\left(4\,z\,y^2+4\,z\right)\,x+2\,t\,y^3-10\,y^2-10\,t\,y+2 \tag{22}$$

NewSparseMultivariatePolynomial(Integer, OrderedVariableList ([x, y, z, t]))

```
f3 :=  2*y*z*t+x*t^2-x-2*z
```

$$\left(t^2-1\right)\,x+2\,t\,z\,y-2\,z \tag{23}$$

NewSparseMultivariatePolynomial(Integer, OrderedVariableList ([x, y, z, t]))

```
f4 :=   -x*z^3+ 4*y*z^2*t+ 4*x*z*t^2+ 2*y*t^3+ 4*x*z+ 4*z^2-10*y*t- 10*t^2+2
```

$$\left(-z^3+\left(4\,t^2+4\right)\,z\right)\,x+\left(4\,t\,z^2+2\,t^3-10\,t\right)\,y+4\,z^2-10\,t^2+2 \tag{24}$$

NewSparseMultivariatePolynomial(Integer, OrderedVariableList([x, y, z, t]))

```
lf := [f1, f2, f3, f4]
```

$$\left[(2\,t\,y-2)\,x+z\,y^2-z,\ -z\,x^3+(4\,t\,y+4)\,x^2+\left(4\,z\,y^2+4\,z\right)\,x+2\,t\,y^3-10\,y^2-10\,t\,y+2,\right.$$
$$\left.\left(t^2-1\right)\,x+2\,t\,z\,y-2\,z,\ \left(-z^3+\left(4\,t^2+4\right)\,z\right)\,x+\left(4\,t\,z^2+2\,t^3-10\,t\right)\,y+4\,z^2-10\,t^2+2\right] \tag{25}$$

List(NewSparseMultivariatePolynomial(Integer, OrderedVariableList([x, y, z, t])))

First of all, let us solve this system in the sense of Kalkbrener.

```
zeroSetSplit(lf)$T
```

$$\left[\left\{t^2-1,\ z^8-16\,z^6+256\,z^2-256,\ t\,y-1,\ \left(z^3-8\,z\right)\,x-8\,z^2+16\right\},\ \left\{3\,t^2+1,\ z^2-7\,t^2-1,\right.\right.$$
$$\left.\left.y+t,\ x+z\right\},\ \left\{t^8-10\,t^6+10\,t^2-1,\ z,\ \left(t^3-5\,t\right)\,y-5\,t^2+1,\ x\right\},\ \left\{t^2+3,\ z^2-4,\ y+t,\ x-z\right\}\right] \tag{26}$$

List(RegularTriangularSet(Integer, IndexedExponents(OrderedVariableList([x, y, z, t])), OrderedVariableList([x, y, z, t]), NewSparseMultivariatePolynomial(Integer, OrderedVariableList([x, y, z, t]))))

And now in the sense of Lazard (or Wu and other authors).

```
lts2 := zeroSetSplit(lf,false)$T
```

$$\left[\left\{t^8-10\,t^6+10\,t^2-1,\ z,\ \left(t^3-5\,t\right)\,y-5\,t^2+1,\ x\right\},\ \left\{t^2-1,\ z^8-16\,z^6+256\,z^2-256,\ t\,y-1,\right.\right.$$
$$\left.\left.\left(z^3-8\,z\right)\,x-8\,z^2+16\right\},\ \left\{3\,t^2+1,\ z^2-7\,t^2-1,\ y+t,\ x+z\right\},\ \left\{t^2+3,\ z^2-4,\ y+t,\ x-z\right\}\right] \tag{27}$$

List(RegularTriangularSet(Integer, IndexedExponents(OrderedVariableList([x, y, z, t])), OrderedVariableList([x, y, z, t]), NewSparseMultivariatePolynomial(Integer, OrderedVariableList([x, y, z, t]))))

Up to the ordering of the components, both decompositions are identical.

Let us check that each component has a finite number of solutions.

```
[coHeight(ts) for ts in lts2]
```

$$[0,\ 0,\ 0,\ 0] \tag{28}$$

List(NonNegativeInteger)

Let us count the degrees of each component,

```
degrees := [degree(ts) for ts in lts2]
```

$$[8,\ 16,\ 4,\ 4] \tag{29}$$

List(NonNegativeInteger)

and compute their sum.

```
reduce(+,degrees)
```

$$32 \tag{30}$$

PositiveInteger

We study now the options of the zeroSetSplit operation. As we have seen yet, there is an optional second argument which is a boolean value. If this value is true (this is the default) then the decomposition is computed in the sense of Kalkbrener, otherwise it is computed in the sense of Lazard.

There is a second boolean optional argument that can be used (in that case the first optional argument must be present). This second option allows you to get some information during the computations.

Therefore, we need to understand a little what is going on during the computations. An important feature of the algorithm is that the intermediate computations are managed in some sense like the processes of a Unix system. Indeed, each intermediate computation may generate other intermediate computations and the management of all these computations is a crucial task for the efficiency. Thus any intermediate computation may be suspended, killed or resumed, depending on algebraic considerations that determine priorities for these processes. The goal is of course to go as fast as possible towards the final decomposition which means to avoid as much as possible unnecessary computations.

To follow the computations, one needs to set to `true` the second argument. Then a lot of numbers and letters are displayed. Between a `[` and a `]` one has the state of the processes at a given time. Just after `[` one can see the number of processes. Then each process is represented by two numbers between `<` and `>`. A process consists of a list of polynomial **ps** and a triangular set **ts**; its goal is to compute the common zeros of **ps** that belong to the regular-zeros set of **ts**. After the processes, the number between pipes gives the total number of polynomials in all the sets `ps`. Finally, the number between braces gives the number of components of a decomposition that are already computed. This number may decrease.

Let us take a third example (Czapor-Geddes-Wang) to see how this information is displayed.

Define a polynomial system.

```
u : R := 2
```

$$2 \tag{31}$$

Integer

```
q1 := 2*(u-1)^2+ 2*(x-z*x+z^2)+ y^2*(x-1)^2- 2*u*x+ 2*y*t*(1-x)*(x-z)+ 2*u*z*t*(t-y)+ _
    u^2*t^2*(1-2*z)+ 2*u*t^2*(z-x)+ 2*u*t*y*(z-1)+ 2*u*z*x*(y+1)+ (u^2-2*u)*z^2*t^2+ _
    2*u^2*z^2+ 4*u*(1-u)*z+ t^2*(z-x)^2
```

$$\left(y^2 - 2\,t\,y + t^2\right) x^2 + \left(-2\,y^2 + \left(\left(2\,t + 4\right) z + 2\,t\right) y + \left(-2\,t^2 + 2\right) z - 4\,t^2 - 2\right) x + y^2 + \left(-2\,t\,z - 4\,t\right) y + \left(t^2 + 10\right) z^2 - 8\,z + 4\,t^2 + 2 \tag{32}$$

NewSparseMultivariatePolynomial(Integer, OrderedVariableList ([x, y, z, t]))

```
q2 := t*(2*z+1)*(x-z)+ y*(z+2)*(1-x)+ u*(u-2)*t+ u*(1-2*u)*z*t+ u*y*(x+u-z*x-1)+ _
    u*(u+1)*z^2*t
```

$$\left(-3\,z\,y + 2\,t\,z + t\right) x + \left(z + 4\right) y + 4\,t\,z^2 - 7\,t\,z \tag{33}$$

NewSparseMultivariatePolynomial(Integer, OrderedVariableList ([x, y, z, t]))

```
q3 := -u^2*(z-1)^2+ 2*z*(z-x)-2*(x-1)
```

$$\left(-2\,z - 2\right) x - 2\,z^2 + 8\,z - 2 \tag{34}$$

NewSparseMultivariatePolynomial(Integer, OrderedVariableList([x, y, z, t]))

```
q4 :=   u^2+4*(z-x^2)+3*y^2*(x-1)^2- 3*t^2*(z-x)^2 _
    +3*u^2*t^2*(z-1)^2+u^2*z*(z-2)+6*u*t*y*(z+x+z*x-1)
```

$$\left(3\,y^2-3\,t^2-4\right)\,x^2+\left(-6\,y^2+\left(12\,t\,z+12\,t\right)\,y+6\,t^2\,z\right)\,x+3\,y^2 \\ +\left(12\,t\,z-12\,t\right)\,y+\left(9\,t^2+4\right)\,z^2+\left(-24\,t^2-4\right)\,z+12\,t^2+4 \tag{35}$$

NewSparseMultivariatePolynomial(Integer, OrderedVariableList([x, y, z, t]))

```
lq := [q1, q2, q3, q4]
```

$$\left[\left(y^2-2\,t\,y+t^2\right)\,x^2+\left(-2\,y^2+\left(\left(2\,t+4\right)\,z+2\,t\right)\,y+\left(-2\,t^2+2\right)\,z-4\,t^2-2\right)\,x+y^2 \\ +\left(-2\,t\,z-4\,t\right)\,y+\left(t^2+10\right)\,z^2-8\,z+4\,t^2+2,\ \left(-3\,z\,y+2\,t\,z+t\right)\,x+\left(z+4\right)\,y+4\,t\,z^2-7\,t\,z, \\ \left(-2\,z-2\right)\,x-2\,z^2+8\,z-2,\ \left(3\,y^2-3\,t^2-4\right)\,x^2+\left(-6\,y^2+\left(12\,t\,z+12\,t\right)\,y+6\,t^2\,z\right)\,x \\ +3\,y^2+\left(12\,t\,z-12\,t\right)\,y+\left(9\,t^2+4\right)\,z^2+\left(-24\,t^2-4\right)\,z+12\,t^2+4\right] \tag{36}$$

List(NewSparseMultivariatePolynomial(Integer, OrderedVariableList([x, y, z, t])))

Let us try the information option.

```
zeroSetSplit(lq,true,true)$T;
```

Between a sequence of processes, thus between a `]` and a `[` you can see capital letters `W, G, I` and lower case letters `i, w`. Each time a capital letter appears a non-trivial computation has been performed and its result is put in a hash-table. Each time a lower case letter appears a needed result has been found in an hash-table. The use of these hash-tables generally speed up the computations. However, on very large systems, it may happen that these hash-tables become too big to be handled by your FriCAS configuration. Then in these exceptional cases, you may prefer getting a result (even if it takes a long time) than getting nothing. Hence you need to know how to prevent the **RSEGSET** constructor from using these hash-tables. In that case you will be using the zeroSetSplit with five arguments. The first one is the input system **lp** as above. The second one is a boolean value `hash?` which is `true` iff you want to use hash-tables. The third one is boolean value `clos?` which is `true` iff you want to solve your system in the sense of Kalkbrener, the other way remaining that of Lazard. The fourth argument is boolean value `info?` which is `true` iff you want to display information during the computations. The last one is boolean value `prep?` which is `true` iff you want to use some heuristics that are performed on the input system before starting the real algorithm. The value of this flag is `true` when you are using zeroSetSplit with less than five arguments. Note that there is no available signature for zeroSetSplit with four arguments.

We finish this section by some remarks about both ways of solving, in the sense of Kalkbrener or in the sense of Lazard. For problems with a finite number of solutions, there are theoretically equivalent and the resulting decompositions are identical, up to the ordering of the components. However, when solving in the sense of Lazard, the algorithm behaves differently. In that case, it becomes more incremental than in the sense of Kalkbrener. That means the polynomials of the input system are considered one after another whereas in the sense of Kalkbrener the input system is treated more globally.

This makes an important difference in positive dimension. Indeed when solving in the sense of Kalkbrener, the *Primeidealkettensatz* of Krull is used. That means any regular triangular containing more polynomials than the input system can be deleted. This is not possible when solving in the sense of Lazard. This explains why Kalkbrener's decompositions usually contain less components than those of Lazard. However, it may happen with some examples that the incremental process (that cannot

be used when solving in the sense of Kalkbrener) provide a more efficient way of solving than the global one even if the *Primeidealkettensatz* is used. Thus just try both, with the various options, before concluding that you cannot solve your favorite system with zeroSetSplit. There exist more options at the development level that are not listed above.

9.72 RomanNumeral

The Roman numeral package was added to FriCAS in MCMLXXXVI for use in denoting higher order derivatives.

For example, let `f` be a symbolic operator.

```
f := operator 'f
```

$$f \tag{4}$$

BasicOperator

This is the seventh derivative of `f` with respect to `x`.

```
D(f x,x,7)
```

$$f^{(vii)}(x) \tag{5}$$

Expression (Integer)

You can have integers printed as Roman numerals by declaring variables to be of type **RomanNumeral** (abbreviation **ROMAN**).

```
a := roman(1978 - 1965)
```

$$XIII \tag{6}$$

RomanNumeral

This package now has a small but devoted group of followers that claim this domain has shown its efficacy in many other contexts. They claim that Roman numerals are every bit as useful as ordinary integers. In a sense, they are correct, because Roman numerals form a ring and you can therefore construct polynomials with Roman numeral coefficients, matrices over Roman numerals, etc..

```
x : UTS(ROMAN,'x,0) := x
```

$$x \tag{7}$$

UnivariateTaylorSeries (RomanNumeral, x, 0)

Was Fibonacci Italian or ROMAN?

```
recip(1 - x - x^2)
```

$$I + x + II\,x^2 + III\,x^3 + V\,x^4 + VIII\,x^5 + XIII\,x^6 + XXI\,x^7 + O\left(x^8\right) \tag{8}$$

Union(UnivariateTaylorSeries (RomanNumeral, x, 0), ...)

You can also construct fractions with Roman numeral numerators and denominators, as this matrix Hilberticus illustrates.

```
m : MATRIX FRAC ROMAN
```

```
m := matrix [[1/(i + j) for i in 1..3] for j in 1..3]
```

$$\begin{bmatrix} \frac{I}{II} & \frac{I}{III} & \frac{I}{IV} \\ \frac{I}{III} & \frac{I}{IV} & \frac{I}{V} \\ \frac{I}{IV} & \frac{I}{V} & \frac{I}{VI} \end{bmatrix} \tag{10}$$

Matrix(Fraction (RomanNumeral))

Note that the inverse of the matrix has integral **ROMAN** entries.

```
inverse m
```

$$\begin{bmatrix} LXXII & -CCXL & CLXXX \\ -CCXL & CM & -DCCXX \\ CLXXX & -DCCXX & DC \end{bmatrix} \tag{11}$$

Union(Matrix(Fraction(RomanNumeral)), ...)

Unfortunately, the spoil-sports say that the fun stops when the numbers get big—mostly because the Romans didn't establish conventions about representing very large numbers.

```
y := factorial 10
```

$$3628800 \tag{12}$$

PositiveInteger

You work it out!

```
roman y
```

$$((((I))))((((I))))((((I)))) \ (((I)))(((I)))(((I)))(((I)))(((I)))(((I))) \ ((I))((I)) \ MMMMMMMMDCCC \tag{13}$$

RomanNumeral

Issue the system command `)show RomanNumeral` to display the full list of operations defined by **RomanNumeral**.

9.73 Segment

The **Segment** domain provides a generalized interval type.

Segments are created using the "`..`" construct by indicating the (included) end points.

```
s := 3..10
```

$$3\,..\,10 \tag{4}$$

Segment(PositiveInteger)

The first end point is called the low and the second is called high.

```
low(s)
```

$$3 \tag{5}$$

PositiveInteger

These names are used even though the end points might belong to an unordered set.

```
high(s)
```

$$10 \tag{6}$$

PositiveInteger

In addition to the end points, each segment has an integer "increment." An increment can be specified using the "by" construct.

```
t := 10..3 by -2
```

$$10\,..\,3 \texttt{ by } -2 \tag{7}$$

Segment(PositiveInteger)

This part can be obtained using the incr function.

```
incr(s)
```

$$1 \tag{8}$$

PositiveInteger

Unless otherwise specified, the increment is 1.

```
incr(t)
```

$$-2 \tag{9}$$

Integer

A single value can be converted to a segment with equal end points. This happens if segments and single values are mixed in a list.

```
l := [1..3, 5, 9, 15..11 by -1]
```

$$[1\,..\,3,\ 5\,..\,5,\ 9\,..\,9,\ 15\,..\,11 \texttt{ by } -1] \tag{10}$$

List (Segment(PositiveInteger))

If the underlying type is an ordered ring, it is possible to perform additional operations. The expand operation creates a list of points in a segment.

```
expand(s)
```

$$[3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10] \tag{11}$$

List (Integer)

If `k > 0`, then `expand(l..h by k)` creates the list `[l, l+k, ..., lN]` where `lN <= h < lN+k`. If `k < 0`, then `lN >= h > lN+k`.

```
expand(t)
```

$$[10,\ 8,\ 6,\ 4] \tag{12}$$

List (Integer)

It is also possible to expand a list of segments. This is equivalent to appending lists obtained by expanding each segment individually.

```
expand(l)
```

$$[1,\ 2,\ 3,\ 5,\ 9,\ 15,\ 14,\ 13,\ 12,\ 11] \tag{13}$$

List (Integer)

For more information on related topics, see '`SegmentBinding`' on page 564 and '`UniversalSegment`' on page 602. Issue the system command `)show Segment` to display the full list of operations defined by **Segment**.

9.74 SegmentBinding

The **SegmentBinding** type is used to indicate a range for a named symbol.

First give the symbol, then an "=" and finally a segment of values.

```
x = a..b
```

$$x = (a\,..\,b) \tag{4}$$

SegmentBinding(Symbol)

This is used to provide a convenient syntax for arguments to certain operations.

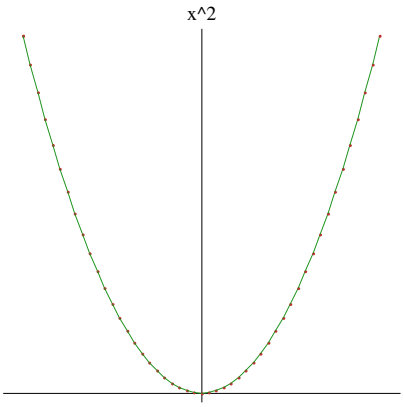
```
sum(i^2, i = 0..n)
```

$$\frac{2\,n^3 + 3\,n^2 + n}{6} \tag{5}$$

Fraction (Polynomial(Integer))

The draw operation uses a **SegmentBinding** argument as a range of coordinates. This is an example of a two-dimensional parametrized plot; other draw options use more than one **SegmentBinding** argument.

```
draw(x^2, x = -2..2)
```



The left-hand side must be of type **Symbol** but the right-hand side can be a segment over any type.

```
sb := y = 1/2..3/2
```

$$y = \left(\left(\frac{1}{2}\right) .. \left(\frac{3}{2}\right)\right) \tag{6}$$

SegmentBinding(Fraction(Integer))

The left- and right-hand sides can be obtained using the variable and segment operations.

```
variable(sb)
```

$$y \tag{7}$$

Symbol

```
segment(sb)
```

$$\left(\frac{1}{2}\right) .. \left(\frac{3}{2}\right) \tag{8}$$

Segment(Fraction(Integer))

For more information on related topics, see '`Segment`' on page 562 and '`UniversalSegment`' on page 602. Issue the system command `)show SegmentBinding` to display the full list of operations defined by **SegmentBinding**.

9.75 Set

The **Set** domain allows one to represent explicit finite sets of values. These are similar to lists, but duplicate elements are not allowed. Sets can be created by giving a fixed set of values ...

```
s := set [x^2-1, y^2-1, z^2-1]
```

$$\{x^2 - 1,\ y^2 - 1,\ z^2 - 1\} \tag{4}$$

Set(Polynomial(Integer))

or by using a collect form, just as for lists. In either case, the set is formed from a finite collection of values.

```
t := set [x^i - i+1 for i in 2..10 | prime? i]
```

$$\{x^2 - 1,\ x^3 - 2,\ x^5 - 4,\ x^7 - 6\} \tag{5}$$

Set(Polynomial(Integer))

The basic operations on sets are intersect, union, difference, and symmetricDifference.

```
i := intersect(s,t)
```

$$\{x^2 - 1\} \tag{6}$$

Set(Polynomial(Integer))

```
u := union(s,t)
```

$$\{x^2 - 1,\ y^2 - 1,\ z^2 - 1,\ x^3 - 2,\ x^5 - 4,\ x^7 - 6\} \tag{7}$$

Set(Polynomial(Integer))

The set `difference(s,t)` contains those members of `s` which are not in `t`.

```
difference(s,t)
```

$$\{y^2 - 1,\ z^2 - 1\} \tag{8}$$

Set(Polynomial(Integer))

The set `symmetricDifference(s,t)` contains those elements which are in `s` or `t` but not in both.

```
symmetricDifference(s,t)
```

$$\{y^2 - 1,\ z^2 - 1,\ x^3 - 2,\ x^5 - 4,\ x^7 - 6\} \tag{9}$$

Set(Polynomial(Integer))

Set membership is tested using the member? operation.

```
member?(y, s)
```

$$\texttt{false} \tag{10}$$

Boolean

```
member?((y+1)*(y-1), s)
```

$$\texttt{true} \tag{11}$$

Boolean

The subset? function determines whether one set is a subset of another.

```
subset?(i, s)
```

$$\texttt{true} \tag{12}$$

Boolean

```
subset?(u, s)
```

$$\texttt{false} \tag{13}$$

Boolean

When the base type is finite, the absolute complement of a set is defined. This finds the set of all multiplicative generators of **PrimeField 11**—the integers mod 11.

```
gs := set [g for i in 1..11 | primitive?(g := i::PF 11)]
```

$$\{2,\ 6,\ 7,\ 8\} \tag{14}$$

Set(PrimeField(11))

The following values are not generators.

```
complement gs
```

$$\{1,\ 3,\ 4,\ 5,\ 9,\ 10,\ 0\} \tag{15}$$

Set(PrimeField(11))

Often the members of a set are computed individually; in addition, values can be inserted or removed from a set over the course of a computation. There are two ways to do this:

```
a := set [i^2 for i in 1..5]
```

$$\{1,\ 4,\ 9,\ 16,\ 25\} \tag{16}$$

Set(PositiveInteger)

One is to view a set as a data structure and to apply updating operations.

```
insert!(32, a)
```

$$\{1,\ 4,\ 9,\ 16,\ 25,\ 32\} \tag{17}$$

Set(PositiveInteger)

```
remove!(25, a)
```

$$\{1,\ 4,\ 9,\ 16,\ 32\} \tag{18}$$

Set(PositiveInteger)

```
a
```

$$\{1,\ 4,\ 9,\ 16,\ 32\} \tag{19}$$

Set(PositiveInteger)

The other way is to view a set as a mathematical entity and to create new sets from old.

```
b := b0 := set [i^2 for i in 1..5]
```

$$\{1,\ 4,\ 9,\ 16,\ 25\} \tag{20}$$

Set(PositiveInteger)

```
b := union(b, {32})
```

$$\{1,\ 4,\ 9,\ 16,\ 25,\ 32\} \tag{21}$$

Set(PositiveInteger)

```
b := difference(b, {25})
```

$$\{1,\ 4,\ 9,\ 16,\ 32\} \tag{22}$$

Set(PositiveInteger)

```
b0
```

$$\{1,\ 4,\ 9,\ 16,\ 25\} \tag{23}$$

Set(PositiveInteger)

For more information about lists, see '`List`' on page 490. Issue the system command `)show Set` to display the full list of operations defined by **Set**.

9.76 SingleInteger

The **SingleInteger** domain is intended to provide support in FriCAS for machine integer arithmetic. It is generally much faster than (bignum) **Integer** arithmetic but suffers from a limited range of values. Since FriCAS can be implemented on top of various dialects of Lisp, the actual representation of small integers may not correspond exactly to the host machine's integer representation.

The underlying Lisp primitives treat machine-word sized computations specially.

You can discover the minimum and maximum values in your implementation by using min and max.

```
min()$SingleInteger
```

$$-\ 4611686018427387904 \tag{4}$$

SingleInteger

```
max()$SingleInteger
```

$$4611686018427387903 \tag{5}$$

SingleInteger

To avoid confusion with **Integer**, which is the default type for integers, you usually need to work with declared variables (Section 2.3 on page 74) ...

```
a := 1234 :: SingleInteger
```

$$1234 \tag{6}$$

SingleInteger

or use package calling (Section 2.9 on page 89).

```
b := 124$SingleInteger
```

$$124 \tag{7}$$

SingleInteger

You can add, multiply and subtract **SingleInteger** objects, and ask for the greatest common divisor (gcd).

```
gcd(a,b)
```

$$2 \tag{8}$$

SingleInteger

The least common multiple (lcm) is also available.

```
lcm(a,b)
```

$$76508 \tag{9}$$

SingleInteger

Operations mulmod, addmod, submod, and invmod are similar—they provide arithmetic modulo a given small integer. Here is `5 * 6 mod 13`.

```
mulmod(5,6,13)$SingleInteger
```

$$4 \tag{10}$$

SingleInteger

To reduce a small integer modulo a prime, use positiveRemainder.

```
positiveRemainder(37,13)$SingleInteger
```

$$11 \tag{11}$$

SingleInteger

Operations And, Or, xor, and Not provide bit level operations on small integers.

```
And(3,4)$SingleInteger
```

$$0 \tag{12}$$

SingleInteger

Use `shift(int,numToShift)` to shift bits, where `i` is shifted left if `numToShift` is positive, right if negative.

```
shift(1,4)$SingleInteger
```

$$16 \tag{13}$$

SingleInteger

```
shift(31,-1)$SingleInteger
```

$$15 \tag{14}$$

SingleInteger

Many other operations are available for small integers, including many of those provided for **Integer**. To see the other operations, use the Browse HyperDoc facility (Section 14 on page 687). Issue the system command `)show SingleInteger` to display the full list of operations defined by **SingleInteger**..

9.77 SparseTable

The **SparseTable** domain provides a general purpose table type with default entries. Here we create a table to save strings under integer keys. The value `"Try again!"` is returned if no other value has been stored for a key.

```
t: SparseTable(Integer, String, "Try again!") := table()
```

$$\texttt{table}() \tag{4}$$

SparseTable(Integer , String , "Try again!")

Entries can be stored in the table.

```
t.3 := "Number three"
```

$$\texttt{"Number three"} \tag{5}$$

String

```
t.4 := "Number four"
```

```
                                "Number four"                                (6)
                                                                      String
```

These values can be retrieved as usual, but if a lookup fails the default entry will be returned.

```
t.3
```

```
                                "Number three"                               (7)
                                                                      String
```

```
t.2
```

```
                                "Try again!"                                 (8)
                                                                      String
```

To see which values are explicitly stored, the keys and entries functions can be used.

```
keys t
```

```
                                   [4, 3]                                    (9)
                                                              List ( Integer )
```

```
entries t
```

```
                          ["Number four", "Number three"]                    (10)
                                                              List ( String )
```

If a specific table representation is required, the **GeneralSparseTable** constructor should be used. The domain **SparseTable(K, E, dflt)** is equivalent to **GeneralSparseTable(K,E, Table(K,E), dflt)**. For more information, see '`Table`' on page 587 and '`GeneralSparseTable`' on page 434. Issue the system command `)show SparseTable` to display the full list of operations defined by **SparseTable**.

9.78 SquareFreeRegularTriangularSet

The **SquareFreeRegularTriangularSet** domain constructor implements square-free regular triangular sets. See the **RegularTriangularSet** domain constructor for general regular triangular sets. Let T be a regular triangular set consisting of polynomials *t1, ..., tm* ordered by increasing main variables. The regular triangular set T is square-free if T is empty or if *t1, ..., tm-1* is square-free and if the polynomial *tm* is square-free as a univariate polynomial with coefficients in the tower of simple extensions associated with *t1, ..., tm-1*.

The main interest of square-free regular triangular sets is that their associated towers of simple extensions are product of fields. Consequently, the saturated ideal of a square-free regular triangular set is radical. This property simplifies some of the operations related to regular triangular sets. However, building square-free regular triangular sets is generally more expensive than building general regular triangular sets.

Like the **RegularTriangularSet** domain constructor, the **SquareFreeRegularTriangularSet** domain constructor also implements a method for solving polynomial systems by means of regular triangular sets. This is in fact the same method with some adaptations to take into account the fact that the computed regular chains are square-free. Note that it is also possible to pass from a decomposition into general regular triangular sets to a decomposition into square-free regular triangular sets. This conversion is used internally by the **LazardSetSolvingPackage** package constructor.

N.B. When solving polynomial systems with the **SquareFreeRegularTriangularSet** domain constructor or the **LazardSetSolvingPackage** package constructor, decompositions have no redundant components. See also **LexTriangularPackage** and **ZeroDimensionalSolvePackage** for the case of algebraic systems with a finite number of (complex) solutions.

We shall explain now how to use the constructor **SquareFreeRegularTriangularSet**.

This constructor takes four arguments. The first one, **R**, is the coefficient ring of the polynomials; it must belong to the category **GcdDomain**. The second one, **E**, is the exponent monoid of the polynomials; it must belong to the category **OrderedAbelianMonoidSup**. The third one, **V**, is the ordered set of variables; it must belong to the category **OrderedSet**. The last one is the polynomial ring; it must belong to the category **RecursivePolynomialCategory(R,E,V)**. The abbreviation for **SquareFreeRegularTriangularSet** is **SREGSET**.

Note that the way of understanding triangular decompositions is detailed in the example of the **RegularTriangularSet** constructor.

Let us illustrate the use of this constructor with one example (Donati-Traverso). Define the coefficient ring.

```
R := Integer
```

$$Integer \tag{4}$$

Type

Define the list of variables,

```
ls : List Symbol := [x,y,z,t]
```

$$[x,\ y,\ z,\ t] \tag{5}$$

List (Symbol)

and make it an ordered set;

```
V := OVAR(ls)
```

$$\text{OrderedVariableList}([x,\ y,\ z,\ t]) \tag{6}$$

Type

then define the exponent monoid.

```
E := IndexedExponents V
```

$$\text{IndexedExponents}(\text{OrderedVariableList}([x,\ y,\ z,\ t])) \tag{7}$$

Type

Define the polynomial ring.

```
P := NSMP(R, V)
```

$$\text{NewSparseMultivariatePolynomial}(Integer, \text{OrderedVariableList}([x, y, z, t])) \tag{8}$$

Type

Let the variables be polynomial.

```
x: P := 'x
```

$$x \tag{9}$$

NewSparseMultivariatePolynomial(Integer , OrderedVariableList ([x, y, z, t]))

```
y: P := 'y
```

$$y \tag{10}$$

NewSparseMultivariatePolynomial(Integer , OrderedVariableList ([x, y, z, t]))

```
z: P := 'z
```

$$z \tag{11}$$

NewSparseMultivariatePolynomial(Integer , OrderedVariableList ([x, y, z, t]))

```
t: P := 't
```

$$t \tag{12}$$

NewSparseMultivariatePolynomial(Integer , OrderedVariableList ([x, y, z, t]))

Now call the **SquareFreeRegularTriangularSet** domain constructor.

```
ST := SREGSET(R,E,V,P)
```

$$\begin{aligned}&\text{SquareFreeRegularTriangularSet}(Integer, \text{IndexedExponents}(\text{OrderedVariableList}([x,\\ &y, z, t])), \text{OrderedVariableList}([x, y, z,\\ &t]), \text{NewSparseMultivariatePolynomial}(Integer, \text{OrderedVariableList}([x, y, z, t])))\end{aligned} \tag{13}$$

Type

Define a polynomial system.

```
p1 := x ^ 31 - x ^ 6 - x - y
```

$$x^{31} - x^6 - x - y \tag{14}$$

NewSparseMultivariatePolynomial(Integer , OrderedVariableList ([x, y, z, t]))

```
p2 := x ^ 8  - z
```

$$x^8 - z \tag{15}$$

NewSparseMultivariatePolynomial(Integer, OrderedVariableList([x, y, z, t]))

```
p3 := x ^ 10 - t
```

$$x^{10} - t \tag{16}$$

NewSparseMultivariatePolynomial(Integer, OrderedVariableList([x, y, z, t]))

```
lp := [p1, p2, p3]
```

$$\left[x^{31} - x^6 - x - y,\ x^8 - z,\ x^{10} - t\right] \tag{17}$$

List(NewSparseMultivariatePolynomial(Integer, OrderedVariableList([x, y, z, t])))

First of all, let us solve this system in the sense of Kalkbrener.

```
zeroSetSplit(lp)$ST
```

$$\left[\left\{z^5 - t^4,\ t\,z\,y^2 + 2\,z^3\,y - t^8 + 2\,t^5 + t^3 - t^2,\ \left(t^4 - t\right)\,x - t\,y - z^2\right\}\right] \tag{18}$$

List(SquareFreeRegularTriangularSet(Integer, IndexedExponents(OrderedVariableList([x, y, z, t])), OrderedVariableList([x, y, z, t]), NewSparseMultivariatePolynomial(Integer, OrderedVariableList([x, y, z, t]))))

And now in the sense of Lazard (or Wu and other authors).

```
zeroSetSplit(lp,false)$ST
```

$$\left[\left\{z^5 - t^4,\ t\,z\,y^2 + 2\,z^3\,y - t^8 + 2\,t^5 + t^3 - t^2,\ \left(t^4 - t\right)\,x - t\,y - z^2\right\},\right.$$
$$\left.\left\{t^3 - 1,\ z^5 - t,\ t\,y + z^2,\ z\,x^2 - t\right\},\ \{t,\ z,\ y,\ x\}\right] \tag{19}$$

List(SquareFreeRegularTriangularSet(Integer, IndexedExponents(OrderedVariableList([x, y, z, t])), OrderedVariableList([x, y, z, t]), NewSparseMultivariatePolynomial(Integer, OrderedVariableList([x, y, z, t]))))

Now to see the difference with the **RegularTriangularSet** domain constructor, we define:

```
T := REGSET(R,E,V,P)
```

$$\text{RegularTriangularSet}(Integer, \text{IndexedExponents}(\text{OrderedVariableList}([x,\ y,\ z,\ t])), \text{OrderedVariableList}([x,\ y,\ z,\ t]), \text{NewSparseMultivariatePolynomial}(Integer, \text{OrderedVariableList}([x,\ y,\ z,\ t]))) \tag{20}$$

Type

and compute:

```
lts := zeroSetSplit(lp,false)$T
```

$$\left[\left\{z^5 - t^4,\ t\,z\,y^2 + 2\,z^3\,y - t^8 + 2\,t^5 + t^3 - t^2,\ \left(t^4 - t\right)\,x - t\,y - z^2\right\},\right.$$
$$\left.\left\{t^3 - 1,\ z^5 - t,\ t\,z\,y^2 + 2\,z^3\,y + 1,\ z\,x^2 - t\right\},\ \{t,\ z,\ y,\ x\}\right] \tag{21}$$

List (RegularTriangularSet (Integer , IndexedExponents(OrderedVariableList ([x, y, z, t])) , OrderedVariableList ([x, y, z , t]) , NewSparseMultivariatePolynomial(Integer , OrderedVariableList ([x, y, z, t]))))

If you look at the second set in both decompositions in the sense of Lazard, you will see that the polynomial with main variable **y** is not the same.

Let us understand what has happened. We define:

```
ts := lts.2
```

$$\left\{t^3-1,\ z^5-t,\ t\,z\,y^2+2\,z^3\,y+1,\ z\,x^2-t\right\} \tag{22}$$

RegularTriangularSet (Integer , IndexedExponents(OrderedVariableList ([x, y, z, t])) , OrderedVariableList ([x, y, z, t]) , NewSparseMultivariatePolynomial(Integer , OrderedVariableList ([x, y, z, t])))

```
pol := select(ts,'y)$T
```

$$t\,z\,y^2+2\,z^3\,y+1 \tag{23}$$

Union(NewSparseMultivariatePolynomial(Integer , OrderedVariableList ([x, y, z, t])) , ...)

```
tower := collectUnder(ts,'y)$T
```

$$\left\{t^3-1,\ z^5-t\right\} \tag{24}$$

RegularTriangularSet (Integer , IndexedExponents(OrderedVariableList ([x, y, z, t])) , OrderedVariableList ([x, y, z, t]) , NewSparseMultivariatePolynomial(Integer , OrderedVariableList ([x, y, z, t])))

```
pack := RegularTriangularSetGcdPackage(R,E,V,P,T)
```

$$\begin{aligned}&\text{RegularTriangularSetGcdPackage}(Integer, \text{IndexedExponents}(\text{OrderedVariableList}([x,\\ &\quad y,\ z,\ t])), \text{OrderedVariableList}([x,\ y,\ z,\\ &\quad t]), \text{NewSparseMultivariatePolynomial}(Integer, \text{OrderedVariableList}([x,\ y,\ z,\\ &\quad t])), \text{RegularTriangularSet}(Integer, \text{IndexedExponents}(\text{OrderedVariableList}([x,\\ &\quad y,\ z,\ t])), \text{OrderedVariableList}([x,\ y,\ z,\\ &\quad t]), \text{NewSparseMultivariatePolynomial}(Integer, \text{OrderedVariableList}([x,\ y,\ z,\ t]))))\end{aligned} \tag{25}$$

Type

Then we compute:

```
toseSquareFreePart(pol,tower)$pack
```

$$\left[\left[val=t\,y+z^2,\ tower=\left\{t^3-1,\ z^5-t\right\}\right]\right] \tag{26}$$

List (Record(val : NewSparseMultivariatePolynomial(Integer , OrderedVariableList ([x, y, z, t])) , tower : RegularTriangularSet (Integer , IndexedExponents(OrderedVariableList ([x, y, z, t])) , OrderedVariableList ([x, y, z, t]) , NewSparseMultivariatePolynomial(Integer , OrderedVariableList ([x, y, z, t])))))

9.79 SquareMatrix

The top level matrix type in FriCAS is **Matrix** (see 'Matrix' on page 503), which provides basic arithmetic and linear algebra functions. However, since the matrices can be of any size, it is not true that any pair can be added or multiplied. Thus **Matrix** has little algebraic structure.

Sometimes you want to use matrices as coefficients for polynomials or in other algebraic contexts. In this case, **SquareMatrix** should be used. The domain **SquareMatrix(n,R)** gives the ring of n by n square matrices over R.

Since **SquareMatrix** is not normally exposed at the top level, you must expose it before it can be used.

```
)set expose add constructor SquareMatrix
```

```
   SquareMatrix is now explicitly exposed in frame initial
```

Once SQMATRIX has been exposed, values can be created using the squareMatrix function.

```
m := squareMatrix [[1,-%i],[%i,4]]
```

$$\begin{bmatrix} 1 & -i \\ i & 4 \end{bmatrix} \tag{4}$$

SquareMatrix(2, Complex(Integer))

The usual arithmetic operations are available.

```
m*m - m
```

$$\begin{bmatrix} 1 & -4\,i \\ 4\,i & 13 \end{bmatrix} \tag{5}$$

SquareMatrix(2, Complex(Integer))

Square matrices can be used where ring elements are required. For example, here is a matrix with matrix entries.

```
mm := squareMatrix [[m, 1], [1-m, m^2]]
```

$$\begin{bmatrix} \begin{bmatrix} 1 & -i \\ i & 4 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 0 & i \\ -i & -3 \end{bmatrix} & \begin{bmatrix} 2 & -5\,i \\ 5\,i & 17 \end{bmatrix} \end{bmatrix} \tag{6}$$

SquareMatrix(2, SquareMatrix(2, Complex(Integer)))

Or you can construct a polynomial with square matrix coefficients.

```
p := (x + m)^2
```

$$x^2 + \begin{bmatrix} 2 & -2\,i \\ 2\,i & 8 \end{bmatrix} x + \begin{bmatrix} 2 & -5\,i \\ 5\,i & 17 \end{bmatrix} \tag{7}$$

Polynomial(SquareMatrix(2, Complex(Integer)))

This value can be converted to a square matrix with polynomial coefficients.

```
p::SquareMatrix(2, ?)
```

$$\begin{bmatrix} x^2+2\,x+2 & -2\,i\,x-5\,i \\ 2\,i\,x+5\,i & x^2+8\,x+17 \end{bmatrix} \tag{8}$$

SquareMatrix(2, Polynomial(Complex(Integer)))

For more information on related topics, see Section 2.2.4 on page 73, Section 2.11 on page 94, and '`Matrix`' on page 503. Issue the system command `)show SquareMatrix` to display the full list of operations defined by **SquareMatrix**.

9.80 Stream

A **Stream** object is represented as a list whose last element contains the wherewithal to create the next element, should it ever be required. Let `ints` be the infinite stream of non-negative integers.

```
ints := [i for i in 0..]
```

$$[0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ \ldots] \tag{4}$$

Stream(NonNegativeInteger)

By default, ten stream elements are calculated. This number may be changed to something else by the system command `)set streams calculate`. For the display purposes of this book, we have chosen a smaller value. More generally, you can construct a stream by specifying its initial value and a function which, when given an element, creates the next element.

```
f : List INT -> List INT
```

```
f x == [x.1 + x.2, x.1]
```

```
fibs := [i.2 for i in [stream(f, [1, 1])]]
```

```
   Compiling function f with type List(Integer) -> List(Integer)
```

$$[1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ \ldots] \tag{7}$$

Stream(Integer)

You can create the stream of odd non-negative integers by either filtering them from the integers, or by evaluating an expression for each integer.

```
[i for i in ints | odd? i]
```

$$[1,\ 3,\ 5,\ 7,\ 9,\ 11,\ 13,\ \ldots] \tag{8}$$

Stream(NonNegativeInteger)

```
odds := [2*i+1 for i in ints]
```

$$[1,\ 3,\ 5,\ 7,\ 9,\ 11,\ 13,\ \ldots] \tag{9}$$

Stream(NonNegativeInteger)

You can accumulate the initial segments of a stream using the scan operation.

```
scan(0,+,odds)
```

$$[1,\ 4,\ 9,\ 16,\ 25,\ 36,\ 49,\ \ldots] \tag{10}$$

Stream(NonNegativeInteger)

The corresponding elements of two or more streams can be combined in this way.

```
[i*j for i in ints for j in odds]
```

$$[0,\ 3,\ 10,\ 21,\ 36,\ 55,\ 78,\ \ldots] \tag{11}$$

Stream(NonNegativeInteger)

```
map(*,ints,odds)
```

$$[0,\ 3,\ 10,\ 21,\ 36,\ 55,\ 78,\ \ldots] \tag{12}$$

Stream(NonNegativeInteger)

Many operations similar to those applicable to lists are available for streams.

```
first ints
```

$$0 \tag{13}$$

NonNegativeInteger

```
rest ints
```

$$[1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ \ldots] \tag{14}$$

Stream(NonNegativeInteger)

```
fibs 20
```

$$6765 \tag{15}$$

PositiveInteger

The packages **StreamFunctions1**, **StreamFunctions2** and **StreamFunctions3** export some useful stream manipulation operations. For more information, see Section 5.5 on page 143, Section 8.9 on page 282, '`ContinuedFraction`' on page 375, and '`List`' on page 490. Issue the system command `)show Stream` to display the full list of operations defined by **Stream**.

9.81 String

The type **String** provides character strings. Character strings provide all the operations for a one-dimensional array of characters, plus additional operations for manipulating text. For more information on related topics, see '`Character`' on page 363 and '`CharacterClass`' on page 365. You can also issue the system command `)show String` to display the full list of operations defined by **String**.

String values can be created using double quotes.

```
hello := "Hello, I'm FriCAS!"
```

"Hello, I'm FriCAS!" (4)

String

Note, however, that double quotes and underscores must be preceded by an extra underscore.

```
said  := "Jane said, _"Look!_""
```

"Jane said, "Look!"" (5)

String

```
saw   := "She saw exactly one underscore: __."
```

"She saw exactly one underscore: _." (6)

String

It is also possible to use new to create a string of any size filled with a given character. Since there are many new functions it is necessary to indicate the desired type.

```
gasp: String := new(32, char "x")
```

"xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx" (7)

String

The length of a string is given by #.

```
#gasp
```

32 (8)

PositiveInteger

Indexing operations allow characters to be extracted or replaced in strings. For any string `s`, indices lie in the range `1..#s`.

```
hello.2
```

e (9)

Character

Indexing is really just the application of a string to a subscript, so any application syntax works.

```
hello 2
```

$$\texttt{e} \tag{10}$$

Character

```
hello(2)
```

$$\texttt{e} \tag{11}$$

Character

If it is important not to modify a given string, it should be copied before any updating operations are used.

```
hullo := copy hello
```

$$\texttt{"Hello, I'm FriCAS!"} \tag{12}$$

String

```
hullo.2 := char "u"; [hello, hullo]
```

$$[\texttt{"Hello, I'm FriCAS!"}, \texttt{"Hullo, I'm FriCAS!"}] \tag{13}$$

List (String)

Operations are provided to split and join strings. The concat operation allows several strings to be joined together.

```
saidsaw := concat ["alpha","---","omega"]
```

$$\texttt{"alpha---omega"} \tag{14}$$

String

There is a version of concat that works with two strings.

```
concat("hello ","goodbye")
```

$$\texttt{"hello goodbye"} \tag{15}$$

String

Juxtaposition can also be used to concatenate strings.

```
"This " "is " "several " "strings " "concatenated."
```

$$\texttt{"This is several strings concatenated."} \tag{16}$$

String

Substrings are obtained by giving an index range.

```
hello(1..5)
```

"Hello" (17)

String

```
hello(8..)
```

"I'm FriCAS!" (18)

String

A string can be split into several substrings by giving a separation character or character class.

```
split(hello, char " ")
```

["Hello,", "I'm", "FriCAS!"] (19)

List (String)

```
other := complement alphanumeric();
```

CharacterClass

```
split(saidsaw, other)
```

["alpha", "omega"] (21)

List (String)

Unwanted characters can be trimmed from the beginning or end of a string using the operations trim, leftTrim and rightTrim.

```
trim     ("## ++ relax ++ ##", char "#")
```

" ++ relax ++ " (22)

String

Each of these functions takes a string and a second argument to specify the characters to be discarded.

```
trim     ("## ++ relax ++ ##", other)
```

"relax" (23)

String

The second argument can be given either as a single character or as a character class.

```
leftTrim ("## ++ relax ++ ##", other)
```

"relax ++ ##" (24)

String

```
rightTrim("## ++ relax ++ ##", other)
```

"## ++ relax" (25)

String

Strings can be changed to upper case or lower case using the operations upperCase, upperCase!, lowerCase and lowerCase!.

```
upperCase hello
```

"HELLO, I'M FRICAS!" (26)

String

The versions with the exclamation mark change the original string, while the others produce a copy.

```
lowerCase hello
```

"hello, i'm fricas!" (27)

String

Some basic string matching is provided. The function prefix? tests whether one string is an initial prefix of another.

```
prefix?("He", "Hello")
```

true (28)

Boolean

```
prefix?("Her", "Hello")
```

false (29)

Boolean

A similar function, suffix?, tests for suffixes.

```
suffix?("", "Hello")
```

true (30)

Boolean

```
suffix?("LO", "Hello")
```

false (31)

Boolean

The function substring? tests for a substring given a starting position.

```
substring?("ll", "Hello", 3)
```

```
                                    true                                    (32)
                                                                        Boolean
substring?("ll", "Hello", 4)
                                    false                                   (33)
                                                                        Boolean
```

A number of position functions locate things in strings. If the first argument to position is a string, then `position(s,t,i)` finds the location of `s` as a substring of `t` starting the search at position `i`.

```
n := position("nd", "underground",   1)
                                     2                                      (34)
                                                                    PositiveInteger
n := position("nd", "underground", n+1)
                                    10                                      (35)
                                                                    PositiveInteger
```

If `s` is not found, then `0` is returned (`minIndex(s)-1` in **IndexedString**).

```
n := position("nd", "underground", n+1)
                                     0                                      (36)
                                                                    NonNegativeInteger
```

To search for a specific character or a member of a character class, a different first argument is used.

```
position(char "d", "underground", 1)
                                     3                                      (37)
                                                                    PositiveInteger
position(hexDigit(), "underground", 1)
                                     3                                      (38)
                                                                    PositiveInteger
```

9.82 StringTable

This domain provides a table type in which the keys are known to be strings, so special techniques can be used. Other than performance, the type **StringTable(S)** should behave exactly the same way as **Table(String,S)**. See '`Table`' on page 587 for general information about tables. Issue the system command `)show StringTable` to display the full list of operations defined by **StringTable**.

This creates a new table whose keys are strings.

```
t: StringTable(Integer) := table()
```

$$\texttt{table}() \tag{4}$$

StringTable(Integer)

The value associated with each string key is the number of characters in the string.

```
for s in split("My name is Ian Watt.",char " ")
  repeat
    t.s := #s
```

```
for key in keys t repeat output [key, t.key]
```

```
   ["Watt.", 5]
   ["Ian", 3]
   ["is", 2]
   ["name", 4]
   ["My", 2]
```

9.83 Symbol

Symbols are one of the basic types manipulated by FriCAS. The **Symbol** domain provides ways to create symbols of many varieties. Issue the system command `)show Symbol` to display the full list of operations defined by **Symbol**.

The simplest way to create a symbol is to "single quote" an identifier.

```
X: Symbol := 'x
```

$$x \tag{4}$$

Symbol

This gives the symbol even if `x` has been assigned a value. If `x` has not been assigned a value, then it is possible to omit the quote.

```
XX: Symbol := x
```

$$x \tag{5}$$

Symbol

Declarations must be used when working with symbols, because otherwise the interpreter tries to place values in a more specialized type **Variable**.

```
A := 'a
```

$$a \tag{6}$$

Variable (a)

```
B := b
```

$$b \tag{7}$$

Variable (b)

The normal way of entering polynomials uses this fact.

```
x^2 + 1
```

$$x^2 + 1 \tag{8}$$

Polynomial(Integer)

Another convenient way to create symbols is to convert a string. This is useful when the name is to be constructed by a program.

```
"Hello"::Symbol
```

$$Hello \tag{9}$$

Symbol

Sometimes it is necessary to generate new unique symbols, for example, to name constants of integration. The expression `new()` generates a symbol starting with `%`.

```
new()$Symbol
```

$$\%A \tag{10}$$

Symbol

Successive calls to new produce different symbols.

```
new()$Symbol
```

$$\%B \tag{11}$$

Symbol

The expression `new("s")` produces a symbol starting with `%s`.

```
new("xyz")$Symbol
```

$$\%xyz0 \tag{12}$$

Symbol

A symbol can be adorned in various ways. The most basic thing is applying a symbol to a list of subscripts.

```
X[i,j]
```

$$x_{i,\,j} \tag{13}$$

Symbol

Somewhat less pretty is to attach subscripts, superscripts or arguments.

```
U := subscript(u, [1,2,1,2])
```

$$u_{1,\,2,\,1,\,2} \tag{14}$$

Symbol

```
V := superscript(v, [n])
```

$$v^{n} \tag{15}$$

Symbol

```
P := argscript(p, [t])
```

$$p(t) \tag{16}$$

Symbol

It is possible to test whether a symbol has scripts using the scripted? test.

```
scripted? U
```

```
true
```
(17)

Boolean

```
scripted? X
```

```
false
```
(18)

Boolean

If a symbol is not scripted, then it may be converted to a string.

```
string X
```

```
"x"
```
(19)

String

The basic parts can always be extracted using the name and scripts operations.

```
name U
```

$$u \tag{20}$$

Symbol

```
scripts U
```

$$[sub = [1,\ 2,\ 1,\ 2]\,,\ sup = []\,,\ presup = []\,,\ presub = []\,,\ args = []] \tag{21}$$

Record(sub: List(OutputForm), sup: List(OutputForm), presup: List(OutputForm), presub: List(OutputForm), args: List(OutputForm))

```
name X
```

$$x \tag{22}$$

Symbol

```
scripts X
```

$$[sub = [],\ sup = [],\ presup = [],\ presub = [],\ args = []] \tag{23}$$

Record(sub: List(OutputForm), sup: List(OutputForm), presup: List(OutputForm), presub: List(OutputForm), args: List(OutputForm))

The most general form is obtained using the script operation. This operation takes an argument which is a list containing, in this order, lists of subscripts, superscripts, presuperscripts, presubscripts and arguments to a symbol.

```
M := script(Mammoth, [[i,j],[k,l],[0,1],[2],[u,v,w]])
```

$$^{0,\,1}_{2}Mammoth^{k,\,l}_{i,\,j}(u, v, w) \tag{24}$$

Symbol

```
scripts M
```

$$[sub = [i,\ j],\ sup = [k,\ l],\ presup = [0,\ 1],\ presub = [2],\ args = [u,\ v,\ w]] \tag{25}$$

Record(sub: List(OutputForm), sup: List(OutputForm), presup: List(OutputForm), presub: List(OutputForm), args: List(OutputForm))

If trailing lists of scripts are omitted, they are assumed to be empty.

```
N := script(Nut, [[i,j],[k,l],[0,1]])
```

$$^{0,\,1}Nut^{k,\,l}_{i,\,j} \tag{26}$$

Symbol

```
scripts N
```

$$[sub = [i,\ j],\ sup = [k,\ l],\ presup = [0,\ 1],\ presub = [],\ args = []] \tag{27}$$

Record(sub: List(OutputForm), sup: List(OutputForm), presup: List(OutputForm), presub: List(OutputForm), args: List(OutputForm))

9.84 Table

The **Table** constructor provides a general structure for associative storage. This type provides hash tables in which data objects can be saved according to keys of any type. For a given table, specific types must be chosen for the keys and entries.

In this example the keys to the table are polynomials with integer coefficients. The entries in the table are strings.

```
t: Table(Polynomial Integer, String) := table()
```

$$\texttt{table}() \tag{4}$$

Table(Polynomial(Integer), String)

To save an entry in the table, the setelt! operation is used. This can be called directly, giving the table a key and an entry.

```
setelt!(t, x^2 - 1, "Easy to factor")
```

$$\texttt{"Easy to factor"} \tag{5}$$

String

Alternatively, you can use assignment syntax.

```
t(x^3 + 1) := "Harder to factor"
```

$$\texttt{"Harder to factor"} \tag{6}$$

String

```
t(x)         := "The easiest to factor"
```

$$\texttt{"The easiest to factor"} \tag{7}$$

String

Entries are retrieved from the table by calling the elt operation.

```
elt(t, x)
```

$$\texttt{"The easiest to factor"} \tag{8}$$

String

This operation is called when a table is "applied" to a key using this or the following syntax.

```
t.x
```

$$\texttt{"The easiest to factor"} \tag{9}$$

String

```
t x
```

$$\texttt{"The easiest to factor"} \tag{10}$$

String

Parentheses are used only for grouping. They are needed if the key is an infixed expression.

```
t.(x^2 - 1)
```

```
"Easy to factor"
```
(11)

String

Note that the elt operation is used only when the key is known to be in the table—otherwise an error is generated.

```
t (x^3 + 1)
```

```
"Harder to factor"
```
(12)

String

You can get a list of all the keys to a table using the keys operation.

```
keys t
```

$$[x,\ x^3+1,\ x^2-1]$$ (13)

List(Polynomial(Integer))

If you wish to test whether a key is in a table, the search operation is used. This operation returns either an entry or `"failed"`.

```
search(x, t)
```

```
"The easiest to factor"
```
(14)

Union(String, ...)

```
search(x^2, t)
```

```
"failed"
```
(15)

Union("failed", ...)

The return type is a union so the success of the search can be tested using `case`.

```
search(x^2, t) case "failed"
```

```
true
```
(16)

Boolean

The remove! operation is used to delete values from a table.

```
remove!(x^2-1, t)
```

```
"Easy to factor"
```
(17)

Union(String, ...)

If an entry exists under the key, then it is returned. Otherwise remove! returns `"failed"`.

```
remove!(x-1, t)
```

```
                                    "failed"                                    (18)
                                                                  Union(" failed ", ...)
```

The number of key-entry pairs can be found using the # operation.

```
#t
```

```
                                       2                                        (19)
                                                                  PositiveInteger
```

Just as keys returns a list of keys to the table, a list of all the entries can be obtained using the members operation.

```
members t
```

```
                    ["The easiest to factor", "Harder to factor"]               (20)
                                                                  List ( String )
```

A number of useful operations take functions and map them on to the table to compute the result. Here we count the entries which have "Hard" as a prefix.

```
count(s: String +-> prefix?("Hard", s), t)
```

```
                                       1                                        (21)
                                                                  PositiveInteger
```

Other table types are provided to support various needs.

- **AssociationList** gives a list with a table view. This allows new entries to be appended onto the front of the list to cover up old entries. This is useful when table entries need to be stacked or when frequent list traversals are required. See '`AssociationList`' on page 341 for more information.
- **EqTable** gives tables in which keys are considered equal only when they are in fact the same instance of a structure. See '`EqTable`' on page 397 for more information.
- **StringTable** should be used when the keys are known to be strings. See '`StringTable`' on page 583 for more information.
- **SparseTable** provides tables with default entries, so lookup never fails. The **GeneralSparseTable** constructor can be used to make any table type behave this way. See '`SparseTable`' on page 570 for more information.
- **KeyedAccessFile** allows values to be saved in a file, accessed as a table. See '`KeyedAccessFile`' on page 457 for more information.

Issue the system command `)show Table` to display the full list of operations defined by **Table**.

9.85 TextFile

The domain **TextFile** allows FriCAS to read and write character data and exchange text with other programs. This type behaves in FriCAS much like a **File** of strings, with additional operations to cause new lines. We give an example of how to produce an upper case copy of a file. This is the file from which we read the text.

```
f1: TextFile := open("/etc/group", "input")
```

"/etc/group" (4)

TextFile

This is the file to which we write the text.

```
f2: TextFile := open("/tmp/MOTD", "output")
```

"/tmp/MOTD" (5)

TextFile

Entire lines are handled using the readLine! and writeLine! operations.

```
l := readLine! f1
```

"root:x:0:" (6)

String

```
writeLine!(f2, upperCase l)
```

"ROOT:X:0:" (7)

String

Use the endOfFile? operation to check if you have reached the end of the file.

```
while not endOfFile? f1 repeat
    s := readLine! f1
    writeLine!(f2, upperCase s)
```

The file `f1` is exhausted and should be closed.

```
close! f1
```

"/etc/group" (9)

TextFile

It is sometimes useful to write lines a bit at a time. The write! operation allows this.

```
write!(f2, "-The-")
```

"-The-" (10)

String

```
write!(f2, "-End-")
```

$$\text{"-End-"} \tag{11}$$

String

This ends the line. This is done in a machine-dependent manner.

```
writeLine! f2
```

$$\text{""} \tag{12}$$

String

```
close! f2
```

$$\text{"/tmp/MOTD"} \tag{13}$$

TextFile

Finally, clean up.

```
)system rm /tmp/MOTD
```

For more information on related topics, see '`File`' on page 412, '`KeyedAccessFile`' on page 457, and '`Library`' on page 474. Issue the system command `)show TextFile` to display the full list of operations defined by **TextFile**.

9.86 TwoDimensionalArray

The **TwoDimensionalArray** domain is used for storing data in a two-dimensional data structure indexed by row and by column. Such an array is a homogeneous data structure in that all the entries of the array must belong to the same FriCAS domain (although see Section 2.6 on page 83). Each array has a fixed number of rows and columns specified by the user and arrays are not extensible. In FriCAS, the indexing of two-dimensional arrays is one-based. This means that both the "first" row of an array and the "first" column of an array are given the index `1`. Thus, the entry in the upper left corner of an array is in position `(1,1)`.

The operation new creates an array with a specified number of rows and columns and fills the components of that array with a specified entry. The arguments of this operation specify the number of rows, the number of columns, and the entry. This creates a five-by-four array of integers, all of whose entries are zero.

```
arr : ARRAY2 INT := new(5,4,0)
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4}$$

TwoDimensionalArray(Integer)

The entries of this array can be set to other integers using the operation setelt!.

Issue this to set the element in the upper left corner of this array to 17.

```
setelt!(arr, 1, 1, 17)
```

$$17 \tag{5}$$

PositiveInteger

Now the first element of the array is 17.

```
arr
```

$$\begin{bmatrix} 17 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{6}$$

TwoDimensionalArray(Integer)

Likewise, elements of an array are extracted using the operation elt.

```
elt(arr,1,1)
```

$$17 \tag{7}$$

PositiveInteger

Another way to use these two operations is as follows. This sets the element in position (3,2) of the array to 15.

```
arr(3,2) := 15
```

$$15 \tag{8}$$

PositiveInteger

This extracts the element in position (3,2) of the array.

```
arr(3,2)
```

$$15 \tag{9}$$

PositiveInteger

The operations elt and setelt! come equipped with an error check which verifies that the indices are in the proper ranges. For example, the above array has five rows and four columns, so if you ask for the entry in position (6,2) with arr(6,2) FriCAS displays an error message. If there is no need for an error check, you can call the operations qelt and qsetelt! which provide the same functionality but without the error check. Typically, these operations are called in well-tested programs.

The operations row and column extract rows and columns, respectively, and return objects of **OneDimensionalArray** with the same underlying element type.

```
row(arr,1)
```

$$[17,\ 0,\ 0,\ 0] \tag{10}$$

OneDimensionalArray(Integer)

```
column(arr,1)
```

$$[17,\ 0,\ 0,\ 0,\ 0] \tag{11}$$

OneDimensionalArray(Integer)

You can determine the dimensions of an array by calling the operations nrows and ncols, which return the number of rows and columns, respectively.

```
nrows(arr)
```

$$5 \tag{12}$$

PositiveInteger

```
ncols(arr)
```

$$4 \tag{13}$$

PositiveInteger

To apply an operation to every element of an array, use map. This creates a new array. This expression negates every element.

```
map(-,arr)
```

$$\begin{bmatrix} -17 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -15 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{14}$$

TwoDimensionalArray(Integer)

This creates an array where all the elements are doubled.

```
map((x +-> x + x),arr)
```

$$\begin{bmatrix} 34 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 30 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{15}$$

TwoDimensionalArray(Integer)

To change the array destructively, use map! instead of map. If you need to make a copy of any array, use copy.

```
arrc := copy(arr)
```

$$\begin{bmatrix} 17 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 15 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{16}$$

TwoDimensionalArray(Integer)

```
map!(-,arrc)
```

$$\begin{bmatrix} -17 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -15 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{17}$$

TwoDimensionalArray(Integer)

```
arrc
```

$$\begin{bmatrix} -17 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -15 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{18}$$

TwoDimensionalArray(Integer)

```
arr
```

$$\begin{bmatrix} 17 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 15 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{19}$$

TwoDimensionalArray(Integer)

Use member? to see if a given element is in an array.

```
member?(17,arr)
```

```
true
```
(20)

Boolean

```
member?(10317,arr)
```

```
false
```
(21)

Boolean

To see how many times an element appears in an array, use count.

```
count(17,arr)
```

$$1 \tag{22}$$

PositiveInteger

```
count(0,arr)
```

$$18 \tag{23}$$

PositiveInteger

For more information about the operations available for **TwoDimensionalArray**, issue `)show TwoDimensionalArray`. For information on related topics, see '`Matrix`' on page 503 and '`OneDimensionalArray`' on page 517.

9.87 UnivariatePolynomial

The domain constructor **UnivariatePolynomial** (abbreviated **UP**) creates domains of univariate polynomials in a specified variable. For example, the domain **UP(a1,POLY FRAC INT)** provides polynomials in the single variable `a1` whose coefficients are general polynomials with rational number coefficients.

Restriction:

FriCAS does not allow you to create types where **UnivariatePolynomial** is contained in the coefficient type of **Polynomial**. Therefore, **UP(x,POLY INT)** is legal but **POLY UP(x,INT)** is not.

UP(x,INT) is the domain of polynomials in the single variable `x` with integer coefficients.

```
(p,q) : UP(x,INT)
```

```
p := (3*x-1)^2 * (2*x + 8)
```

$$18\,x^3 + 60\,x^2 - 46\,x + 8 \tag{5}$$

UnivariatePolynomial(x, Integer)

```
q := (1 - 6*x + 9*x^2)^2
```

$$81\,x^4 - 108\,x^3 + 54\,x^2 - 12\,x + 1 \tag{6}$$

UnivariatePolynomial(x, Integer)

The usual arithmetic operations are available for univariate polynomials.

```
p^2 + p*q
```

$$1458\,x^7 + 3240\,x^6 - 7074\,x^5 + 10584\,x^4 - 9282\,x^3 + 4120\,x^2 - 878\,x + 72 \tag{7}$$

UnivariatePolynomial(x, Integer)

The operation leadingCoefficient extracts the coefficient of the term of highest degree.

```
leadingCoefficient p
```

$$18 \tag{8}$$

PositiveInteger

The operation degree returns the degree of the polynomial. Since the polynomial has only one variable, the variable is not supplied to operations like degree.

```
degree p
```

$$3 \tag{9}$$

PositiveInteger

The reductum of the polynomial, the polynomial obtained by subtracting the term of highest order, is returned by reductum.

```
reductum p
```

$$60\,x^2 - 46\,x + 8 \tag{10}$$

UnivariatePolynomial(x, Integer)

The operation gcd computes the greatest common divisor of two polynomials.

```
gcd(p,q)
```

$$9\,x^2 - 6\,x + 1 \tag{11}$$

UnivariatePolynomial(x, Integer)

The operation lcm computes the least common multiple.

```
lcm(p,q)
```

$$162\,x^5 + 432\,x^4 - 756\,x^3 + 408\,x^2 - 94\,x + 8 \tag{12}$$

UnivariatePolynomial(x, Integer)

The operation resultant computes the resultant of two univariate polynomials. In the case of p and q, the resultant is 0 because they share a common root.

```
resultant(p,q)
```

$$0 \tag{13}$$

NonNegativeInteger

To compute the derivative of a univariate polynomial with respect to its variable, use D.

```
D p
```

$$54\,x^2 + 120\,x - 46 \tag{14}$$

UnivariatePolynomial(x, Integer)

Univariate polynomials can also be used as if they were functions. To evaluate a univariate polynomial at some point, apply the polynomial to the point.

```
p(2)
```

$$300 \tag{15}$$

PositiveInteger

The same syntax is used for composing two univariate polynomials, i.e. substituting one polynomial for the variable in another. This substitutes q for the variable in p.

```
p(q)
```

$$\begin{aligned} &9565938\,x^{12} - 38263752\,x^{11} + 70150212\,x^{10} - 77944680\,x^9 + 58852170\,x^8 - 32227632\,x^7 \\ &+ 13349448\,x^6 - 4280688\,x^5 + 1058184\,x^4 - 192672\,x^3 + 23328\,x^2 - 1536\,x + 40 \end{aligned} \tag{16}$$

UnivariatePolynomial(x, Integer)

This substitutes p for the variable in q.

```
q(p)
```

$$\begin{aligned} &8503056\,x^{12} + 113374080\,x^{11} + 479950272\,x^{10} + 404997408\,x^9 \\ &- 1369516896\,x^8 - 626146848\,x^7 + 2939858712\,x^6 - 2780728704\,x^5 \\ &+ 1364312160\,x^4 - 396838872\,x^3 + 69205896\,x^2 - 6716184\,x + 279841 \end{aligned} \tag{17}$$

UnivariatePolynomial(x, Integer)

To obtain a list of coefficients of the polynomial, use coefficients.

```
l := coefficients p
```

$$[18,\ 60,\ -46,\ 8] \tag{18}$$

List(Integer)

From this you can use gcd and reduce to compute the content of the polynomial.

```
reduce(gcd,l)
```

$$2 \tag{19}$$

PositiveInteger

Alternatively (and more easily), you can just call content.

```
content p
```

$$2 \tag{20}$$

PositiveInteger

Note that the operation coefficients omits the zero coefficients from the list. Sometimes it is useful to convert a univariate polynomial to a vector whose i^{th} position contains the degree `i-1` coefficient of the polynomial.

```
ux := (x^4+2*x+3)::UP(x,INT)
```

$$x^4 + 2\,x + 3 \tag{21}$$

UnivariatePolynomial(x, Integer)

To get a complete vector of coefficients, use the operation vectorise, which takes a univariate polynomial and an integer denoting the length of the desired vector.

```
vectorise(ux,5)
```

$$[3,\ 2,\ 0,\ 0,\ 1] \tag{22}$$

Vector(Integer)

It is common to want to do something to every term of a polynomial, creating a new polynomial in the process. This is a function for iterating across the terms of a polynomial, squaring each term.

```
squareTerms(p) ==
  reduce(+,[t^2 for t in monomials p])
```

Recall what p looked like.

```
p
```

$$18\,x^3 + 60\,x^2 - 46\,x + 8 \tag{24}$$

UnivariatePolynomial(x, Integer)

We can demonstrate **squareTerms** on p.

```
squareTerms p
```

```
   Compiling function squareTerms with type UnivariatePolynomial(x,Integer) ->
      UnivariatePolynomial(x,Integer)
```

$$324\,x^6 + 3600\,x^4 + 2116\,x^2 + 64 \tag{25}$$

UnivariatePolynomial(x, Integer)

When the coefficients of the univariate polynomial belong to a field,[7] it is possible to compute quotients and remainders.

[7] For example, when the coefficients are rational numbers, as opposed to integers. The important property of a field is that non-zero elements can be divided and produce another element. The quotient of the integers 2 and 3 is not another integer.

```
(r,s) : UP(a1,FRAC INT)
```

```
r := a1^2 - 2/3
```

$$a1^2 - \frac{2}{3} \tag{27}$$

UnivariatePolynomial(a1, Fraction(Integer))

```
s := a1 + 4
```

$$a1 + 4 \tag{28}$$

UnivariatePolynomial(a1, Fraction(Integer))

When the coefficients are rational numbers or rational expressions, the operation quo computes the quotient of two polynomials.

```
r quo s
```

$$a1 - 4 \tag{29}$$

UnivariatePolynomial(a1, Fraction(Integer))

The operation rem computes the remainder.

```
r rem s
```

$$\frac{46}{3} \tag{30}$$

UnivariatePolynomial(a1, Fraction(Integer))

The operation divide can be used to return a record of both components.

```
d := divide(r, s)
```

$$\left[quotient = a1 - 4,\ remainder = \frac{46}{3} \right] \tag{31}$$

Record(quotient: UnivariatePolynomial(a1, Fraction(Integer)), remainder: UnivariatePolynomial(a1, Fraction(Integer)))

Now we check the arithmetic!

```
r - (d.quotient * s + d.remainder)
```

$$0 \tag{32}$$

UnivariatePolynomial(a1, Fraction(Integer))

It is also possible to integrate univariate polynomials when the coefficients belong to a field.

```
integrate r
```

$$\frac{1}{3}\,a1^3 - \frac{2}{3}\,a1 \tag{33}$$

Type: UnivariatePolynomial(a1, Fraction(Integer))

```
integrate s
```

$$\frac{1}{2}\,a1^2 + 4\,a1 \tag{34}$$

Type: UnivariatePolynomial(a1, Fraction(Integer))

One application of univariate polynomials is to see expressions in terms of a specific variable. We start with a polynomial in a1 whose coefficients are quotients of polynomials in b1 and b2.

```
t : UP(a1,FRAC POLY INT)
```

Since in this case we are not talking about using multivariate polynomials in only two variables, we use **Polynomial**. We also use **Fraction** because we want fractions.

```
t := a1^2 - a1/b2 + (b1^2-b1)/(b2+3)
```

$$a1^2 - \frac{1}{b2}\,a1 + \frac{b1^2 - b1}{b2 + 3} \tag{36}$$

Type: UnivariatePolynomial(a1, Fraction(Polynomial(Integer)))

We push all the variables into a single quotient of polynomials.

```
u : FRAC POLY INT := t
```

$$\frac{a1^2\,b2^2 + \left(b1^2 - b1 + 3\,a1^2 - a1\right)\,b2 - 3\,a1}{b2^2 + 3\,b2} \tag{37}$$

Type: Fraction(Polynomial(Integer))

Alternatively, we can view this as a polynomial in the variable This is a *mode-directed* conversion: you indicate as much of the structure as you care about and let FriCAS decide on the full type and how to do the transformation.

```
u :: UP(b1,?)
```

$$\frac{1}{b2 + 3}\,b1^2 - \frac{1}{b2 + 3}\,b1 + \frac{a1^2\,b2 - a1}{b2} \tag{38}$$

Type: UnivariatePolynomial(b1, Fraction(Polynomial(Integer)))

See Section 8.2 on page 260 for a discussion of the factorization facilities in FriCAS for univariate polynomials. For more information on related topics, see Section 1.9 on page 48, Section 2.7 on page 84, '`Polynomial`' on page 531, '`MultivariatePolynomial`' on page 512, and '`DistributedMultivariatePolynomial`' on page 394. Issue the system command `)show UnivariatePolynomial` to display the full list of operations defined by **UnivariatePolynomial**.

9.88 UniversalSegment

The **UniversalSegment** domain generalizes **Segment** by allowing segments without a "high" end point.

```
pints  := 1..
```

$$1..$$ (4)

UniversalSegment(PositiveInteger)

```
nevens := (0..) by -2
```

$$0.. \text{ by } -2$$ (5)

UniversalSegment(NonNegativeInteger)

Values of type **Segment** are automatically converted to type **UniversalSegment** when appropriate.

```
useg: UniversalSegment(Integer) := 3..10
```

$$3..10$$ (6)

UniversalSegment(Integer)

The operation hasHi is used to test whether a segment has a `high` end point.

```
hasHi pints
```

$$\texttt{false}$$ (7)

Boolean

```
hasHi nevens
```

$$\texttt{false}$$ (8)

Boolean

```
hasHi useg
```

$$\texttt{true}$$ (9)

Boolean

All operations available on type **Segment** apply to **UniversalSegment**, with the proviso that expansions produce streams rather than lists. This is to accommodate infinite expansions.

```
expand pints
```

$$[1, 2, 3, 4, 5, 6, 7, \ldots]$$ (10)

Stream(Integer)

```
expand nevens
```

$$[0, -2, -4, -6, -8, -10, -12, \ldots] \tag{11}$$

Stream(Integer)

```
expand [1, 3, 10..15, 100..]
```

$$[1, 3, 10, 11, 12, 13, 14, \ldots] \tag{12}$$

Stream(Integer)

For more information on related topics, see 'Segment' on page 562, 'SegmentBinding' on page 564, 'List' on page 490, and 'Stream' on page 577. Issue the system command)show UniversalSegment to display the full list of operations defined by **UniversalSegment**.

9.89 Vector

The **Vector** domain is used for storing data in a one-dimensional indexed data structure. A vector is a homogeneous data structure in that all the components of the vector must belong to the same FriCAS domain. Each vector has a fixed length specified by the user; vectors are not extensible. This domain is similar to the **OneDimensionalArray** domain, except that when the components of a **Vector** belong to a **Ring**, arithmetic operations are provided. For more examples of operations that are defined for both **Vector** and **OneDimensionalArray**, see 'OneDimensionalArray' on page 517.

As with the **OneDimensionalArray** domain, a **Vector** can be created by calling the operation new, its components can be accessed by calling the operations elt and qelt, and its components can be reset by calling the operations setelt! and qsetelt!. This creates a vector of integers of length 5 all of whose components are 12.

```
u : VECTOR INT := new(5,12)
```

$$[12, 12, 12, 12, 12] \tag{4}$$

Vector(Integer)

This is how you create a vector from a list of its components.

```
v : VECTOR INT := vector([1,2,3,4,5])
```

$$[1, 2, 3, 4, 5] \tag{5}$$

Vector(Integer)

Indexing for vectors begins at 1. The last element has index equal to the length of the vector, which is computed by #.

```
#(v)
```

$$5 \tag{6}$$

PositiveInteger

This is the standard way to use elt to extract an element. Functionally, it is the same as if you had typed `elt(v,2)`.

```
v.2
```

$$2 \tag{7}$$

PositiveInteger

This is the standard way to use setelt! to change an element. It is the same as if you had typed `setelt!(v, 3, 99)`.

```
v.3 := 99
```

$$99 \tag{8}$$

PositiveInteger

Now look at `v` to see the change. You can use qelt and qsetelt! (instead of elt and setelt!, respectively) but *only* when you know that the index is within the valid range.

```
v
```

$$[1,\ 2,\ 99,\ 4,\ 5] \tag{9}$$

Vector(Integer)

When the components belong to a **Ring**, FriCAS provides arithmetic operations for **Vector**. These include left and right scalar multiplication.

```
5 * v
```

$$[5,\ 10,\ 495,\ 20,\ 25] \tag{10}$$

Vector(Integer)

```
v * 7
```

$$[7,\ 14,\ 693,\ 28,\ 35] \tag{11}$$

Vector(Integer)

```
w : VECTOR INT := vector([2,3,4,5,6])
```

$$[2,\ 3,\ 4,\ 5,\ 6] \tag{12}$$

Vector(Integer)

Addition and subtraction are also available.

```
v + w
```

$$[3,\ 5,\ 103,\ 9,\ 11] \tag{13}$$

Vector(Integer)

Of course, when adding or subtracting, the two vectors must have the same length or an error message is displayed.

```
v - w
```

$$[-1, -1, 95, -1, -1] \tag{14}$$

Vector(Integer)

For more information about other aggregate domains, see the following: '`List`' on page 490, '`Matrix`' on page 503, '`OneDimensionalArray`' on page 517, '`Set`' on page 565, '`Table`' on page 587, and '`TwoDimensionalArray`' on page 592. Issue the system command `)show Vector` to display the full list of operations defined by **Vector**.

9.90 Void

When an expression is not in a value context, it is given type **Void**. For example, in the expression

```
r := (a; b; if c then d else e; f)
```

values are used only from the subexpressions `c` and `f`: all others are thrown away. The subexpressions `a`, `b`, `d` and `e` are evaluated for side-effects only and have type **Void**. There is a unique value of type **Void**.

You will most often see results of type **Void** when you declare a variable.

```
a : Integer
```

Usually no output is displayed for **Void** results. You can force the display of a rather ugly object by issuing `)set message void on`.

```
)set message void on
```

```
b : Fraction Integer
```

$$"()" \tag{5}$$

```
)set message void off
```

All values can be converted to type **Void**.

```
3::Void
```

Once a value has been converted to **Void**, it cannot be recovered.

```
% :: PositiveInteger
```

```
   Cannot convert the value from type Void to PositiveInteger .
```

9.91 WuWenTsunTriangularSet

The **WuWenTsunTriangularSet** domain constructor implements the characteristic set method of Wu Wen Tsun. This algorithm computes a list of triangular sets from a list of polynomials such that the algebraic variety defined by the given list of polynomials decomposes into the union of the regular-zero sets of the computed triangular sets. The constructor takes four arguments. The first one, **R**, is the coefficient ring of the polynomials; it must belong to the category **IntegralDomain**. The second one, **E**, is the exponent monoid of the polynomials; it must belong to the category **OrderedAbelianMonoidSup**. The third one, **V**, is the ordered set of variables; it must belong to the category **OrderedSet**. The last one is the polynomial ring; it must belong to the category **RecursivePolynomialCategory(R,E,V)**. The abbreviation for **WuWenTsunTriangularSet** is **WUTSET**.

Let us illustrate the facilities by an example.

Define the coefficient ring.

```
R := Integer
```

$$Integer \tag{4}$$

Type

Define the list of variables,

```
ls : List Symbol := [x,y,z,t]
```

$$[x,\ y,\ z,\ t] \tag{5}$$

List (Symbol)

and make it an ordered set;

```
V := OVAR(ls)
```

$$\text{OrderedVariableList}([x,\ y,\ z,\ t]) \tag{6}$$

Type

then define the exponent monoid.

```
E := IndexedExponents V
```

$$\text{IndexedExponents}(\text{OrderedVariableList}([x,\ y,\ z,\ t])) \tag{7}$$

Type

Define the polynomial ring.

```
P := NSMP(R, V)
```

$$\text{NewSparseMultivariatePolynomial}(Integer, \text{OrderedVariableList}([x,\ y,\ z,\ t])) \tag{8}$$

Type

Let the variables be polynomial.

```
x: P := 'x
```

$$x \tag{9}$$

NewSparseMultivariatePolynomial(Integer, OrderedVariableList([x, y, z, t]))

```
y: P := 'y
```

$$y \tag{10}$$

NewSparseMultivariatePolynomial(Integer, OrderedVariableList([x, y, z, t]))

```
z: P := 'z
```

$$z \tag{11}$$

NewSparseMultivariatePolynomial(Integer, OrderedVariableList([x, y, z, t]))

```
t: P := 't
```

$$t \tag{12}$$

NewSparseMultivariatePolynomial(Integer, OrderedVariableList([x, y, z, t]))

Now call the **WuWenTsunTriangularSet** domain constructor.

```
T := WUTSET(R,E,V,P)
```

$$\text{WuWenTsunTriangularSet}(Integer, \text{IndexedExponents}(\text{OrderedVariableList}([x, y, z, t])), \text{OrderedVariableList}([x, y, z, t]), \text{NewSparseMultivariatePolynomial}(Integer, \text{OrderedVariableList}([x, y, z, t]))) \tag{13}$$

Type

Define a polynomial system.

```
p1 := x ^ 31 - x ^ 6 - x - y
```

$$x^{31} - x^6 - x - y \tag{14}$$

NewSparseMultivariatePolynomial(Integer, OrderedVariableList([x, y, z, t]))

```
p2 := x ^ 8  - z
```

$$x^8 - z \tag{15}$$

NewSparseMultivariatePolynomial(Integer, OrderedVariableList([x, y, z, t]))

```
p3 := x ^ 10 - t
```

$$x^{10} - t \tag{16}$$

Type: NewSparseMultivariatePolynomial(Integer, OrderedVariableList([x, y, z, t]))

```
lp := [p1, p2, p3]
```

$$\left[x^{31} - x^6 - x - y,\ x^8 - z,\ x^{10} - t\right] \tag{17}$$

Type: List(NewSparseMultivariatePolynomial(Integer, OrderedVariableList([x, y, z, t])))

Compute a characteristic set of the system.

```
characteristicSet(lp)$T
```

$$\left\{z^5 - t^4,\ t^4\,z^2\,y^2 + 2\,t^3\,z^4\,y + \left(-t^7 + 2\,t^4 - t\right)\,z^6 + t^6\,z,\ \left(t^3 - 1\right)\,z^3\,x - z^3\,y - t^3\right\} \tag{18}$$

Type: Union(WuWenTsunTriangularSet(Integer, IndexedExponents(OrderedVariableList([x, y, z, t])), OrderedVariableList([x, y, z, t]), NewSparseMultivariatePolynomial(Integer, OrderedVariableList([x, y, z, t]))), ...)

Solve the system.

```
zeroSetSplit(lp)$T
```

$$\begin{aligned}\Big[&\{t, z, y, x\},\ \left\{t^3 - 1,\ z^5 - t^4,\ z^3\,y + t^3,\ z\,x^2 - t\right\},\ \left\{z^5 - t^4,\right.\\ &\left.t^4\,z^2\,y^2 + 2\,t^3\,z^4\,y + \left(-t^7 + 2\,t^4 - t\right)\,z^6 + t^6\,z,\ \left(t^3 - 1\right)\,z^3\,x - z^3\,y - t^3\right\}\Big]\end{aligned} \tag{19}$$

Type: List(WuWenTsunTriangularSet(Integer, IndexedExponents(OrderedVariableList([x, y, z, t])), OrderedVariableList([x, y, z, t]), NewSparseMultivariatePolynomial(Integer, OrderedVariableList([x, y, z, t]))))

The **RegularTriangularSet** and **SquareFreeRegularTriangularSet** domain constructors, and the **LazardSetSolvingPackage**, **SquareFreeRegularTriangularSet** and **ZeroDimensionalSolvePackage** package constructors also provide operations to compute triangular decompositions of algebraic varieties. These five constructor use a special kind of characteristic sets, called regular triangular sets. These special characteristic sets have better properties than the general ones. Regular triangular sets and their related concepts are presented in the paper "On the Theories of Triangular sets" By P. Aubry, D. Lazard and M. Moreno Maza (to appear in the Journal of Symbolic Computation). The decomposition algorithm (due to the third author) available in the four above constructors provide generally better timings than the characteristic set method. In fact, the **WUTSET** constructor remains interesting for the purpose of manipulating characteristic sets whereas the other constructors are more convenient for solving polynomial systems.

Note that the way of understanding triangular decompositions is detailed in the example of the **RegularTriangularSet** constructor.

9.92 XPBWPolynomial

Initialisations

```
a:Symbol := 'a
```

$$a \tag{4}$$

Symbol

```
b:Symbol := 'b
```

$$b \tag{5}$$

Symbol

```
RN    := Fraction(Integer)
```

$$\text{Fraction}(Integer) \tag{6}$$

Type

```
word   := FreeMonoid Symbol
```

$$\text{FreeMonoid}(Symbol) \tag{7}$$

Type

```
lword := LyndonWord(Symbol)
```

$$\text{LyndonWord}(Symbol) \tag{8}$$

Type

```
base  := PoincareBirkhoffWittLyndonBasis Symbol
```

$$\text{PoincareBirkhoffWittLyndonBasis}(Symbol) \tag{9}$$

Type

```
dpoly := XDistributedPolynomial(Symbol, RN)
```

$$\text{XDistributedPolynomial}(Symbol, \text{Fraction}(Integer)) \tag{10}$$

Type

```
rpoly := XRecursivePolynomial(Symbol, RN)
```

$$\text{XRecursivePolynomial}(Symbol, \text{Fraction}(Integer)) \tag{11}$$

Type

```
lpoly := LiePolynomial(Symbol, RN)
```

$$\text{LiePolynomial}(Symbol, \text{Fraction}(Integer)) \tag{12}$$

Type

```
poly  := XPBWPolynomial(Symbol, RN)
```

$$\text{XPBWPolynomial}(Symbol, \text{Fraction}(Integer)) \tag{13}$$

Type

```
liste : List lword := LyndonWordsList([a,b], 6)
```

$$[[a], [b], [a\,b], [a^2\,b], [a\,b^2], [a^3\,b], [a^2\,b^2], [a\,b^3], [a^4\,b], [a^3\,b^2], [a^2\,b\,a\,b], [a^2\,b^3], [a\,b\,a\,b^2], [a\,b^4], [a^5\,b], [a^4\,b^2], [a^3\,b\,a\,b], [a^3\,b^3], [a^2\,b\,a\,b^2], [a^2\,b^2\,a\,b], [a^2\,b^4], [a\,b\,a\,b^3], [a\,b^5]] \tag{14}$$

List (LyndonWord(Symbol))

Let's make some polynomials

```
0$poly
```

$$0 \tag{15}$$

XPBWPolynomial(Symbol, Fraction(Integer))

```
1$poly
```

$$1 \tag{16}$$

XPBWPolynomial(Symbol, Fraction(Integer))

```
p : poly := a
```

$$[a] \tag{17}$$

XPBWPolynomial(Symbol, Fraction(Integer))

```
q : poly := b
```

$$[b] \tag{18}$$

XPBWPolynomial(Symbol, Fraction(Integer))

```
pq: poly := p*q
```

$$[a\,b] + [b]\ [a] \tag{19}$$

XPBWPolynomial(Symbol, Fraction(Integer))

Coerce to distributed polynomial

```
pq :: dpoly
```

$$a\,b \tag{20}$$

XDistributedPolynomial(Symbol, Fraction(Integer))

Check some polynomial operations

```
mirror pq
```

$$[b]\ [a] \tag{21}$$

XPBWPolynomial(Symbol, Fraction(Integer))

```
listOfTerms pq
```

$$[[k = [b]\ [a]\,,\ c = 1]\,,\ [k = [a\,b]\,,\ c = 1]] \tag{22}$$

List (Record(k: PoincareBirkhoffWittLyndonBasis(Symbol), c: Fraction (Integer)))

```
reductum pq
```

$$[a\,b] \tag{23}$$

XPBWPolynomial(Symbol, Fraction(Integer))

```
leadingMonomial pq
```

$$[b]\ [a] \tag{24}$$

XPBWPolynomial(Symbol, Fraction(Integer))

```
coefficients pq
```

$$[1,\ 1] \tag{25}$$

List (Fraction (Integer))

```
leadingTerm pq
```

$$[k = [b]\ [a]\,,\ c = 1] \tag{26}$$

Record(k: PoincareBirkhoffWittLyndonBasis(Symbol), c: Fraction (Integer))

```
degree pq
```

$$2 \tag{27}$$

PositiveInteger

```
pq4:=exp(pq,4)
```

$$1 + [a\,b] + [b]\ [a] + \frac{1}{2}\ [a\,b]\ [a\,b] + \frac{1}{2}\ \left[a\,b^2\right]\ [a] + \frac{1}{2}\ [b]\ \left[a^2\,b\right] + \frac{3}{2}\ [b]\ [a\,b]\ [a] + \frac{1}{2}\ [b]\ [b]\ [a]\ [a] \tag{28}$$

XPBWPolynomial(Symbol, Fraction(Integer))

```
log(pq4,4) - pq
```

$$0 \tag{29}$$

XPBWPolynomial(Symbol, Fraction(Integer))

Calculations with verification in **XDistributedPolynomial**.

```
lp1 :lpoly := LiePoly liste.10
```

$$[a^3\,b^2] \tag{30}$$

LiePolynomial(Symbol, Fraction(Integer))

```
lp2 :lpoly := LiePoly liste.11
```

$$[a^2\,b\,a\,b] \tag{31}$$

LiePolynomial(Symbol, Fraction(Integer))

```
lp  :lpoly := [lp1, lp2]
```

$$[a^3\,b^2\,a^2\,b\,a\,b] \tag{32}$$

LiePolynomial(Symbol, Fraction(Integer))

```
lpd1: dpoly := lp1
```

$$a^3\,b^2 - 2\,a^2\,b\,a\,b - a^2\,b^2\,a + 4\,a\,b\,a\,b\,a - a\,b^2\,a^2 - 2\,b\,a\,b\,a^2 + b^2\,a^3 \tag{33}$$

XDistributedPolynomial(Symbol, Fraction(Integer))

```
lpd2: dpoly := lp2
```

$$a^2\,b\,a\,b - a^2\,b^2\,a - 3\,a\,b\,a^2\,b + 4\,a\,b\,a\,b\,a - a\,b^2\,a^2 + 2\,b\,a^3\,b - 3\,b\,a^2\,b\,a + b\,a\,b\,a^2 \tag{34}$$

XDistributedPolynomial(Symbol, Fraction(Integer))

```
lpd : dpoly := lpd1 * lpd2 - lpd2 * lpd1
```

$$
\begin{aligned}
&a^3\,b^2\,a^2\,b\,a\,b - a^3\,b^2\,a^2\,b^2\,a - 3\,a^3\,b^2\,a\,b\,a^2\,b + 4\,a^3\,b^2\,a\,b\,a\,b\,a - a^3\,b^2\,a\,b^2\,a^2 \\
&+ 2\,a^3\,b^3\,a^3\,b - 3\,a^3\,b^3\,a^2\,b\,a + a^3\,b^3\,a\,b\,a^2 - a^2\,b\,a\,b\,a^3\,b^2 + 3\,a^2\,b\,a\,b\,a^2\,b^2\,a \\
&+ 6\,a^2\,b\,a\,b\,a\,b\,a^2\,b - 12\,a^2\,b\,a\,b\,a\,b\,a\,b\,a + 3\,a^2\,b\,a\,b\,a\,b^2\,a^2 - 4\,a^2\,b\,a\,b^2\,a^3\,b \\
&+ 6\,a^2\,b\,a\,b^2\,a^2\,b\,a - a^2\,b\,a\,b^3\,a^3 + a^2\,b^2\,a^4\,b^2 - 3\,a^2\,b^2\,a^3\,b\,a\,b + 3\,a^2\,b^2\,a^2\,b\,a^2\,b \\
&- 2\,a^2\,b^2\,a\,b\,a^3\,b + 3\,a^2\,b^2\,a\,b\,a^2\,b\,a - 3\,a^2\,b^2\,a\,b\,a\,b\,a^2 + a^2\,b^2\,a\,b^2\,a^3 \\
&+ 3\,a\,b\,a^2\,b\,a^3\,b^2 - 6\,a\,b\,a^2\,b\,a^2\,b\,a\,b - 3\,a\,b\,a^2\,b\,a^2\,b^2\,a + 12\,a\,b\,a^2\,b\,a\,b\,a\,b\,a \\
&- 3\,a\,b\,a^2\,b\,a\,b^2\,a^2 - 6\,a\,b\,a^2\,b^2\,a\,b\,a^2 + 3\,a\,b\,a^2\,b^3\,a^3 - 4\,a\,b\,a\,b\,a\,b^4 \\
&+ 12\,a\,b\,a\,b\,a\,b^3\,b\,a\,b - 12\,a\,b\,a\,b\,a\,b^2\,b\,a^2\,b + 8\,a\,b\,a\,b\,a\,b\,a\,b^3\,b - 12\,a\,b\,a\,b\,a\,b\,a\,b^2\,b\,a \\
&+ 12\,a\,b\,a\,b\,a\,b\,a\,b\,a\,b^2 - 4\,a\,b\,a\,b\,a\,b^2\,a^3 + a\,b^2\,a^5\,b^2 - 3\,a\,b^2\,a^4\,b\,a\,b + 3\,a\,b^2\,a^3\,b\,a^2\,b \\
&- 2\,a\,b^2\,a^2\,b\,a^3\,b + 3\,a\,b^2\,a^2\,b\,a^2\,b\,a - 3\,a\,b^2\,a^2\,b\,a\,b\,a^2 + a\,b^2\,a^2\,b^2\,a^3 - 2\,b\,a^3\,b\,a^3\,b^2 \\
&+ 4\,b\,a^3\,b\,a^2\,b\,a\,b + 2\,b\,a^3\,b\,a^2\,b^2\,a - 8\,b\,a^3\,b\,a\,b\,a\,b\,a + 2\,b\,a^3\,b\,a\,b^2\,a^2 + 4\,b\,a^3\,b^2\,a\,b\,a^2 \\
&- 2\,b\,a^3\,b^3\,a^3 + 3\,b\,a^2\,b\,a^4\,b^2 - 6\,b\,a^2\,b\,a^3\,b\,a\,b - 3\,b\,a^2\,b\,a^3\,b^2\,a + 12\,b\,a^2\,b\,a^2\,b\,a\,b\,a \\
&- 3\,b\,a^2\,b\,a^2\,b^2\,a^2 - 6\,b\,a^2\,b\,a\,b\,a\,b\,a^2 + 3\,b\,a^2\,b\,a\,b^2\,a^3 - b\,a\,b\,a^5\,b^2 + 3\,b\,a\,b\,a^4\,b^2\,a \\
&+ 6\,b\,a\,b\,a^3\,b\,a^2\,b - 12\,b\,a\,b\,a^3\,b\,a\,b\,a + 3\,b\,a\,b\,a^3\,b^2\,a^2 - 4\,b\,a\,b\,a^2\,b\,a^3\,b \\
&+ 6\,b\,a\,b\,a^2\,b\,a^2\,b\,a - b\,a\,b\,a^2\,b^2\,a^3 + b^2\,a^5\,b\,a\,b - b^2\,a^5\,b^2\,a - 3\,b^2\,a^4\,b\,a^2\,b \\
&+ 4\,b^2\,a^4\,b\,a\,b\,a - b^2\,a^4\,b^2\,a^2 + 2\,b^2\,a^3\,b\,a^3\,b - 3\,b^2\,a^3\,b\,a^2\,b\,a + b^2\,a^3\,b\,a\,b\,a^2
\end{aligned}
\tag{35}
$$

XDistributedPolynomial(Symbol, Fraction(Integer))

```
lp :: dpoly - lpd
```

$$0 \tag{36}$$

XDistributedPolynomial(Symbol, Fraction(Integer))

Calculations with verification in **XRecursivePolynomial**.

```
p := 3 * lp
```

$$3\,\left[a^3\,b^2\,a^2\,b\,a\,b\right] \tag{37}$$

XPBWPolynomial(Symbol, Fraction(Integer))

```
q := lp1
```

$$\left[a^3\,b^2\right] \tag{38}$$

XPBWPolynomial(Symbol, Fraction(Integer))

```
pq:= p * q
```

$$3\,\left[a^3\,b^2\,a^2\,b\,a\,b\right]\,\left[a^3\,b^2\right] \tag{39}$$

XPBWPolynomial(Symbol, Fraction(Integer))

```
pr:rpoly := p :: rpoly
```

$$(((((-3\,b\,a+3\,a\,b)\,b\,a+((9\,b\,a-12\,a\,b)\,a+3\,a\,a\,b)\,b)\,a+((-6\,b\,a+9\,a\,b)\,a-3\,a\,a\,b)\,a\,b)\,b\,b\,a \tag{40}$$
$$+((((3\,b\,b\,a-9\,a\,b\,b)\,a+((-18\,b\,a+36\,a\,b)\,a-9\,a\,a\,b)\,b)\,a+((12\,b\,a-18\,a\,b)\,a\,a+3\,a\,a\,a\,b)\,b)\,b\,a+(((-3\,b\,b\,a+9\,b\,a\,b)\,a-9\,b\,a\,a\,b)\,a+($$
$$+(((((-9\,b\,b\,a+(18\,b\,a+9\,a\,b)\,b)\,a+(-36\,a\,b\,a+9\,a\,a\,b)\,b)\,a+(18\,a\,a\,b\,a-9\,a\,a\,a\,b)\,b)\,b\,a+(((12\,b\,b\,a-36\,b\,a\,b)\,a+36\,b\,a\,a\,b)\,a+(((-2$$
$$+(((-3\,b\,b\,a+9\,b\,a\,b)\,a-9\,b\,a\,a\,b)\,a+(((6\,b\,a-9\,a\,b)\,a+9\,a\,a\,b)\,a-3\,a\,a\,a\,b)\,b)\,a\,a\,b)\,b)\,a$$
$$+((((((6\,b\,b\,a+(-12\,b\,a-6\,a\,b)\,b)\,a+(24\,a\,b\,a-6\,a\,a\,b)\,b)\,a+(-12\,a\,a\,b\,a+6\,a\,a\,a\,b)\,b)\,b\,a+(((-9\,b\,b\,a+(18\,b\,a+9\,a\,b)\,b)\,a+(-36\,a\,b\,a$$
$$+(((3\,b\,b\,a-9\,a\,b\,b)\,a+((-18\,b\,a+36\,a\,b)\,a-9\,a\,a\,b)\,b)\,a+((12\,b\,a-18\,a\,b)\,a\,a+3\,a\,a\,a\,b)\,b)\,a\,a\,b)\,a$$
$$+(((-3\,b\,a+3\,a\,b)\,b\,a+((9\,b\,a-12\,a\,b)\,a+3\,a\,a\,b)\,b)\,a+((-6\,b\,a+9\,a\,b)\,a-3\,a\,a\,b)\,a\,b)\,a\,a\,a\,b$$

XRecursivePolynomial(Symbol, Fraction(Integer))

```
qr:rpoly := q :: rpoly
```

$$((1\,b\,b\,a+(-2\,b\,a-a\,b)\,b)\,a+(4\,a\,b\,a-a\,a\,b)\,b)\,a+(-2\,a\,a\,b\,a+1\,a\,a\,a\,b)\,b \tag{41}$$

XRecursivePolynomial(Symbol, Fraction(Integer))

```
pq :: rpoly - pr*qr
```

$$0 \tag{42}$$

XRecursivePolynomial(Symbol, Fraction(Integer))

9.93 XPolynomial

The **XPolynomial** domain constructor implements multivariate polynomials whose set of variables is **Symbol**. These variables do not commute. The only parameter of this constructor is the coefficient ring which may be non-commutative. However, coefficients and variables commute. The representation of the polynomials is recursive. The abbreviation for **XPolynomial** is **XPOLY**.

Other constructors like **XPolynomialRing**, **XRecursivePolynomial**, **XDistributedPolynomial**, **LiePolynomial** and **XPBWPolynomial** implement multivariate polynomials in non-commutative variables.

We illustrate now some of the facilities of the **XPOLY** domain constructor.

Define a polynomial ring over the integers.

```
poly := XPolynomial(Integer)
```

$$\text{XPolynomial}(Integer) \tag{4}$$

Type

Define a first polynomial,

```
pr: poly := 2*x + 3*y-5
```

$$-5+2\,x+3\,y \tag{5}$$

XPolynomial(Integer)

and a second one.

```
pr2: poly := pr*pr
```

$$25 + (-20 + 4\,x + 6\,y)\;x + (-30 + 6\,x + 9\,y)\;y \tag{6}$$

XPolynomial(Integer)

Rewrite **pr** in a distributive way,

```
pd  := expand pr
```

$$-5 + 2\,x + 3\,y \tag{7}$$

XDistributedPolynomial(Symbol, Integer)

compute its square,

```
pd2 := pd*pd
```

$$25 - 20\,x - 30\,y + 4\,x^2 + 6\,x\,y + 6\,y\,x + 9\,y^2 \tag{8}$$

XDistributedPolynomial(Symbol, Integer)

and checks that:

```
expand(pr2) - pd2
```

$$0 \tag{9}$$

XDistributedPolynomial(Symbol, Integer)

We define:

```
qr :=  pr^3
```

$$\begin{aligned} &-125 + (150 + (-60 + 8\,x + 12\,y)\;x + (-90 + 12\,x + 18\,y)\;y)\;x \\ &+ (225 + (-90 + 12\,x + 18\,y)\;x + (-135 + 18\,x + 27\,y)\;y)\;y \end{aligned} \tag{10}$$

XPolynomial(Integer)

and:

```
qd :=  pd^3
```

$$\begin{aligned} &-125 + 150\,x + 225\,y - 60\,x^2 - 90\,x\,y - 90\,y\,x - 135\,y^2 + 8\,x^3 \\ &+ 12\,x^2\,y + 12\,x\,y\,x + 18\,x\,y^2 + 12\,y\,x^2 + 18\,y\,x\,y + 18\,y^2\,x + 27\,y^3 \end{aligned} \tag{11}$$

XDistributedPolynomial(Symbol, Integer)

We truncate **qd** at degree **3**:

```
trunc(qd,2)
```

$$-125 + 150\,x + 225\,y - 60\,x^2 - 90\,x\,y - 90\,y\,x - 135\,y^2 \tag{12}$$

XDistributedPolynomial(Symbol, Integer)

The same for **qr**:

```
trunc(qr,2)
```

$$-125+(150-60\,x-90\,y)\;x+(225-90\,x-135\,y)\;y \tag{13}$$

XPolynomial(Integer)

We define:

```
Word := FreeMonoid Symbol
```

$$\text{FreeMonoid}(Symbol) \tag{14}$$

Type

and:

```
w: Word := x*y^2
```

$$x\,y^2 \tag{15}$$

FreeMonoid(Symbol)

The we can compute the right-quotient of **qr** by **r**:

```
rquo(qr,w)
```

$$18 \tag{16}$$

XPolynomial(Integer)

and the shuffle-product of **pr** by **r**:

```
sh(pr,w::poly)
```

$$2\,x\,y\,y\,x+(2\,x\,y\,x+((-5+4\,x+3\,y)\;x+9\,x\,y)\;y)\;y \tag{17}$$

XPolynomial(Integer)

9.94 XPolynomialRing

The **XPolynomialRing** domain constructor implements generalized polynomials with coefficients from an arbitrary **Ring** (not necessarily commutative) and whose exponents are words from an arbitrary **OrderedMonoid** (not necessarily commutative too). Thus these polynomials are (finite) linear combinations of words.

This constructor takes two arguments. The first one is a **Ring** and the second is an **OrderedMonoid**. The abbreviation for **XPolynomialRing** is **XPR**.

Other constructors like **XPolynomial**, **XRecursivePolynomial**, **XDistributedPolynomial**, **LiePolynomial** and **XPBWPolynomial** implement multivariate polynomials in non-commutative variables.

We illustrate now some of the facilities of the **XPR** domain constructor.

Define the free ordered monoid generated by the symbols.

```
Word := FreeMonoid(Symbol)
```

$$\text{FreeMonoid}(Symbol) \tag{4}$$

Type

Define the linear combinations of these words with integer coefficients.

```
poly:= XPR(Integer,Word)
```

$$\text{XPolynomialRing}(Integer, \text{FreeMonoid}(Symbol)) \tag{5}$$

Type

Then we define a first element from **poly**.

```
p:poly := 2 * x - 3 * y + 1
```

$$1 + 2\,x - 3\,y \tag{6}$$

XPolynomialRing(Integer, FreeMonoid(Symbol))

And a second one.

```
q:poly := 2 * x + 1
```

$$1 + 2\,x \tag{7}$$

XPolynomialRing(Integer, FreeMonoid(Symbol))

We compute their sum,

```
p + q
```

$$2 + 4\,x - 3\,y \tag{8}$$

XPolynomialRing(Integer, FreeMonoid(Symbol))

their product,

```
p * q
```

$$1 + 4\,x - 3\,y + 4\,x^2 - 6\,y\,x \tag{9}$$

XPolynomialRing(Integer, FreeMonoid(Symbol))

and see that variables do not commute.

```
(p +q)^2 -p^2 -q^2 - 2*p*q
```

$$-6\,x\,y + 6\,y\,x \tag{10}$$

XPolynomialRing(Integer, FreeMonoid(Symbol))

Now we define a ring of square matrices,

```
M := SquareMatrix(2,Fraction Integer)
```

$$\text{SquareMatrix}(2, \text{Fraction}(Integer)) \tag{11}$$

Type

and the linear combinations of words with these matrices as coefficients.

```
poly1:= XPR(M,Word)
```

$$\text{XPolynomialRing}(\text{SquareMatrix}(2, \text{Fraction}(Integer)), \text{FreeMonoid}(Symbol)) \tag{12}$$

Type

Define a first matrix,

```
m1:M := matrix [[i*j^2 for i in 1..2] for j in 1..2]
```

$$\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix} \tag{13}$$

SquareMatrix(2, Fraction(Integer))

a second one,

```
m2:M := m1 - 5/4
```

$$\begin{bmatrix} -\frac{1}{4} & 2 \\ 4 & \frac{27}{4} \end{bmatrix} \tag{14}$$

SquareMatrix(2, Fraction(Integer))

and a third one.

```
m3: M := m2^2
```

$$\begin{bmatrix} \frac{129}{16} & 13 \\ 26 & \frac{857}{16} \end{bmatrix} \tag{15}$$

SquareMatrix(2, Fraction(Integer))

Define a polynomial,

```
pm:poly1    := m1*x + m2*y + m3*z - 2/3
```

$$\begin{bmatrix} -\frac{2}{3} & 0 \\ 0 & -\frac{2}{3} \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix} x + \begin{bmatrix} -\frac{1}{4} & 2 \\ 4 & \frac{27}{4} \end{bmatrix} y + \begin{bmatrix} \frac{129}{16} & 13 \\ 26 & \frac{857}{16} \end{bmatrix} z \tag{16}$$

XPolynomialRing(SquareMatrix(2, Fraction(Integer)), FreeMonoid(Symbol))

a second one,

```
qm:poly1 := pm - m1*x
```

$$\begin{bmatrix} -\frac{2}{3} & 0 \\ 0 & -\frac{2}{3} \end{bmatrix} + \begin{bmatrix} -\frac{1}{4} & 2 \\ 4 & \frac{27}{4} \end{bmatrix} y + \begin{bmatrix} \frac{129}{16} & 13 \\ 26 & \frac{857}{16} \end{bmatrix} z \tag{17}$$

XPolynomialRing(SquareMatrix(2, Fraction(Integer)), FreeMonoid(Symbol))

and the following power.

```
qm^3
```

$$\begin{bmatrix} -\frac{8}{27} & 0 \\ 0 & -\frac{8}{27} \end{bmatrix} + \begin{bmatrix} -\frac{1}{3} & \frac{8}{3} \\ \frac{16}{3} & 9 \end{bmatrix} y + \begin{bmatrix} \frac{43}{4} & \frac{52}{3} \\ \frac{104}{3} & \frac{857}{12} \end{bmatrix} z + \begin{bmatrix} -\frac{129}{8} & -26 \\ -52 & -\frac{857}{8} \end{bmatrix} y^2 + \begin{bmatrix} -\frac{3199}{32} & -\frac{831}{4} \\ -\frac{831}{2} & -\frac{26467}{32} \end{bmatrix} y\,z$$
$$+ \begin{bmatrix} -\frac{3199}{32} & -\frac{831}{4} \\ -\frac{831}{2} & -\frac{26467}{32} \end{bmatrix} z\,y + \begin{bmatrix} -\frac{103169}{128} & -\frac{6409}{4} \\ -\frac{6409}{2} & -\frac{820977}{128} \end{bmatrix} z^2 + \begin{bmatrix} \frac{3199}{64} & \frac{831}{8} \\ \frac{831}{4} & \frac{26467}{64} \end{bmatrix} y^3 + \begin{bmatrix} \frac{103169}{256} & \frac{6409}{8} \\ \frac{6409}{4} & \frac{820977}{256} \end{bmatrix} y^2 z \tag{18}$$
$$+ \begin{bmatrix} \frac{103169}{256} & \frac{6409}{8} \\ \frac{6409}{4} & \frac{820977}{256} \end{bmatrix} y\,z\,y + \begin{bmatrix} \frac{3178239}{1024} & \frac{795341}{128} \\ \frac{795341}{64} & \frac{25447787}{1024} \end{bmatrix} y\,z^2 + \begin{bmatrix} \frac{103169}{256} & \frac{6409}{8} \\ \frac{6409}{4} & \frac{820977}{256} \end{bmatrix} z\,y^2$$
$$+ \begin{bmatrix} \frac{3178239}{1024} & \frac{795341}{128} \\ \frac{795341}{64} & \frac{25447787}{1024} \end{bmatrix} z\,y\,z + \begin{bmatrix} \frac{3178239}{1024} & \frac{795341}{128} \\ \frac{795341}{64} & \frac{25447787}{1024} \end{bmatrix} z^2\,y + \begin{bmatrix} \frac{98625409}{4096} & \frac{12326223}{256} \\ \frac{12326223}{128} & \frac{788893897}{4096} \end{bmatrix} z^3$$

XPolynomialRing(SquareMatrix(2, Fraction(Integer)), FreeMonoid(Symbol))

9.95 ZeroDimensionalSolvePackage

The **ZeroDimensionalSolvePackage** package constructor provides operations for computing symbolically the complex or real roots of zero-dimensional algebraic systems.

The package provides **no** multiplicity information (i.e. some returned roots may be double or higher) but only distinct roots are returned.

Complex roots are given by means of univariate representations of irreducible regular chains. These representations are computed by the `univariateSolve` operation (by calling the **InternalRationalUnivariateRepresentationPackage** package constructor which does the job).

Real roots are given by means of tuples of coordinates lying in the **RealClosure** of the coefficient ring. They are computed by the `realSolve` and `positiveSolve` operations. The former computes all the solutions of the input system with real coordinates whereas the later concentrate on the solutions with (strictly) positive coordinates. In both cases, the computations are performed by the **RealClosure** constructor.

Both computations of complex roots and real roots rely on triangular decompositions. These decompositions can be computed in two different ways. First, by a applying the `zeroSetSplit` operation from the **REGSET** domain constructor. In that case, no Groebner bases are computed. This strategy is used by default. Secondly, by applying the `zeroSetSplit` from **LEXTRIPK**. To use this later strategy with the operations `univariateSolve`, `realSolve` and `positiveSolve` one just needs to use an extra boolean argument.

Note that the way of understanding triangular decompositions is detailed in the example of the **RegularTriangularSet** constructor.

The **ZeroDimensionalSolvePackage** constructor takes three arguments. The first one **R** is the coefficient ring; it must belong to the categories **OrderedRing**, **EuclideanDomain**, **CharacteristicZero** and **RealConstant**. This means essentially that **R** is **Integer** or **Fraction(Integer)**. The

second argument **ls** is the list of variables involved in the systems to solve. The third one MUST BE **concat(ls,s)** where **s** is an additional symbol used for the univariate representations. The abbreviation for **ZeroDimensionalSolvePackage** is **ZDSOLVE**.

We illustrate now how to use the constructor **ZDSOLVE** by two examples: the *Arnborg and Lazard* system and the *L-3* system (Aubry and Moreno Maza). Note that the use of this package is also demonstrated in the example of the **LexTriangularPackage** constructor.

Define the coefficient ring.

```
R := Integer
```

$$Integer \tag{4}$$

Type

Define the lists of variables:

```
ls : List Symbol := [x,y,z,t]
```

$$[x,\ y,\ z,\ t] \tag{5}$$

List (Symbol)

and:

```
ls2 : List Symbol := [x,y,z,t,new()$Symbol]
```

$$[x,\ y,\ z,\ t,\ \%A] \tag{6}$$

List (Symbol)

Call the package:

```
pack := ZDSOLVE(R,ls,ls2)
```

$$\text{ZeroDimensionalSolvePackage}(Integer, [x,\ y,\ z,\ t], [x,\ y,\ z,\ t,\ \%A]) \tag{7}$$

Type

Define a polynomial system (Arnborg-Lazard)

```
p1 := x^2*y*z + x*y^2*z + x*y*z^2 + x*y*z + x*y + x*z + y*z
```

$$x\,y\,z^2 + \left(x\,y^2 + \left(x^2 + x + 1\right)\,y + x\right)\,z + x\,y \tag{8}$$

Polynomial(Integer)

```
p2 := x^2*y^2*z + x*y^2*z^2 + x^2*y*z + x*y*z + y*z + x + z
```

$$x\,y^2\,z^2 + \left(x^2\,y^2 + \left(x^2 + x + 1\right)\,y + 1\right)\,z + x \tag{9}$$

Polynomial(Integer)

```
p3 := x^2*y^2*z^2 + x^2*y^2*z + x*y^2*z + x*y*z + x*z + z + 1
```

$$x^2\,y^2\,z^2+\left(\left(x^2+x\right)\,y^2+x\,y+x+1\right)\,z+1 \tag{10}$$

Polynomial(Integer)

```
lp := [p1, p2, p3]
```

$$\left[x\,y\,z^2+\left(x\,y^2+\left(x^2+x+1\right)\,y+x\right)\,z+x\,y,\ x\,y^2\,z^2+\left(x^2\,y^2+\left(x^2+x+1\right)\,y+1\right)\,z+x,\right. \tag{11}$$
$$\left.x^2\,y^2\,z^2+\left(\left(x^2+x\right)\,y^2+x\,y+x+1\right)\,z+1\right]$$

List (Polynomial(Integer))

Note that these polynomials do not involve the variable **t**; we will use it in the second example.

First compute a decomposition into regular chains (i.e. regular triangular sets).

```
triangSolve(lp)$pack
```

$$\left[\left\{z^{20}-6\,z^{19}-41\,z^{18}+71\,z^{17}+106\,z^{16}+92\,z^{15}+197\,z^{14}+145\,z^{13}+257\,z^{12}+278\,z^{11}\right. \tag{12}$$
$$+201\,z^{10}+278\,z^9+257\,z^8+145\,z^7+197\,z^6+92\,z^5+106\,z^4+71\,z^3-41\,z^2-6\,z+1,$$
$$\left(14745844\,z^{19}+50357474\,z^{18}-130948857\,z^{17}-185261586\,z^{16}-180077775\,z^{15}-338007307\,z^{14}-275379623\,z^{13}-453190404\,z^{12}-474\right.$$
$$+1917314\,z^{19}+6508991\,z^{18}-16973165\,z^{17}-24000259\,z^{16}-23349192\,z^{15}-43786426\,z^{14}$$
$$-35696474\,z^{13}-58724172\,z^{12}-61480792\,z^{11}-47452440\,z^{10}-62378085\,z^9-55776527\,z^8-33940618\,z^7$$
$$-42233406\,z^6-21122875\,z^5-22958177\,z^4-13504569\,z^3+8448317\,z^2+1195888\,z-202934,$$
$$\left.\left.\left(z^3-2\,z\right)\,y^2+\left(-z^3-z^2-2\,z-1\right)\,y-z^2-z+1\right)\,x+z^2-1\right\}\right]$$

List (RegularChain(Integer , [x, y, z, t]))

We can see easily from this decomposition (consisting of a single regular chain) that the input system has 20 complex roots.

Then we compute a univariate representation of this regular chain.

```
univariateSolve(lp)$pack
```

$$\left[\left[complexRoots=\mathsf{?}^{12}-12\,\mathsf{?}^{11}+24\,\mathsf{?}^{10}+4\,\mathsf{?}^9-9\,\mathsf{?}^8+27\,\mathsf{?}^7-21\,\mathsf{?}^6+27\,\mathsf{?}^5-9\,\mathsf{?}^4+4\,\mathsf{?}^3+24\,\mathsf{?}^2-12\,\mathsf{?}+1,\right. \tag{13}$$
$$coordinates=\left[63\,x+62\,\%A^{11}-721\,\%A^{10}+1220\,\%A^9+705\,\%A^8-285\,\%A^7+1512\,\%A^6-735\,\%A^5+1401\,\%A^4-21\,\%A^3+215\,\%A^2+\right.$$
$$63\,y-75\,\%A^{11}+890\,\%A^{10}-1682\,\%A^9-516\,\%A^8+588\,\%A^7-1953\,\%A^6+1323\,\%A^5-1815\,\%A^4+426\,\%A^3-243\,\%A^2-1801\,\%A+67$$
$$\left.z-\%A\right]\Big],\ \left[complexRoots=\mathsf{?}^6+\mathsf{?}^5+\mathsf{?}^4+\mathsf{?}^3+\mathsf{?}^2+\mathsf{?}+1,\ coordinates=\left[x-\%A^5,\right.\right.$$
$$\left.\left.y-\%A^3,\ z-\%A\right]\right],\ \left[complexRoots=\mathsf{?}^2+5\,\mathsf{?}+1,\ coordinates=[x-1,\ y-1,\ z-\%A]\right]\Big]$$

List (Record(complexRoots: SparseUnivariatePolynomial(Integer), coordinates: List (Polynomial(Integer))))

We see that the zeros of our regular chain are split into three components. This is due to the use of univariate polynomial factorization.

Each of these components consist of two parts. The first one is an irreducible univariate polynomial **p(?)** which defines a simple algebraic extension of the field of fractions of **R**. The second one consists of multivariate polynomials **pol1(x,%A)**, **pol2(y,%A)** and **pol3(z,%A)**. Each of these polynomials involve two variables: one is an indeterminate **x**, **y** or **z** of the input system **lp** and the other is **%A** which represents any root of **p(?)**. Recall that this **%A** is the last element of the third parameter of **ZDSOLVE**. Thus any complex root **?** of **p(?)** leads to a solution of the input system **lp** by replacing

%A by this **?** in **pol1(x,%A)**, **pol2(y,%A)** and **pol3(z,%A)**. Note that the polynomials **pol1(x,%A)**, **pol2(y,%A)** and **pol3(z,%A)** have degree one w.r.t. **x**, **y** or **z** respectively. This is always the case for all univariate representations. Hence the operation **univariateSolve** replaces a system of multivariate polynomials by a list of univariate polynomials, what justifies its name. Another example of univariate representations illustrates the **LexTriangularPackage** package constructor.

We now compute the solutions with real coordinates:

```
lr := realSolve(lp)$pack;
```

List (List (RealClosure(Fraction (Integer))))

The number of real solutions for the input system is:

```
# lr
```

$$8 \tag{15}$$

PositiveInteger

Each of these real solutions is given by a list of elements in **RealClosure(R)**. In these 8 lists, the first element is a value of **z**, the second of **y** and the last of **x**. This is logical since by setting the list of variables of the package to **[x,y,z,t]** we mean that the elimination ordering on the variables is $\mathbf{t} < \mathbf{z} < \mathbf{y} < \mathbf{x}$. Note that each system treated by the **ZDSOLVE** package constructor needs only to be zero-dimensional w.r.t. the variables involved in the system it-self and not necessarily w.r.t. all the variables used to define the package.

We can approximate these real numbers as follows. This computation takes between 30 sec. and 5 min, depending on your machine.

```
[[approximate(r,1/1000000) for r in point] for point in lr];
```

List (List (Fraction (Integer)))

We can also concentrate on the solutions with real (strictly) positive coordinates:

```
lpr := positiveSolve(lp)$pack
```

$$[] \tag{17}$$

List (List (RealClosure(Fraction (Integer))))

Thus we have checked that the input system has no solution with strictly positive coordinates.

Let us define another polynomial system (*L-3*).

```
f0 := x^3 + y + z + t- 1
```

$$z + y + x^3 + t - 1 \tag{18}$$

Polynomial(Integer)

```
f1 := x + y^3 + z + t -1
```

$$z + y^3 + x + t - 1 \tag{19}$$

Polynomial(Integer)

```
f2 := x + y + z^3 + t-1
```

$$z^3 + y + x + t - 1 \tag{20}$$

Polynomial(Integer)

```
f3 := x + y + z + t^3 -1
```

$$z + y + x + t^3 - 1 \tag{21}$$

Polynomial(Integer)

```
lf := [f0, f1, f2, f3]
```

$$\left[z + y + x^3 + t - 1,\ z + y^3 + x + t - 1,\ z^3 + y + x + t - 1,\ z + y + x + t^3 - 1\right] \tag{22}$$

List (Polynomial(Integer))

First compute a decomposition into regular chains (i.e. regular triangular sets).

```
lts := triangSolve(lf)$pack
```

$$
\begin{aligned}
&\left[\left\{t^2+t+1,\ z^3-z-t^3+t,\ \left(3\,z+3\,t^3-3\right)\,y^2+\left(3\,z^2+\left(6\,t^3-6\right)\,z+3\,t^6-6\,t^3+3\right)\,y+\left(3\,t^3-3\right)\,z^2\right. \qquad (23)\\
&\left.+\left(3\,t^6-6\,t^3+3\right)\,z+t^9-3\,t^6+5\,t^3-3\,t,\ x+y+z\right\},\ \left\{t^{16}-6\,t^{13}+9\,t^{10}+4\,t^7+15\,t^4-54\,t^2+27,\right.\\
&\left(4907232\,t^{15}+40893984\,t^{14}-115013088\,t^{13}+22805712\,t^{12}+36330336\,t^{11}+162959040\,t^{10}-159859440\,t^9-156802608\,t^8+117168768\right.\\
&+48\,t^{54}-912\,t^{51}+8232\,t^{48}-72\,t^{46}-46848\,t^{45}+1152\,t^{43}+186324\,t^{42}-3780\,t^{40}-543144\,t^{39}\\
&-3168\,t^{38}-21384\,t^{37}+1175251\,t^{36}+41184\,t^{35}+278003\,t^{34}-1843242\,t^{33}-301815\,t^{32}-1440726\,t^{31}\\
&+1912012\,t^{30}+1442826\,t^{29}+4696262\,t^{28}-922481\,t^{27}-4816188\,t^{26}-10583524\,t^{25}-208751\,t^{24}\\
&+11472138\,t^{23}+16762859\,t^{22}-857663\,t^{21}-19328175\,t^{20}-18270421\,t^{19}+4914903\,t^{18}+22483044\,t^{17}\\
&+12926517\,t^{16}-8605511\,t^{15}-17455518\,t^{14}-5014597\,t^{13}+8108814\,t^{12}+8465535\,t^{11}+190542\,t^{10}\\
&-4305624\,t^9-2226123\,t^8+661905\,t^7+1169775\,t^6+226260\,t^5-209952\,t^4-141183\,t^3+27216\,t,\\
&\left(3\,z+3\,t^3-3\right)\,y^2+\left(3\,z^2+\left(6\,t^3-6\right)\,z+3\,t^6-6\,t^3+3\right)\,y+\left(3\,t^3-3\right)\,z^2+\left(3\,t^6-6\,t^3+3\right)\,z+t^9\\
&-3\,t^6+5\,t^3-3\,t,\ x+y+z+t^3-1\},\ \left\{t,\ z-1,\ y^2-1,\ x+y\right\},\ \left\{t-1,\ z,\ y^2-1,\right.\\
&\left.x+y\right\},\ \left\{t-1,\ z^2-1,\ z\,y+1,\ x\right\},\ \left\{t^{16}-6\,t^{13}+9\,t^{10}+4\,t^7+15\,t^4-54\,t^2+27,\right.\\
&\left(4907232\,t^{29}+40893984\,t^{28}-115013088\,t^{27}-1730448\,t^{26}-168139584\,t^{25}+738024480\,t^{24}-195372288\,t^{23}+315849456\,t^{22}-25672792\right.\\
&-48\,t^{68}+1152\,t^{65}-13560\,t^{62}+360\,t^{60}+103656\,t^{59}-7560\,t^{57}-572820\,t^{56}+71316\,t^{54}+2414556\,t^{53}\\
&+2736\,t^{52}-402876\,t^{51}-7985131\,t^{50}-49248\,t^{49}+1431133\,t^{48}+20977409\,t^{47}+521487\,t^{46}-2697635\,t^{45}\\
&-43763654\,t^{44}-3756573\,t^{43}-2093410\,t^{42}+71546495\,t^{41}+19699032\,t^{40}+35025028\,t^{39}-89623786\,t^{38}\\
&-77798760\,t^{37}-138654191\,t^{36}+87596128\,t^{35}+235642497\,t^{34}+349607642\,t^{33}-93299834\,t^{32}\\
&-551563167\,t^{31}-630995176\,t^{30}+186818962\,t^{29}+995427468\,t^{28}+828416204\,t^{27}-393919231\,t^{26}-1076617485\,t^{25}\\
&-1609479791\,t^{24}+595738126\,t^{23}+1198787136\,t^{22}+4342832069\,t^{21}-2075938757\,t^{20}-4390835799\,t^{19}-4822843033\,t^{18}\\
&+6932747678\,t^{17}+6172196808\,t^{16}+1141517740\,t^{15}-4981677585\,t^{14}-9819815280\,t^{13}-7404299976\,t^{12}-157295760\,t^{11}\\
&+29124027630\,t^{10}+14856038208\,t^9-16184101410\,t^8-26935440354\,t^7-3574164258\,t^6+10271338974\,t^5+11191425264\,t^4\\
&+6869861262\,t^3-9780477840\,t^2-3586674168\,t+2884297248,\\
&\left(3\,z^3+\left(6\,t^3-6\right)\,z^2+\left(6\,t^6-12\,t^3+3\right)\,z+2\,t^9-6\,t^6+t^3+3\,t\right)\,y+\left(3\,t^3-3\right)\,z^3+\left(6\,t^6-12\,t^3+6\right)\,z^2+\left(4\,t^9-12\,t^6+11\,t^3-3\right)\,z\\
&+t^{12}-4\,t^9+5\,t^6-2\,t^3,\ x+y+z+t^3-1\},\ \left\{t-1,\ z^2-1,\right.\\
&\left.y,\ x+z\right\},\ \left\{t^8+t^7+t^6-2\,t^5-2\,t^4-2\,t^3+19\,t^2+19\,t-8,\right.\\
&\left(2395770\,t^7+3934440\,t^6-3902067\,t^5-10084164\,t^4-1010448\,t^3+32386932\,t^2+22413225\,t-10432368\right)\,z-463519\,t^7\\
&+3586833\,t^6+9494955\,t^5-8539305\,t^4-33283098\,t^3+35479377\,t^2+46263256\,t-17419896,\\
&\left(3\,z^4+\left(9\,t^3-9\right)\,z^3+\left(12\,t^6-24\,t^3+9\right)\,z^2+\left(-152\,t^3+219\,t-67\right)\,z-41\,t^6+57\,t^4+25\,t^3-57\,t+16\right)\,y+\left(3\,t^3-3\right)\,z^4\\
&+\left(9\,t^6-18\,t^3+9\right)\,z^3+\left(-181\,t^3+270\,t-89\right)\,z^2+\left(-92\,t^6+135\,t^4+49\,t^3-135\,t+43\right)\,z+27\,t^7-27\,t^6-54\,t^4+396\,t^3\\
&-486\,t+144,\ x+y+z+t^3-1\},\ \left\{t,\ z-t^3+1,\ y-1,\ x-1\right\},\\
&\left.\left\{t-1,\ z,\ y,\ x\right\},\ \left\{t,\ z-1,\ y,\ x\right\},\ \left\{t,\ z,\ y-1,\ x\right\},\ \left\{t,\ z,\ y,\ x-1\right\}\right]
\end{aligned}
$$

List(RegularChain(Integer, [x, y, z, t]))

Then we compute a univariate representation.

```
univariateSolve(lf)$pack
```

$$[[complexRoots = ?,\ coordinates = [x-1,\ y-1,\ z+1,\ t-\%A]],\ [complexRoots = ?,\ coordinates = [x,\ y-1,\ z,\ t-\%A]],\ [complexRoots = ?-1,\ coordinates = [x,\ y,\ z,\ t-\%A]],\ [complexRoots = ?,\ coordinates = [x-1,\ y,\ z,\ t-\%A]],\ [complexRoots = ?,\ coordinates = [x,\ y,\ z-1,\ t-\%A]],\ [complexRoots = ?,\ coordinates = [x+1,\ y-1,\ z,\ t-1]],\ [complexRoots = ?-2,\ coordinates = [x-1,\ y+1,\ z,\ t-1]],\ [complexRoots = ?-1,\ coordinates = [x-1,\ y+1,\ z-1,\ t]],\ [complexRoots = ?+1,\ coordinates = [x+1,\ y-1,\ z-1,\ t]],\ [complexRoots = ?^6-2\,?^3+3\,?^2-3,\ coordinates = [2\,x+\%A^3+\%A-1,\ 2\,y+\%A^3+\%A-1,\ z-\%A,\ t-\%A]],\ [complexRoots = ?^5+3\,?^3-2\,?^2+3\,?-3,\ coordinates = [x-\%A,\ y-\%A,\ z+\%A^3+2\,\%A-1,\ t-\%A]],\ [complexRoots = ?^4-?^3-2\,?^2+3,\ coordinates = [x+\%A^3-\%A-1,\ y+\%A^3-\%A-1,\ z-\%A^3+2\,\%A+1,\ t-\%A]],\ [complexRoots = ?+1,\ coordinates = [x-1,\ y-1,\ z,\ t-\%A]],\ [complexRoots = ?^6+2\,?^3+3\,?^2-3,\ coordinates = [2\,x-\%A^3-\%A-1,\ y+\%A,\ 2\,z-\%A^3-\%A-1,\ t+\%A]],\ [complexRoots = ?^6+12\,?^4+20\,?^3-45\,?^2-42\,?-953,\ coordinates = [12609\,x+23\,\%A^5+49\,\%A^4-46\,\%A^3+362\,\%A^2-5015\,\%A-8239,\ 25218\,y+23\,\%A^5+49\,\%A^4-46\,\%A^3+362\,\%A^2+7594\,\%A-8239,\ 25218\,z+23\,\%A^5+49\,\%A^4-46\,\%A^3+362\,\%A^2+7594\,\%A-8239,\ 12609\,t+23\,\%A^5+49\,\%A^4-46\,\%A^3+362\,\%A^2-5015\,\%A-8239]],\ [complexRoots = ?^5+12\,?^3-16\,?^2+48\,?-96,\ coordinates = [8\,x+\%A^3+8\,\%A-8,\ 2\,y-\%A,\ 2\,z-\%A,\ 2\,t-\%A]],\ [complexRoots = ?^5+?^4-5\,?^3-3\,?^2+9\,?+3,\ coordinates = [2\,x-\%A^3+2\,\%A-1,\ 2\,y+\%A^3-4\,\%A+1,\ 2\,z-\%A^3+2\,\%A-1,\ 2\,t-\%A^3+2\,\%A-1]],\ [complexRoots = ?^4-3\,?^3+4\,?^2-6\,?+13,\ coordinates = [9\,x-2\,\%A^3+4\,\%A^2-\%A+2,\ 9\,y+\%A^3-2\,\%A^2+5\,\%A-1,\ 9\,z+\%A^3-2\,\%A^2+5\,\%A-1,\ 9\,t+\%A^3-2\,\%A^2-4\,\%A-1]],\ [complexRoots = ?^4-11\,?^2+37,\ coordinates = [3\,x-\%A^2+7,\ 6\,y+\%A^2+3\,\%A-7,\ 3\,z-\%A^2+7,\ 6\,t+\%A^2-3\,\%A-7]],\ [complexRoots = ?+1,\ coordinates = [x-1,\ y,\ z-1,\ t+1]],\ [complexRoots = ?+2,\ coordinates = [x,\ y-1,\ z-1,\ t+1]],\ [complexRoots = ?,\ coordinates = [x,\ y+1,\ z-1,\ t-1]],\ [complexRoots = ?-2,\ coordinates = [x,\ y-1,\ z+1,\ t-1]],\ [complexRoots = ?,\ coordinates = [x+1,\ y,\ z-1,\ t-1]],\ [complexRoots = ?-2,\ coordinates = [x-1,\ y,\ z+1,\ t-1]],\ [complexRoots = ?^4+5\,?^3+16\,?^2+30\,?+57,\ coordinates = [151\,x+15\,\%A^3+54\,\%A^2+104\,\%A+93,\ 151\,y-10\,\%A^3-36\,\%A^2-19\,\%A-62,\ 151\,z-5\,\%A^3-18\,\%A^2-85\,\%A-31,\ 151\,t-5\,\%A^3-18\,\%A^2-85\,\%A-31]],\ [complexRoots = ?^4-?^3-2\,?^2+3,\ coordinates = [x-\%A^3+2\,\%A+1,\ y+\%A^3-\%A-1,\ z-\%A,\ t+\%A^3-\%A-1]],\ [complexRoots = ?^4+2\,?^3-8\,?^2+48,\ coordinates = [8\,x-\%A^3+4\,\%A-8,\ 2\,y+\%A,\ 8\,z+\%A^3-8\,\%A+8,\ 8\,t-\%A^3+4\,\%A-8]],\ [complexRoots = ?^5+?^4-2\,?^3-4\,?^2+5\,?+8,\ coordinates = [3\,x+\%A^3-1,\ 3\,y+\%A^3-1,\ 3\,z+\%A^3-1,\ t-\%A]],\ [complexRoots = ?^3+3\,?-1,\ coordinates = [x-\%A,\ y-\%A,\ z-\%A,\ t-\%A]]] \tag{24}$$

List(Record(complexRoots: SparseUnivariatePolynomial(Integer), coordinates: List(Polynomial(Integer))))

Note that this computation is made from the input system **lf**. However it is possible to reuse a pre-computed regular chain as follows:

```
ts := lts.1
```

$$\{t^2+t+1,\ z^3-z-t^3+t,\ (3\,z+3\,t^3-3)\,y^2+(3\,z^2+(6\,t^3-6)\,z+3\,t^6-6\,t^3+3)\,y+(3\,t^3-3)\,z^2+(3\,t^6-6\,t^3+3)\,z+t^9-3\,t^6+5\,t^3-3\,t,\ x+y+z\} \tag{25}$$

RegularChain(Integer, [x, y, z, t])

```
univariateSolve(ts)$pack
```

$$\begin{aligned}&[[complexRoots = ?^4 + 5\,?^3 + 16\,?^2 + 30\,? + 57,\\ &\ coordinates = [151\,x + 15\,\%A^3 + 54\,\%A^2 + 104\,\%A + 93,\\ &\ 151\,y - 10\,\%A^3 - 36\,\%A^2 - 19\,\%A - 62,\ 151\,z - 5\,\%A^3 - 18\,\%A^2 - 85\,\%A - 31,\\ &\ 151\,t - 5\,\%A^3 - 18\,\%A^2 - 85\,\%A - 31]],\ [complexRoots = ?^4 - ?^3 - 2\,?^2 + 3,\\ &\ coordinates = [x - \%A^3 + 2\,\%A + 1,\ y + \%A^3 - \%A - 1,\ z - \%A,\ t + \%A^3 - \%A - 1]],\\ &\ [complexRoots = ?^4 + 2\,?^3 - 8\,?^2 + 48,\ coordinates = [8\,x - \%A^3 + 4\,\%A - 8,\\ &\ 2\,y + \%A,\ 8\,z + \%A^3 - 8\,\%A + 8,\ 8\,t - \%A^3 + 4\,\%A - 8]]]\end{aligned} \tag{26}$$

List (Record(complexRoots: SparseUnivariatePolynomial(Integer), coordinates : List (Polynomial(Integer))))

```
realSolve(ts)$pack
```

$$[] \tag{27}$$

List (List (RealClosure(Fraction (Integer))))

We compute now the full set of points with real coordinates:

```
lr2 := realSolve(lf)$pack;
```

List (List (RealClosure(Fraction (Integer))))

The number of real solutions for the input system is:

```
#lr2
```

$$27 \tag{29}$$

PositiveInteger

Another example of computation of real solutions illustrates the **LexTriangularPackage** package constructor.

We concentrate now on the solutions with real (strictly) positive coordinates:

```
lpr2 := positiveSolve(lf)$pack
```

$$\left[\left[\%B40,\ -\frac{1}{3}\,\%B40^3 + \frac{1}{3},\ -\frac{1}{3}\,\%B40^3 + \frac{1}{3},\ -\frac{1}{3}\,\%B40^3 + \frac{1}{3}\right]\right] \tag{30}$$

List (List (RealClosure(Fraction (Integer))))

Finally, we approximate the coordinates of this point with 20 exact digits:

```
[approximate(r,1/10^21)::Float for r in lpr2.1]
```

$$[0.32218535462608559291,\ 0.32218535462608559291,\ 0.32218535462608559291,\ 0.32218535462608559291] \tag{31}$$

List (Float)

Part III

Advanced Programming in FriCAS

Chapter 10

Interactive Programming

Programming in the interpreter is easy. So is the use of FriCAS's graphics facility. Both are rather flexible and allow you to use them for many interesting applications. However, both require learning some basic ideas and skills.

All graphics examples in the FriCAS Images section are either produced directly by interactive commands or by interpreter programs. Four of these programs are introduced here. By the end of this chapter you will know enough about graphics and programming in the interpreter to not only understand all these examples, but to tackle interesting and difficult problems on your own. Appendix B lists all the remaining commands and programs used to create these images.

10.1 Drawing Ribbons Interactively

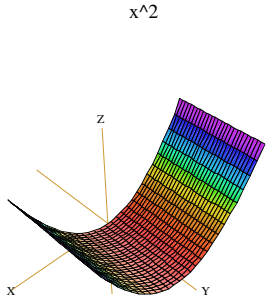
We begin our discussion of interactive graphics with the creation of a useful facility: plotting ribbons of two-dimensional graphs in three-dimensional space. Suppose you want to draw the two-dimensional graphs of n functions $f_i(x), 1 \leq i \leq n$, all over some fixed range of x. One approach is to create a two-dimensional graph for each one, then superpose one on top of the other. What you will more than likely get is a jumbled mess. Even if you make each function a different color, the result is likely to be confusing.

A better approach is to display each of the $f_i(x)$ in three dimensions as a "ribbon" of some appropriate width along the y-direction, laying down each ribbon next to the previous one. A ribbon is simply a function of x and y depending only on x.

We illustrate this for $f_i(x)$ defined as simple powers of x for x ranging between -1 and 1.

Draw the ribbon for $z = x^2$.

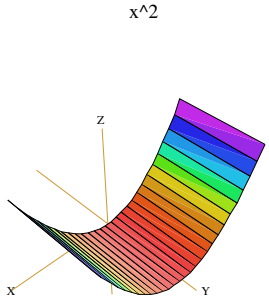
```
draw(x^2,x=-1..1,y=0..1)
```



Now that was easy! What you get is a "wire-mesh" rendition of the ribbon. That's fine for now. Notice that the mesh-size is small in both the x and the y directions. FriCAS normally computes points in both these directions. This is unnecessary. One step is all we need in the y-direction. To have FriCAS economize on y-points, we re-draw the ribbon with option `var2Steps == 1`.

Re-draw the ribbon, but with option `var2Steps == 1` so that only `1` step is computed in the y direction.

```
vp := draw(x^2,x=-1..1,y=0..1,var2Steps==1)
```



The operation has created a viewport, that is, a graphics window on your screen. We assigned the viewport to `vp` and now we manipulate its contents.

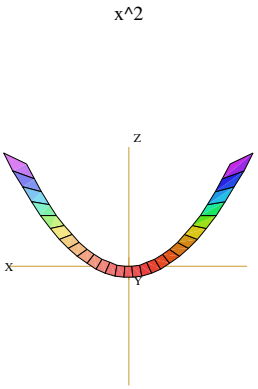
Graphs are objects, like numbers and algebraic expressions. You may want to do some experimenting with graphs. For example, say

```
showRegion(vp, "on")
```

to put a bounding box around the ribbon. Try it! Issue `rotate(vp, -45, 90)` to rotate the figure -45 longitudinal degrees and 90 latitudinal degrees.

Here is a different rotation. This turns the graph so you can view it along the y-axis.

```
rotate(vp, 0, -90)
```



There are many other things you can do. In fact, most everything you can do interactively using the three-dimensional control panel (such as translating, zooming, resizing, coloring, perspective and lighting selections) can also be done directly by operations (see Chapter 7 for more details).

When you are done experimenting, say `reset(vp)` to restore the picture to its original position and settings.

Let's add another ribbon to our picture—one for x^3. Since y ranges from 0 to 1 for the first ribbon, now let y range from 1 to 2. This puts the second ribbon next to the first one.

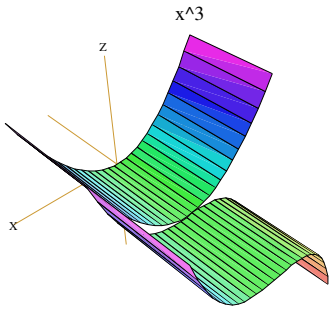
How do you add a second ribbon to the viewport? One method is to extract the "space" component from the viewport using the operation subspace. You can think of the space component as the object inside the window (here, the ribbon). Let's call it `sp`. To add the second ribbon, you draw the second ribbon using the option `space == sp`.

Extract the space component of `vp`.

```
sp := subspace(vp)
```

Add the ribbon for x^3 alongside that for x^2.

```
vp := draw(x^3,x=-1..1,y=1..2,var2Steps==1, space==sp)
```

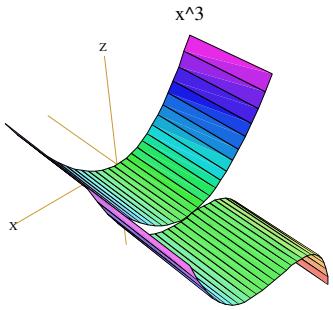


Unless you moved the original viewport, the new viewport covers the old one. You might want to check that the old object is still there by moving the top window.

Let's show quadrilateral polygon outlines on the ribbons and then enclose the ribbons in a box.

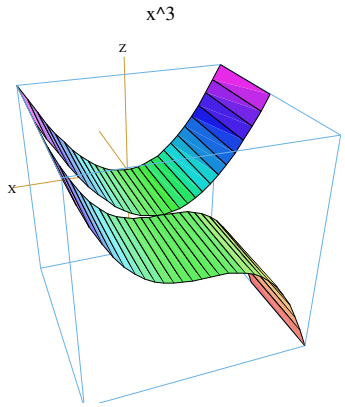
Show quadrilateral polygon outlines.

```
drawStyle(vp,"shade");outlineRender(vp,"on")
```



Enclose the ribbons in a box.

```
rotate(vp,20,-60); showRegion(vp,"on")
```



This process has become tedious! If we had to add two or three more ribbons, we would have to repeat the above steps several more times. It is time to write an interpreter program to help us take care of the details.

10.2 A Ribbon Program

The above approach creates a new viewport for each additional ribbon. A better approach is to build one object composed of all ribbons before creating a viewport. To do this, use makeObject rather than draw. The operations have similar formats, but draw returns a viewport and makeObject returns a space object.

We now create a function **drawRibbons** of two arguments: `flist`, a list of formulas for the ribbons you want to draw, and `xrange`, the range over which you want them drawn. Using this function, you can just say

```
drawRibbons([x^2, x^3], x=-1..1)
```

to do all of the work required in the last section. Here is the **drawRibbons** program. Invoke your favorite editor and create a file called **ribbon.input** containing the following program.

Listing 10.1: The first **drawRibbons** function.

```
1  drawRibbons(flist, xrange) ==
2    sp := createThreeSpace()                -- Create empty space sp.
3    y0 := 0                                 -- The initial ribbon position.
4    for f in flist repeat                   -- For each function f,
5      makeObject(f, xrange, y=y0..y0+1,     --   create and add a ribbon
6         space==sp, var2Steps == 1)         --   for f to the space sp.
7      y0 := y0 + 1                          -- The next ribbon position.
8    vp := makeViewport3D(sp, "Ribbons")     -- Create viewport.
9    drawStyle(vp, "shade")                  -- Select shading style.
10   outlineRender(vp, "on")                 -- Show polygon outlines.
11   showRegion(vp,"on")                     -- Enclose in a box.
12   n := # flist                            -- The number of ribbons
13   zoom(vp,n,1,n)                          -- Zoom in x- and z-directions.
14   rotate(vp,0,75)                         -- Change the angle of view.
15   vp                                      -- Return the viewport.
```

Here are some remarks on the syntax used in the **drawRibbons** function (consult Chapter 6 for more details). Unlike most other programming languages which use semicolons, parentheses, or *begin–end* brackets to delineate the structure of programs, the structure of a FriCAS program is determined by indentation. The first line of a function definition always begins in column 1. All other lines of the function are indented with respect to the first line and form a *pile* (see Section 5.2 on page 126).

The definition of **drawRibbons** consists of a pile of expressions to be executed one after another. Each expression of the pile is indented at the same level. Lines 4-7 designate one single expression: since lines 5-7 are indented with respect to the others, these lines are treated as a continuation of line 4. Also since lines 5 and 7 have the same indentation level, these lines designate a pile within the outer pile.

The last line of a pile usually gives the value returned by the pile. Here it is also the value returned by the function. FriCAS knows this is the last line of the function because it is the last line of the file. In other cases, a new expression beginning in column one signals the end of a function.

The line `drawStyle(vp,"shade")` is given after the viewport has been created to select the drawing style. We have also used the zoom option. Without the zoom, the viewport region would be scaled equally in all three coordinate directions.

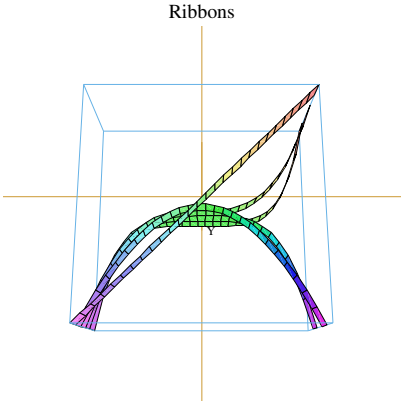
Let's try the function **drawRibbons**. First you must read the file to give FriCAS the function definition.

Read the input file.

```
)read ribbon
```

Draw ribbons for $x, x^2, \ldots, x^5$ for $-1 \leq x \leq 1$

```
drawRibbons([x^i for i in 1..5],x=-1..1)
```



10.3 Coloring and Positioning Ribbons

Before leaving the ribbon example, we make two improvements. Normally, the color given to each point in the space is a function of its height within a bounding box. The points at the bottom of the box are red, those at the top are purple.

To change the normal coloring, you can give an option `colorFunction ==` *function*. When FriCAS goes about displaying the data, it determines the range of colors used for all points within the box. FriCAS then distributes these numbers uniformly over the number of hues. Here we use the simple color function $(x, y) \mapsto i$ for the i^{th} ribbon.

Also, we add an argument `yrange` so you can give the range of `y` occupied by the ribbons. For example, if the `yrange` is given as `y=0..1` and there are 5 ribbons to be displayed, each ribbon would have width 0.2 and would appear in the range $0 \leq y \leq 1$.

Refer to lines 4-9. Line 4 assigns to `yVar` the variable part of the `yrange` (after all, it need not be `y`). Suppose that `yrange` is given as `t = a..b` where `a` and `b` have numerical values. Then line 5 assigns the value of `a` to the variable `y0`. Line 6 computes the width of the ribbon by dividing the difference of `a` and `b` by the number, `num`, of ribbons. The result is assigned to the variable `width`. Note that in the for-loop in line 7, we are iterating in parallel; it is not a nested loop.

Listing 10.2: The final **drawRibbons** function.

```
1  drawRibbons(flist, xrange, yrange) ==
2    sp := createThreeSpace()                          -- Create empty space sp.
3    num := # flist                                    -- The number of ribbons.
4    yVar := variable yrange                           -- The ribbon variable.
5    y0:Float    := low segment yrange                 -- The first ribbon coordinate.
6    width:Float := (high segment yrange - y0)/num     -- The width of a ribbon.
7    for f in flist for color in 1..num repeat         -- For each function f,
8      makeObject(f, xrange, yVar = y0..y0+width,      --   create and add ribbon to
9        var2Steps == 1, _                             --   sp of a different color.
10       colorFunction == (x,y) +-> color, _
11       space == sp)
12     y0 := y0 + width                                -- The next ribbon coordinate.
13   vp := makeViewport3D(sp, "Ribbons")               -- Create viewport.
14   drawStyle(vp, "shade")                            -- Select shading style.
15   outlineRender(vp, "on")                           -- Show polygon outlines.
16   showRegion(vp, "on")                              -- Enclose – in a box.
17   vp                                                -- Return the viewport.
```

10.4 Points, Lines, and Curves

What you have seen so far is a high-level program using the graphics facility. We now turn to the more basic notions of points, lines, and curves in three-dimensional graphs. These facilities use small floats (objects of type **DoubleFloat**) for data. Let us first give names to the small float values 0 and 1. The small float 0.

```
zero := 0.0@DFLOAT
```

$$0.0 \tag{1}$$

DoubleFloat

The small float 1.

```
one := 1.0@DFLOAT
```

$$1.0 \tag{2}$$

DoubleFloat

The "@" sign means "of the type." Thus `zero` is 0.0 of the type **DoubleFloat**. You can also say `0.0::DFLOAT`.

Points can have four small float components: x, y, z coordinates and an optional color. A "curve" is simply a list of points connected by straight line segments. Create the point `origin` with color zero, that is, the lowest color on the color map.

```
origin := point [zero,zero,zero,zero]
```

$$[0.0,\ 0.0,\ 0.0,\ 0.0] \tag{3}$$

Point(DoubleFloat)

Create the point `unit` with color zero.

```
unit := point [one,one,one,zero]
```

$$[1.0,\ 1.0,\ 1.0,\ 0.0] \tag{4}$$

Point(DoubleFloat)

Create the curve (well, here, a line) from `origin` to `unit`.

```
line := [origin, unit]
```

$$[[0.0,\ 0.0,\ 0.0,\ 0.0]\ ,\ [1.0,\ 1.0,\ 1.0,\ 0.0]] \tag{5}$$

List(Point(DoubleFloat))

We make this line segment into an arrow by adding an arrowhead. The arrowhead extends to, say, `p3` on the left, and to, say, `p4` on the right. To describe an arrow, you tell FriCAS to draw the two curves `[p1, p2, p3]` and `[p2, p4]`. We also decide through experimentation on values for `arrowScale`, the ratio of the size of the arrowhead to the stem of the arrow, and `arrowAngle`, the angle between the arrowhead and the arrow.

Invoke your favorite editor and create an input file called **arrows.input**. This input file first defines the values of arrowAngle and arrowScale, then defines the function **makeArrow**(p_1, p_2) to draw an arrow from point p_1 to p_2.

```
arrowAngle := %pi-%pi/10.0@DFLOAT                -- The angle of the arrowhead.
arrowScale := 0.2@DFLOAT                         -- The size of the arrowhead
                                                 --   relative to the stem.
makeArrow(p1, p2) ==
  delta := p2 - p1                               -- The arrow.
  len := arrowScale * length delta               -- The length of the arrowhead.
  theta := atan(delta.1, delta.2)                -- The angle from the x-axis
  c1 := len*cos(theta + arrowAngle)              -- The x-coord of left endpoint.
  s1 := len*sin(theta + arrowAngle)              -- The y-coord of left endpoint.
  c2 := len*cos(theta - arrowAngle)              -- The x-coord of right endpoint.
  s2 := len*sin(theta - arrowAngle)              -- The y-coord of right endpoint.
  z  := p2.3*(1 - arrowScale)                    -- The z-coord of both endpoints.
  p3 := point [p2.1 + c1, p2.2 + s1, z, p2.4]    -- The left endpoint of head.
  p4 := point [p2.1 + c2, p2.2 + s2, z, p2.4]    -- The right endpoint of head.
  [[p1, p2, p3], [p2, p4]]                       -- The arrow as a list of curves.
```

Read the file and then create an arrow from the point origin to the point unit. Read the input file defining **makeArrow**.

```
)read arrows
```

Construct the arrow (a list of two curves).

```
arrow := makeArrow(origin,unit)
```

```
   Compiling function makeArrow with type (Point(DoubleFloat), Point(DoubleFloat))
       -> List(List(Point(DoubleFloat)))
```

$$[[[0.0,\ 0.0,\ 0.0,\ 0.0],\ [1.0,\ 1.0,\ 1.0,\ 0.0],\ [0.6913462860460797,\ 0.842733077659504,\ 0.8,\ 0.0]],\ [[1.0,\ 1.0,\ 1.0,\ 0.0],\ [0.842733077659504,\ 0.6913462860460797,\ 0.8,\ 0.0]]] \tag{9}$$

List (List (Point(DoubleFloat)))

Create an empty object sp of type **ThreeSpace**.

```
sp := createThreeSpace()
```

```
                                3-Space with 0 components                          (10)
```

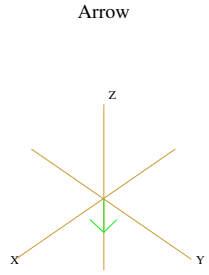
ThreeSpace(DoubleFloat)

Add each curve of the arrow to the space sp.

```
for a in arrow repeat sp := curve(sp,a)
```

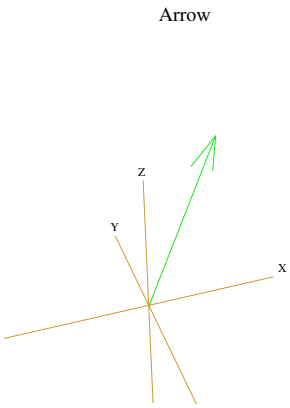
Create a three-dimensional viewport containing that space.

```
vp := makeViewport3D(sp,"Arrow")
```



Here is a better viewing angle.

```
rotate(vp,200,-60)
```



10.5 A Bouquet of Arrows

Let's draw a "bouquet" of arrows. Each arrow is identical. The arrowheads are uniformly placed on a circle parallel to the xy-plane. Thus the position of each arrow differs only by the angle θ, $0 \leq \theta < 2\pi$, between the arrow and the x-axis on the xy-plane.

Our bouquet is rather special: each arrow has a different color. This is arranged by letting the color of each successive arrow be denoted by θ. In this way, the color of arrows ranges from red to green to violet. Here is a program to draw a bouquet of n arrows.

```
1  drawBouquet(n,title) ==
2    z := 0.0@DFLOAT
3    e := 1.0@DFLOAT
4    angle := z                                          -- The initial angle.
5    sp := createThreeSpace()                            -- Create empty space sp.
6    for i in 0..n-1 repeat                              -- For each index i, create:
7      start := point [z,z,z,angle]                      -- point at base of arrow;
8      end   := point [cos angle, sin angle, e, angle]   -- point at tip of arrow;
9      arrow := makeArrow(start,end)                     -- ith arrow.
```

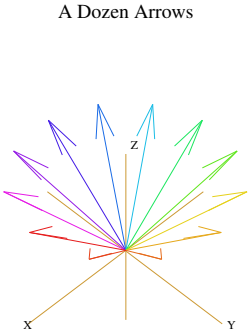
```
10      for a in makeArrow(start,end) repeat          -- For each arrow component,
11        curve(sp,a)                                 --    add the component to sp.
12      angle := angle + 2*%pi/n                      -- The next angle.
13    makeViewport3D(sp,title)                        -- Create the viewport from sp.
```

Read the input file.

```
)read bouquet
```

A bouquet of a dozen arrows.

```
drawBouquet(12,"A Dozen Arrows")
```



10.6 Drawing Complex Vector Fields

We now put our arrows to good use drawing complex vector fields. These vector fields give a representation of complex-valued functions of complex variables. Consider a Cartesian coordinate grid of points (x, y) in the plane, and some complex-valued function f defined on this grid. At every point on this grid, compute the value of $f(x + iy)$ and call it z. Since z has both a real and imaginary value for a given (x, y) grid point, there are four dimensions to plot. What do we do? We represent the values of z by arrows planted at each grid point. Each arrow represents the value of z in polar coordinates (r, θ). The length of the arrow is proportional to r. Its direction is given by θ.

The code for drawing vector fields is in the file **vectors.input**. We discuss its contents from top to bottom.

Before showing you the code, we have two small matters to take care of. First, what if the function has large spikes, say, ones that go off to infinity? We define a variable `clipValue` for this purpose. When `r` exceeds the value of `clipValue`, then the value of `clipValue` is used instead of that for `r`. For convenience, we define a function `clipFun(x)` which uses `clipValue` to "clip" the value of `x`.

```
1  clipValue : DFLOAT := 6                          -- Maximum value allowed.
2  clipFun(x) == min(max(x,-clipValue),clipValue)
```

Notice that we identify `clipValue` as a small float but do not declare the type of the function **clipFun**. As it turns out, **clipFun** is called with a small float value. This declaration ensures that **clipFun** never does a conversion when it is called.

The second matter concerns the possible "poles" of a function, the actual points where the spikes have infinite values. FriCAS uses normal **DoubleFloat** arithmetic which does not directly handle infinite values. If your function has poles, you must adjust your step size to avoid landing directly on them (FriCAS calls error when asked to divide a value by 0, for example).

We set the variables realSteps and imagSteps to hold the number of steps taken in the real and imaginary directions, respectively. Most examples will have ranges centered around the origin. To avoid a pole at the origin, the number of points is taken to be odd.

```
realSteps: INT := 25                                    -- Number of real steps.
imagSteps: INT := 25                                    -- Number of imaginary steps.
)read arrows
```

Now define the function **drawComplexVectorField** to draw the arrows. It is good practice to declare the type of the main function in the file. This one declaration is usually sufficient to ensure that other lower-level functions are compiled with the correct types.

```
C := Complex DoubleFloat
S := Segment DoubleFloat
drawComplexVectorField: (C -> C, S, S) -> VIEW3D
```

The first argument is a function mapping complex small floats into complex small floats. The second and third arguments give the range of real and imaginary values as segments like a..b. The result is a three-dimensional viewport. Here is the full function definition:

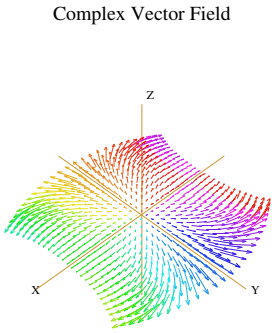
```
drawComplexVectorField(f, realRange,imagRange) ==
  delReal := (high(realRange)-low(realRange))/realSteps  -- The real step size.
  delImag := (high(imagRange)-low(imagRange))/imagSteps  -- The imaginary step size.
  sp := createThreeSpace()                              -- Create empty space sp.
  real := low(realRange)                                -- The initial real value.
  for i in 1..realSteps+1 repeat                        -- Begin real iteration.
    imag := low(imagRange)                              -- The initial imaginary value.
    for j in 1..imagSteps+1 repeat                      -- Begin imaginary iteration.
      z := f complex(real,imag)                         -- The value of f at the point.
      arg := argument z                                 -- The direction of the arrow.
      len := clipFun sqrt norm z                        -- The length of the arrow.
      p1 := point [real, imag, 0.0@DFLOAT, arg]         -- The base point of the arrow.
      scaleLen := delReal * len                         -- The scaled length of the arrow.
      p2 := point [p1.1 + scaleLen*cos(arg),            -- The tip point of the arrow.
                   p1.2 + scaleLen*sin(arg),0.0@DFLOAT, arg]
      arrow := makeArrow(p1, p2)                        -- Create the arrow.
      for a in arrow repeat curve(sp, a)                -- Add arrow to the space sp.
      imag := imag + delImag                            -- The next imaginary value.
    real := real + delReal                              -- The next real value.
  makeViewport3D(sp, "Complex Vector Field")            -- Draw it!
```

As a first example, let us draw f(z) == sin(z). There is no need to create a user function: just pass the sin from **Complex DoubleFloat**. Read the file.

```
)read vectors
```

Draw the complex vector field of sin(x).

```
drawComplexVectorField(sin,-2..2,-2..2)
```



10.7 Drawing Complex Functions

Here is another way to graph a complex function of complex arguments. For each complex value z, compute $f(z)$, again expressing the value in polar coordinates (r, θ). We draw the complex valued function, again considering the (x, y)-plane as the complex plane, using r as the height (or z-coordinate) and θ as the color. This is a standard plot—we learned how to do this in Chapter 7—but here we write a new program to illustrate the creation of polygon meshes, or grids.

Call this function **drawComplex**. It displays the points using the "mesh" of points. The function definition is in three parts.

```
drawComplex: (C -> C, S, S) -> VIEW3D
drawComplex(f, realRange, imagRange) ==                -- The first part.
  delReal := (high(realRange)-low(realRange))/realSteps  -- The real step size.
  delImag := (high(imagRange)-low(imagRange))/imagSteps  -- The imaginary step size.
  llp:List List Point DFLOAT := []                     -- Initial list of list of points llp.
```

Variables `delReal` and `delImag` give the step sizes along the real and imaginary directions as computed by the values of the global variables `realSteps` and `imagSteps`. The mesh is represented by a list of lists of points `llp`, initially empty. Now `[ ]` alone is ambiguous, so to set this initial value you have to tell FriCAS what type of empty list it is. Next comes the loop which builds `llp`.

```
  real := low(realRange)                               -- The initial real value.
  for i in 1..realSteps+1 repeat                       -- Begin real iteration.
    imag := low(imagRange)                             -- The initial imaginary value.
    lp := []$(List Point DFLOAT)                       -- The initial list of points lp.
    for j in 1..imagSteps+1 repeat                     -- Begin imaginary iteration.
      z := f complex(real,imag)                        -- The value of f at the point.
      pt := point [real,imag, clipFun sqrt norm z,     -- Create a point.
                   argument z]
      lp := cons(pt,lp)                                -- Add the point to lp.
      imag := imag + delImag                           -- The next imaginary value.
    real := real + delReal                             -- The next real value.
    llp := cons(lp, llp)                               -- Add lp to llp.
```

The code consists of both an inner and outer loop. Each pass through the inner loop adds one list `lp` of points to the list of lists of points `llp`. The elements of `lp` are collected in reverse order.

```
1  makeViewport3D(mesh(llp), "Complex Function")      -- Create a mesh and display.
```

The operation mesh then creates an object of type **ThreeSpace(DoubleFloat)** from the list of lists of points. This is then passed to makeViewport3D to display the image.

Now add this function directly to your **vectors.input** file and re-read the file using `)read vectors`. We try **drawComplex** using a user-defined function `f`.

Read the file.

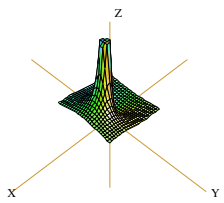
```
)read vectors
```

This one has a pole at $z = 0$.

```
f(z) == exp(1/z)
```

Draw it with an odd number of steps to avoid the pole.

```
drawComplex(f,-2..2,-2..2)
```



10.8 Functions Producing Functions

In Section 6.14 on page 169, you learned how to use the operation function to create a function from symbolic formulas. Here we introduce a similar operation which not only creates functions, but functions from functions.

The facility we need is provided by the package **MakeUnaryCompiledFunction(E,S,T)**. This package produces a unary (one-argument) compiled function from some symbolic data generated by a previous computation.[1] The `E` tells where the symbolic data comes from; the `S` and `T` give FriCAS the source and target type of the function, respectively. The compiled function produced has type `S` $\rightarrow$ `T`. To produce a compiled function with definition `p(x) == expr`, call `compiledFunction(expr, x)` from this package. The function you get has no name. You must assign the function to the variable `p` to give it that name. Do some computation.

```
(x+1/3)^5
```

[1] **MakeBinaryCompiledFunction** is available for binary functions.

$$x^5 + \frac{5}{3}x^4 + \frac{10}{9}x^3 + \frac{10}{27}x^2 + \frac{5}{81}x + \frac{1}{243} \tag{1}$$

Polynomial(Fraction(Integer))

Convert this to an anonymous function of x. Assign it to the variable p to give the function a name.

```
p := compiledFunction(%,x)$MakeUnaryCompiledFunction(POLY FRAC INT,DFLOAT,DFLOAT)
```

```
   Compiling function %A with type DoubleFloat -> DoubleFloat
```

$$\text{theMap(unaryFunction)} \tag{2}$$

(DoubleFloat → DoubleFloat)

Apply the function.

```
p(sin(1.3))
```

$$3.668751115057229 \tag{3}$$

DoubleFloat

For a more sophisticated application, read on.

10.9 Automatic Newton Iteration Formulas

We resume our continuing saga of arrows and complex functions. Suppose we want to investigate the behavior of Newton's iteration function in the complex plane. Given a function f, we want to find the complex values z such that $f(z) = 0$.

The first step is to produce a Newton iteration formula for a given f: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. We represent this formula by a function g that performs the computation on the right-hand side, that is, $x_{n+1} = g(x_n)$.

The type **Expression Integer** (abbreviated **EXPR INT**) is used to represent general symbolic expressions in FriCAS. To make our facility as general as possible, we assume f has this type. Given f, we want to produce a Newton iteration function g which, given a complex point x_n, delivers the next Newton iteration point x_{n+1}.

This time we write an input file called **newton.input**. We need to import **MakeUnaryCompiledFunction** (discussed in the last section), call it with appropriate types, and then define the function newtonStep which references it. Here is the function newtonStep:

```
1  C := Complex DoubleFloat                          -- The complex numbers.
2  E := Expression Integer                           -- The expression domain.
3  complexFunPack:=MakeUnaryCompiledFunction(E,C,C)  -- Package for making functions.
4
5  newtonStep(f) ==                                  -- Newton's iteration function.
6    fun  := complexNumericFunction f                -- Function for f.
7    deriv := complexDerivativeFunction(f,1)         -- Function for f'.
8    (x:C):C +->                                     -- Return the iterator function.
9      x - fun(x)/deriv(x)
10
11 complexNumericFunction f ==                       -- Turn an expression f into a
12   v := theVariableIn f                            --   function.
13   compiledFunction(f, v)$complexFunPack
```

```
14
15 complexDerivativeFunction(f,n) ==                    -- Create an nth derivative
16   v := theVariableIn f                               --    function.
17   df := D(f,v,n)
18   compiledFunction(df, v)$complexFunPack
19
20 theVariableIn f ==                                   -- Returns the variable in f.
21   vl := variables f                                  -- The list of variables.
22   nv := # vl                                         -- The number of variables.
23   nv > 1 => error "Expression is not univariate."
24   nv = 0 => 'x                                       -- Return a dummy variable.
25   first vl
```

Do you see what is going on here? A formula `f` is passed into the function **newtonStep**. First, the function turns `f` into a compiled program mapping complex numbers into complex numbers. Next, it does the same thing for the derivative of `f`. Finally, it returns a function which computes a single step of Newton's iteration.

The function **complexNumericFunction** extracts the variable from the expression `f` and then turns `f` into a function which maps complex numbers into complex numbers. The function **complexDerivativeFunction** does the same thing for the derivative of `f`. The function **theVariableIn** extracts the variable from the expression `f`, calling the function error if `f` has more than one variable. It returns the dummy variable `x` if `f` has no variables.

Let's now apply **newtonStep** to the formula for computing cube roots of two. Read the input file with the definitions.

```
)read newton
```

```
)read vectors
```

The cube root of two.

```
f := x^3 - 2
```

$$x^3 - 2 \tag{19}$$

Polynomial(Integer)

Get Newton's iteration formula.

```
g := newtonStep f
```

```
   Compiling function theVariable with type Polynomial(Integer) -> Symbol
   Compiling function complexNumericFunction with type Polynomial(Integer) -> (
      Complex(DoubleFloat) -> Complex(DoubleFloat))
   Compiling function complexDerivativeFunction with type (Polynomial(Integer),
      PositiveInteger) -> (Complex(DoubleFloat) -> Complex(DoubleFloat))
   Compiling function newtonStep with type Polynomial(Integer) -> (Complex(
      DoubleFloat) -> Complex(DoubleFloat))
   Compiling function %B with type Complex(DoubleFloat) -> Complex(DoubleFloat)
   Compiling function %C with type Complex(DoubleFloat) -> Complex(DoubleFloat)
```

$$\text{theMap}(?) \tag{20}$$

(Complex(DoubleFloat)→ Complex(DoubleFloat))

Let `a` denote the result of applying Newton's iteration once to the complex number `1 + %i`.

```
a := g(1.0 + %i)
```

$$0.6666666666666667 + 0.33333333333333337\,i \tag{21}$$

Complex(DoubleFloat)

Now apply it repeatedly. How fast does it converge?

```
[(a := g(a)) for i in 1..]
```

$$\begin{aligned}[1.1644444444444444 &- 0.7377777777777778\,i,\ 0.9261400469716478 - 0.17463006425584393\,i,\\ &1.3164444838140228 + 0.15690694583015852\,i,\\ &1.2462991025761463 + 0.015454763610132094\,i,\\ &1.259872529653208 - \texttt{3.382716205931127e-4}\,i,\\ &1.259920960928212 + \texttt{2.602353465342268e-8}\,i,\\ &1.259921049894879 - \texttt{3.6751942591616685e-15}\,i,\ \ldots]\end{aligned} \tag{22}$$

Stream(Complex(DoubleFloat))

Check the accuracy of the last iterate.

```
a^3
```

$$2.0000000000000275 - \texttt{1.7502021699542322e-14}\,i \tag{23}$$

Complex(DoubleFloat)

In '`MappingPackage1`' on page 500, we show how functions can be manipulated as objects in FriCAS. A useful operation to consider here is `*`, which means composition. For example `g*g` causes the Newton iteration formula to be applied twice. Correspondingly, `g^n` means to apply the iteration formula `n` times.

Apply `g` twice to the point `1 + %i`.

```
(g*g) (1.0 + %i)
```

$$1.1644444444444444 - 0.7377777777777778\,i \tag{24}$$

Complex(DoubleFloat)

Apply `g` 11 times.

```
(g^11) (1.0 + %i)
```

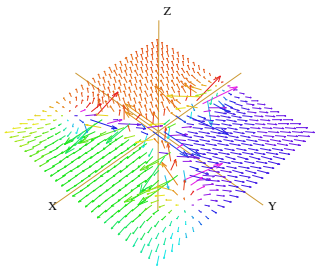
$$1.2599210498948732 \tag{25}$$

Complex(DoubleFloat)

Look now at the vector field and surface generated after two steps of Newton's formula for the cube root of two. The poles in these pictures represent bad starting values, and the flat areas are the regions of convergence to the three roots. The vector field.

```
drawComplexVectorField(g^3,-3..3,-3..3)
```

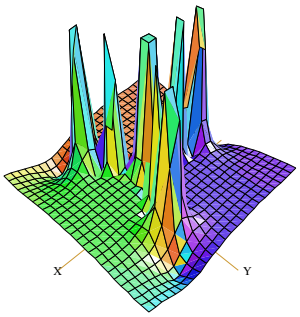
Complex Vector Field



The surface.

```
drawComplex(g^3,-3..3,-3..3)
```

Complex Function



Chapter 11

Packages

Packages provide the bulk of FriCAS's algorithmic library, from numeric packages for computing special functions to symbolic facilities for differential equations, symbolic integration, and limits.

In Chapter 10, we developed several useful functions for drawing vector fields and complex functions. We now show you how you can add these functions to the FriCAS library to make them available for general use.

The way we created the functions in Chapter 10 is typical of how you, as an advanced FriCAS user, may interact with FriCAS. You have an application. You go to your editor and create an input file defining some functions for the application. Then you run the file and try the functions. Once you get them all to work, you will often want to extend them, add new features, perhaps write additional functions.

Eventually, when you have a useful set of functions for your application, you may want to add them to your local FriCAS library. To do this, you embed these function definitions in a package and add that package to the library.

To introduce new packages, categories, and domains into the system, you need to use the FriCAS compiler to convert the constructors into executable machine code. An existing compiler in FriCAS is available on an "as-is" basis. A new, faster compiler will be available in version 2.0 of FriCAS.

Listing 11.1: The DrawComplex package.

```
C      ==> Complex DoubleFloat                    -- All constructors used in a file
S      ==> Segment DoubleFloat                    --    must be spelled out in full
INT    ==> Integer                                --    unless abbreviated by macros
DFLOAT ==> DoubleFloat                            --    like these at the top of
VIEW3D ==> ThreeDimensionalViewport               --    a file.
CURVE  ==> List List Point DFLOAT

)abbrev package DRAWCX DrawComplex                -- Identify kinds and abbreviations
DrawComplex(): Exports == Implementation where    -- Type definition begins here.

  Exports == with                                 -- Export part begins.
    drawComplex: (C -> C,S,S,Boolean) -> VIEW3D   -- Exported Operations
    drawComplexVectorField: (C -> C,S,S) -> VIEW3D
    setRealSteps: INT -> INT
    setImagSteps: INT -> INT
    setClipValue: DFLOAT -> DFLOAT

  Implementation == add                           -- Implementation part begins.
    arrowScale : DFLOAT := (0.2)::DFLOAT          -- (relative size) Local variable 1.
```

```
20      arrowAngle : DFLOAT := pi()-pi()/(20::DFLOAT)     -- Local variable 2.
21      realSteps  : INT := 11                            -- (real steps) Local variable 3.
22      imagSteps  : INT := 11                            -- (imaginary steps) Local variable 4.
23      clipValue  : DFLOAT  := 10::DFLOAT                -- (maximum vector length) Local variable 5.
24
25      setRealSteps(n) == realSteps := n                 -- Exported function definition 1.
26      setImagSteps(n) == imagSteps := n                 -- Exported function definition 2.
27      setClipValue(c) == clipValue := c                 -- Exported function definition 3.
28
29      clipFun: DFLOAT -> DFLOAT                         -- Clip large magnitudes.
30      clipFun(x) == min(max(x, -clipValue), clipValue)  -- Local function definition 1.
31
32      makeArrow: (Point DFLOAT,Point DFLOAT,DFLOAT,DFLOAT) -> CURVE
33      makeArrow(p1, p2, len, arg) == ...                -- Local function definition 2.
34
35      drawComplex(f, realRange, imagRange, arrows?) == ... -- Exported function definition 4.
```

11.1 Names, Abbreviations, and File Structure

Each package has a name and an abbreviation. For a package of the complex draw functions from Chapter 10, we choose the name **DrawComplex** and abbreviation **DRAWCX**.[1] To be sure that you have not chosen a name or abbreviation already used by the system, issue the system command `)show` for both the name and the abbreviation.

Once you have named the package and its abbreviation, you can choose any new filename you like with extension "**.spad**" to hold the definition of your package. We choose the name **drawpak.spad**. If your application involves more than one package, you can put them all in the same file. FriCAS assumes no relationship between the name of a library file, and the name or abbreviation of a package.

Near the top of the "**.spad**" file, list all the abbreviations for the packages using `)abbrev`, each command beginning in column one. Macros giving names to FriCAS expressions can also be placed near the top of the file. The macros are only usable from their point of definition until the end of the file.

Consider the definition of **DrawComplex** in Figure 11.1. After the macro definition

```
S       ==> Segment DoubleFloat
```

the name `S` can be used in the file as a shorthand for **Segment DoubleFloat**.[2] The abbreviation command for the package

```
)abbrev package DRAWCX DrawComplex
```

is given after the macros (although it could precede them).

11.2 Syntax

The definition of a package has the syntax:

[1] An abbreviation can be any string of between two and seven capital letters and digits, beginning with a letter. See Section 2.2.5 on page 73 for more information.

[2] The interpreter also allows `macro` for macro definitions.

PackageForm : *Exports* == *Implementation*

The syntax for defining a package constructor is the same as that for defining any function in FriCAS. In practice, the definition extends over many lines so that this syntax is not practical. Also, the type of a package is expressed by the operator `with` followed by an explicit list of operations. A preferable way to write the definition of a package is with a `where` expression:

The definition of a package usually has the form:
PackageForm : `Exports == Implementation where`
    optional type declarations
    `Exports == with`
        list of exported operations
    `Implementation == add`
        list of function definitions for exported operations

The **DrawComplex** package takes no parameters and exports five operations, each a separate item of a *pile*. Each operation is described as a *declaration*: a name, followed by a colon (":"), followed by the type of the operation. All operations have types expressed as *mappings* with the syntax

source → *target*

11.3 Abstract Datatypes

A constructor as defined in FriCAS is called an *abstract datatype* in the computer science literature. Abstract datatypes separate "specification" (what operations are provided) from "implementation" (how the operations are implemented). The `Exports` (specification) part of a constructor is said to be "public" (it provides the user interface to the package) whereas the `Implementation` part is "private" (information here is effectively hidden—programs cannot take advantage of it).

The `Exports` part specifies what operations the package provides to users. As an author of a package, you must ensure that the `Implementation` part provides a function for each operation in the `Exports` part.[3]

An important difference between interactive programming and the use of packages is in the handling of global variables such as `realSteps` and `imagSteps`. In interactive programming, you simply change the values of variables by *assignment*. With packages, such variables are local to the package—their values can only be set using functions exported by the package. In our example package, we provide two functions setRealSteps and setImagSteps for this purpose.

Another local variable is `clipValue` which can be changed using the exported operation setClipValue. This value is referenced by the internal function clipFun that decides whether to use the computed value of the function at a point or, if the magnitude of that value is too large, the value assigned to `clipValue` (with the appropriate sign).

[3]The **DrawComplex** package enhances the facility described in Chapter 10.7 by allowing a complex function to have arrows emanating from the surface to indicate the direction of the complex argument.

11.4 Capsules

The part to the right of `add` in the `Implementation` part of the definition is called a *capsule*. The purpose of a capsule is:

- to define a function for each exported operation, and
- to define a *local environment* for these functions to run.

What is a local environment? First, what is an environment? Think of the capsule as an input file that FriCAS reads from top to bottom. Think of the input file as having a `)clear all` at the top so that initially no variables or functions are defined. When this file is read, variables such as `realSteps` and `arrowSize` in **DrawComplex** are set to initial values. Also, all the functions defined in the capsule are compiled. These include those that are exported (like `drawComplex`), and those that are not (like `makeArrow`). At the end, you get a set of name-value pairs: variable names (like `realSteps` and `arrowSize`) are paired with assigned values, while operation names (like `drawComplex` and `makeArrow`) are paired with function values.

This set of name-value pairs is called an *environment*. Actually, we call this environment the "initial environment" of a package: it is the environment that exists immediately after the package is first built. Afterwards, functions of this capsule can access or reset a variable in the environment. The environment is called *local* since any changes to the value of a variable in this environment can be seen *only* by these functions.

Only the functions from the package can change the variables in the local environment. When two functions are called successively from a package, any changes caused by the first function called are seen by the second.

Since the environment is local to the package, its names don't get mixed up with others in the system or your workspace. If you happen to have a variable called `realSteps` in your workspace, it does not affect what the **DrawComplex** functions do in any way.

The functions in a package are compiled into machine code. Unlike function definitions in input files that may be compiled repeatedly as you use them with varying argument types, functions in packages have a unique type (generally parameterized by the argument parameters of a package) and a unique compilation residing on disk.

The capsule itself is turned into a compiled function. This so-called *capsule function* is what builds the initial environment spoken of above. If the package has arguments (see below), then each call to the package constructor with a distinct pair of arguments builds a distinct package, each with its own local environment.

11.5 Input Files vs. Packages

A good question at this point would be "Is writing a package more difficult than writing an input file?"

The programs in input files are designed for flexibility and ease-of-use. FriCAS can usually work out all of your types as it reads your program and does the computations you request. Let's say that you define a one-argument function without giving its type. When you first apply the function to a value, this value is understood by FriCAS as identifying the type for the argument parameter. Most of the time FriCAS goes through the body of your function and figures out the target type that you have in

mind. FriCAS sometimes fails to get it right. Then—and only then—do you need a declaration to tell FriCAS what type you want.

Input files are usually written to be read by FriCAS—and by you. Without suitable documentation and declarations, your input files are likely incomprehensible to a colleague—and to you some months later!

Packages are designed for legibility, as well as run-time efficiency. There are few new concepts you need to learn to write packages. Rather, you just have to be explicit about types and type conversions. The types of all functions are pre-declared so that FriCAS—and the reader— knows precisely what types of arguments can be passed to and from the functions (certainly you don't want a colleague to guess or to have to work this out from context!). The types of local variables are also declared. Type conversions are explicit, never automatic.[4]

In summary, packages are more tedious to write than input files. When writing input files, you can casually go ahead, giving some facts now, leaving others for later. Writing packages requires forethought, care and discipline.

11.6 Compiling Packages

Once you have defined the package **DrawComplex**, you need to compile and test it. To compile the package, issue the system command `)compile drawpak`. FriCAS reads the file **drawpak.spad** and compiles its contents into machine binary. If all goes well, the file **DRAWCX.NRLIB** is created in your local directory for the package. To test the package, you must load the package before trying an operation.

Compile the package.

```
)compile drawpak
```

Expose the package.

```
)expose DRAWCX
```

```
   DrawComplex is now explicitly exposed in frame initial
```

Use an odd step size to avoid a pole at the origin.

```
setRealSteps 51
```

$$51 \tag{1}$$

PositiveInteger

```
setImagSteps 51
```

$$51 \tag{2}$$

PositiveInteger

Define **f** to be the Gamma function.

```
f(z) == Gamma(z)
```

[4]There is one exception to this rule: conversions from a subdomain to a domain are automatic. After all, the objects both have the domain as a common type.

Clip values of function with magnitude larger than 7.

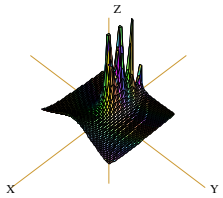
```
setClipValue 7
```

$$7.0 \tag{4}$$

DoubleFloat

Draw the Gamma function.

```
drawComplex(f,-%pi..%pi,-%pi..%pi, false)
```



11.7 Parameters

The power of packages becomes evident when packages have parameters. Usually these parameters are domains and the exported operations have types involving these parameters.

In Chapter 2, you learned that categories denote classes of domains. Although we cover this notion in detail in the next chapter, we now give you a sneak preview of its usefulness.

In Section 6.15 on page 172, we defined functions `bubbleSort(m)` and `insertionSort(m)` to sort a list of integers. If you look at the code for these functions, you see that they may be used to sort *any* structure `m` with the right properties. Also, the functions can be used to sort lists of *any* elements—not just integers. Let us now recall the code for `bubbleSort`.

```
bubbleSort(m) ==
  n := #m
  for i in 1..(n-1) repeat
    for j in n..(i+1) by -1 repeat
      if m.j < m.(j-1) then swap!(m,j,j-1)
  m
```

What properties of "lists of integers" are assumed by the sorting algorithm? In the first line, the operation `#` computes the maximum index of the list. The first obvious property is that `m` must have a finite number of elements. In FriCAS, this is done by telling FriCAS that `m` has the category **finiteAggregate**. As we show later in Section 12.9 on page 665, by using category tests programs can query domains as to the presence or absence of property (attribute) represented by a category.

The operation swap! swaps elements of m. Using Browse, you find that swap! requires its elements to come from a domain of category **IndexedAggregate** which also has category **shallowlyMutable**. This category means that you can change the internal components of m without changing its external structure. Shallowly-mutable data structures include lists, streams, one- and two-dimensional arrays, vectors, and matrices.

The category **IndexedAggregate** designates the class of aggregates whose elements can be accessed by the notation m(s) for suitable selectors s. The category **IndexedAggregate** takes two arguments: Index, a domain of selectors for the aggregate, and Entry, a domain of entries for the aggregate. Since the sort functions access elements by integers, we must choose Index = **Integer**. The most general class of domains for which bubbleSort and insertionSort are defined are those of category **IndexedAggregate(Integer,Entry)** with the two additional categories **shallowlyMutable** and **finiteAggregate**.

Using Browse, you can also discover that FriCAS has many kinds of domains of category **shallowlyMutable**. Those of class **IndexedAggregate(Integer,Entry)** include **Bits**, **FlexibleArray**, **OneDimensionalArray**, **List**, **String**, and **Vector**, and also **HashTable** and **EqTable** with integer keys. Although you may never want to sort all such structures, we nonetheless demonstrate FriCAS's ability to do so.

Another requirement is that **Entry** has an operation <. One way to get this operation is to assume that **Entry** has category **OrderedSet**. By definition, it will then export a < operation. A more general approach is to allow any comparison function f to be used for sorting. This function will be passed as an argument to the sorting functions.

Our sorting package then takes two arguments: a domain S of objects of *any* type, and a domain A, an aggregate of type **IndexedAggregate(Integer, S)** with the above two additional categories. Here is its definition using what are close to the original definitions of bubbleSort and insertionSort for sorting lists of integers. The symbol "!" is added to the ends of the operation names. This uniform naming convention is used for FriCAS operation names that destructively change one or more of their arguments.

```
1  SortPackage(S,A) : Exports == Implementation where
2    S: Object
3    A: IndexedAggregate(Integer,S)
4      with (finiteAggregate; shallowlyMutable)
5
6    Exports == with
7      bubbleSort!: (A,(S,S) -> Boolean) -> A
8      insertionSort!: (A, (S,S) -> Boolean) -> A
9
10   Implementation == add
11     bubbleSort!(m,f) ==
12       n := #m
13       for i in 1..(n-1) repeat
14         for j in n..(i+1) by -1 repeat
15           if f(m.j,m.(j-1)) then swap!(m,j,j-1)
16       m
17     insertionSort!(m,f) ==
18       for i in 2..#m repeat
19         j := i
20         while j > 1 and f(m.j,m.(j-1)) repeat
21           swap!(m,j,j-1)
22           j := (j - 1) pretend PositiveInteger
23       m
```

11.8 Conditionals

When packages have parameters, you can say that an operation is or is not exported depending on the values of those parameters. When the domain of objects S has an < operation, we can supply one-argument versions of bubbleSort and insertionSort which use this operation for sorting. The presence of the operation < is guaranteed when S is an ordered set.

```
Exports == with
    bubbleSort!: (A,(S,S) -> Boolean) -> A
    insertionSort!: (A, (S,S) -> Boolean) -> A

    if S has OrderedSet then
      bubbleSort!: A -> A
      insertionSort!: A -> A
```

In addition to exporting the one-argument sort operations conditionally, we must provide conditional definitions for the operations in the Implementation part. This is easy: just have the one-argument functions call the corresponding two-argument functions with the operation < from S.

```
  Implementation == add
       ...
    if S has OrderedSet then
      bubbleSort!(m) == bubbleSort!(m,<$S)
      insertionSort!(m) == insertionSort!(m,<$S)
```

In Section 6.15 on page 172, we give an alternative definition of bubbleSort using first and rest that is more efficient for a list (for which access to any element requires traversing the list from its first node). To implement a more efficient algorithm for lists, we need the operation setelt! which allows us to destructively change the first and rest of a list. Using Browse, you find that these operations come from category **UnaryRecursiveAggregate**. Several aggregate types are unary recursive aggregates including those of **List** and **AssociationList**. We provide two different implementations for bubbleSort! and insertionSort!: one for list-like structures, another for array-like structures.

```
Implementation == add
       ...
    if A has UnaryRecursiveAggregate(S) then
      bubbleSort!(m,fn) ==
        empty? m => m
        l := m
        while not empty? (r := l.rest) repeat
           r := bubbleSort! r
           x := l.first
           if fn(r.first,x) then
             l.first := r.first
             r.first := x
           l.rest := r
           l := l.rest
         m
       insertionSort!(m,fn) ==
          ...
```

The ordering of definitions is important. The standard definitions come first and then the predicate

```
A has UnaryRecursiveAggregate(S)
```

is evaluated. If true, the special definitions cover up the standard ones.

Another equivalent way to write the capsule is to use an `if-then-else` expression:

```
1      if A has UnaryRecursiveAggregate(S) then
2         ...
3      else
4         ...
```

11.9 Testing

Once you have written the package, embed it in a file, for example, **sortpak.spad**. Be sure to include an `)abbrev` command at the top of the file:

```
)abbrev package SORTPAK SortPackage
```

Now compile the file (using `)compile sortpak.spad`). Expose the constructor. You are then ready to begin testing.

```
)expose SORTPAK
```

```
   SortPackage is now explicitly exposed in frame initial
```

Define a list.

```
l := [1,7,4,2,11,-7,3,2]
```

$$[1,\ 7,\ 4,\ 2,\ 11,\ -7,\ 3,\ 2] \tag{1}$$

List (Integer)

Since the integers are an ordered set, a one-argument operation will do.

```
bubbleSort!(l)
```

$$[-7,\ 1,\ 2,\ 2,\ 3,\ 4,\ 7,\ 11] \tag{2}$$

List (Integer)

Re-sort it using "greater than."

```
bubbleSort!(l,(x,y) +-> x > y)
```

$$[11,\ 7,\ 4,\ 3,\ 2,\ 2,\ 1,\ -7] \tag{3}$$

List (Integer)

Now sort it again using < on integers.

```
bubbleSort!(l, <$Integer)
```

$$[-7,\ 1,\ 2,\ 2,\ 3,\ 4,\ 7,\ 11] \tag{4}$$

List (Integer)

A string is an aggregate of characters so we can sort them as well.

```
bubbleSort! "Mathematical Sciences"
```

```
" MSaaaccceeehiilmnstt"                                    (5)
```

String

Is < defined on booleans?

```
false < true
```

```
true                                                       (6)
```

Boolean

Good! Create a bit string representing ten consecutive boolean values `true`.

```
u : Bits := new(10,true)
```

```
"1111111111"                                               (7)
```

Bits

Set bits 3 through 5 to `false`, then display the result.

```
u(3..5) := false; u
```

```
"1100011111"                                               (8)
```

Bits

Now sort these booleans.

```
bubbleSort! u
```

```
"0001111111"                                               (9)
```

Bits

Create an "eq-table" (see '`EqTable`' on page 397), a table having integers as keys and strings as values.

```
t : EqTable(Integer,String) := table()
```

```
table()                                                    (10)
```

EqTable(Integer, String)

Give the table a first entry.

```
t.1 := "robert"
```

```
"robert"                                                   (11)
```

String

And a second.

```
t.2 := "richard"
```

$$\texttt{"richard"} \tag{12}$$

String

What does the table look like?

```
t
```

$$\texttt{table}(2 = \texttt{"richard"}, 1 = \texttt{"robert"}) \tag{13}$$

EqTable(Integer, String)

Now sort it.

```
bubbleSort! t
```

$$\texttt{table}(2 = \texttt{"robert"}, 1 = \texttt{"richard"}) \tag{14}$$

EqTable(Integer, String)

11.10 How Packages Work

Recall that packages as abstract datatypes are compiled independently and put into the library. The curious reader may ask: "How is the interpreter able to find an operation such as bubbleSort!? Also, how is a single compiled function such as bubbleSort! able to sort data of different types?"

After the interpreter loads the package **SortPackage**, the four operations from the package become known to the interpreter. Each of these operations is expressed as a *modemap* in which the type of the operation is written in terms of symbolic domains.

```
)expose SORTPAK
```

See the modemaps for bubbleSort!.

```
)display op bubbleSort!
```

```
There are 2 exposed functions called bubbleSort! :
   [1] (D1,((D3, D3) -> Boolean)) -> D1 from SortPackage(D3,D1)
            if D3 has TYPE and D1 has Join(IXAGG(INT,D3),ATFINAG,ATSHMUT)
   [2] D1 -> D1 from SortPackage(D2,D1)
            if D2 has ORDSET and D2 has TYPE and D1 has Join(IXAGG(INT,D2),ATFINAG
            ,ATSHMUT)
```

What happens if you ask for `bubbleSort!([1,-5,3])`? There is a unique modemap for an operation named bubbleSort! with one argument. Since `[1,-5,3]` is a list of integers, the symbolic domain `D1` is defined as **List(Integer)**. For some operation to apply, it must satisfy the predicate for some `D2`. What `D2`? The third expression of the `and` requires `D1 has IndexedAggregate(Integer, D2)` with two additional categories. So the interpreter searches for an **IndexedAggregate** among the ancestors of **List (Integer)** (see Section 12.4 on page 662). It finds one: **IndexedAggregate(Integer, Integer)**. The interpreter tries defining `D2` as **Integer**. After substituting for `D1` and `D2`, the predicate evaluates to `true`. An applicable operation has been found!

Now FriCAS builds the package **SortPackage(List(Integer), Integer)**. According to its definition, this package exports the required operation: bubbleSort!: **List Integer→List Integer**. The interpreter

then asks the package for a function implementing this operation. The package gets all the functions it needs (for example, rest and swap!) from the appropriate domains and then it returns a bubbleSort! to the interpreter together with the local environment for bubbleSort!. The interpreter applies the function to the argument `[1,-5,3]`. The bubbleSort! function is executed in its local environment and produces the result.

Chapter 12

Categories

This chapter unravels the mysteries of categories—what they are, how they are related to domains and packages, how they are defined in FriCAS, and how you can extend the system to include new categories of your own.

We assume that you have read the introductory material on domains and categories in Section 2.1.1 on page 66. There you learned that the notion of packages covered in the previous chapter are special cases of domains. While this is in fact the case, it is useful here to regard domains as distinct from packages.

Think of a domain as a datatype, a collection of objects (the objects of the domain). From your "sneak preview" in the previous chapter, you might conclude that categories are simply named clusters of operations exported by domains. As it turns out, categories have a much deeper meaning. Categories are fundamental to the design of FriCAS. They control the interactions between domains and algorithmic packages, and, in fact, between all the components of FriCAS.

Categories form hierarchies as shown on the inside cover pages of this book. The inside front-cover pages illustrate the basic algebraic hierarchy of the FriCAS programming language. The inside back-cover pages show the hierarchy for data structures.

Think of the category structures of FriCAS as a foundation for a city on which superstructures (domains) are built. The algebraic hierarchy, for example, serves as a foundation for constructive mathematical algorithms embedded in the domains of FriCAS. Once in place, domains can be constructed, either independently or from one another.

Superstructures are built for quality—domains are compiled into machine code for run-time efficiency. You can extend the foundation in directions beyond the space directly beneath the superstructures, then extend selected superstructures to cover the space. Because of the compilation strategy, changing components of the foundation generally means that the existing superstructures (domains) built on the changed parts of the foundation (categories) have to be rebuilt—that is, recompiled.

Before delving into some of the interesting facts about categories, let's see how you define them in FriCAS.

12.1 Definitions

A category is defined by a function with exactly the same format as any other function in FriCAS.

The definition of a category has the syntax:

CategoryForm : `Category ==` *Extensions* `[ with` *Exports* `]`

The brackets `[ ]` here indicate optionality.

The first example of a category definition is **SetCategory**, the most basic of the algebraic categories in FriCAS.

```
1  SetCategory(): Category ==
2     Join(Type,CoercibleTo OutputForm) with
3        "=" : (%, %) -> Boolean
```

The definition starts off with the name of the category (**SetCategory**); this is always in column one in the source file. All parts of a category definition are then indented with respect to this first line.

In Chapter 2, we talked about **Ring** as denoting the class of all domains that are rings, in short, the class of all rings. While this is the usual naming convention in FriCAS, it is also common to use the word "Category" at the end of a category name for clarity. The interpretation of the name **SetCategory** is, then, "the category of all domains that are (mathematical) sets."

The name **SetCategory** is followed in the definition by its formal parameters enclosed in parentheses "`()`". Here there are no parameters. As required, the type of the result of this category function is the distinguished name Category.

Then comes the "`==`". As usual, what appears to the right of the "`==`" is a definition, here, a category definition. A category definition always has two parts separated by the reserved word `with`.

The first part tells what categories the category extends. Here, the category extends two categories: **Type**, the category of all domains, and **CoercibleTo(OutputForm)**. The operation **Join** is a system-defined operation that forms a single category from two or more other categories.

Every category other than **Type** is an extension of some other category. If, for example, **SetCategory** extended only the category **Type**, the definition here would read "`Type with ...`". In fact, the `Type` is optional in this line; "`with ...`" suffices.

12.2 Exports

To the right of the `with` is a list of all the *exports* of the category. Each exported operation has a name and a type expressed by a *declaration* of the form "*name*: *type*".

Categories can export symbols, as well as `0` and `1` which denote domain constants.[1] In the current implementation, all other exports are operations with types expressed as *mappings* with the syntax

$$source \rightarrow target$$

The category **SetCategory** has a single export: the operation `=` whose type is given by the mapping `(%, %)` → `Boolean`. The "`%`" in a mapping type always means "the domain." Thus the operation `=` takes two arguments from the domain and returns a value of type **Boolean**.

[1] The numbers `0` and `1` are operation names in FriCAS.

The source part of the mapping here is given by a *tuple* consisting of two or more types separated by commas and enclosed in parentheses. If an operation takes only one argument, you can drop the parentheses around the source type. If the mapping has no arguments, the source part of the mapping is either left blank or written as "()". Here are examples of formats of various operations with some contrived names.

```
someIntegerConstant  :    %
aZeroArgumentOperation:   () -> Integer
aOneArgumentOperation:    Integer -> %
aTwoArgumentOperation:    (Integer,%) -> Void
aThreeArgumentOperation:  (%,Integer,%) -> Fraction(%)
```

12.3 Documentation

The definition of **SetCategory** above is missing an important component: its library documentation. Here is its definition, complete with documentation.

```
1  ++ Description:
2  ++ \spadtype{SetCategory} is the basic category
3  ++ for describing a collection of elements with
4  ++ \spadop{=} (equality) and a \spadfun{coerce}
5  ++ to \spadtype{OutputForm}.
6
7  SetCategory(): Category ==
8    Join(Type, CoercibleTo OutputForm) with
9      "=": (%, %) -> Boolean
10       ++ \spad{x = y} tests if \spad{x} and
11       ++ \spad{y} are equal.
```

Documentary comments are an important part of constructor definitions. Documentation is given both for the category itself and for each export. A description for the category precedes the code. Each line of the description begins in column one with "++". The description starts with the word `Description:`.[2] All lines of the description following the initial line are indented by the same amount.

Mark the name of any constructor (with or without parameters) with `\spadtype` like this

```
\spadtype{Polynomial(Integer)}
```

Similarly, mark an operator name with `\spadop`, a FriCAS operation (function) with `\spadfun`, and a variable or FriCAS expression with `\spad`. Library documentation is given in a TEX-like language so that it can be used both for hard-copy and for Browse. These different wrappings cause operations and types to have mouse-active buttons in Browse. For hard-copy output, wrapped expressions appear in a different font. The above documentation appears in hard-copy as:

> **SetCategory** is the basic category for describing a collection of elements with = (equality) and a coerce to **OutputForm**.

and

[2] Other information such as the author's name, date of creation, and so on, can go in this area as well but are currently ignored by FriCAS.

`x = y` tests if `x` and `y` are equal.

For our purposes in this chapter, we omit the documentation from further category descriptions.

12.4 Hierarchies

A second example of a category is **SemiGroup**, defined by:

```
1 SemiGroup(): Category == SetCategory with
2       "*":  (%,%) -> %
3       "^": (%, PositiveInteger) -> %
```

This definition is as simple as that for **SetCategory**, except that there are two exported operations. Multiple exported operations are written as a *pile*, that is, they all begin in the same column. Here you see that the category mentions another type, **PositiveInteger**, in a signature. Any domain can be used in a signature.

Since categories extend one another, they form hierarchies. Each category other than **Type** has one or more parents given by the one or more categories mentioned before the `with` part of the definition. **SemiGroup** extends **SetCategory** and **SetCategory** extends both **Type** and **CoercibleTo (OutputForm)**. Since **CoercibleTo (OutputForm)** also extends **Type**, the mention of **Type** in the definition is unnecessary but included for emphasis.

12.5 Membership

We say a category designates a class of domains. What class of domains? That is, how does FriCAS know what domains belong to what categories? The simple answer to this basic question is key to the design of FriCAS:

Domains belong to categories by assertion.

When a domain is defined, it is asserted to belong to one or more categories. Suppose, for example, that an author of domain **String** wishes to use the binary operator `*` to denote concatenation. Thus `"hello " * "there"` would produce the string `"hello there"`[3]. The author of **String** could then assert that **String** is a member of **SemiGroup**. According to our definition of **SemiGroup**, strings would then also have the operation `^` defined automatically. Then `"--" ^ 4` would produce a string of eight dashes `"--------"`. Since **String** is a member of **SemiGroup**, it also is a member of **SetCategory** and thus has an operation `=` for testing that two strings are equal.

Now turn to the algebraic category hierarchy inside the front cover of this book. Any domain that is a member of a category extending **SemiGroup** is a member of **SemiGroup** (that is, it *is* a semigroup). In particular, any domain asserted to be a **Ring** is a semigroup since **Ring** extends **Monoid**, that, in turn, extends **SemiGroup**. The definition of **Integer** in FriCAS asserts that **Integer** is a member of category **IntegerNumberSystem**, that, in turn, asserts that it is a member of **EuclideanDomain**.

[3] Actually, concatenation of strings in FriCAS is done by juxtaposition or by using the operation concat. The expression `"hello " "there"` produces the string `"hello there"`.

Now **EuclideanDomain** extends **PrincipalIdealDomain** and so on. If you trace up the hierarchy, you see that **EuclideanDomain** extends **Ring**, and, therefore, **SemiGroup**. Thus **Integer** is a semigroup and also exports the operations * and ^.

12.6 Defaults

We actually omitted the last part of the definition of **SemiGroup** in Section 12.4 on page 662. Here now is its complete FriCAS definition.

```
1  SemiGroup(): Category == SetCategory with
2        "*": (%, %) -> %
3        "^": (%, PositiveInteger) -> %
4      add
5        import RepeatedSquaring(%)
6        x: % ^ n: PositiveInteger == expt(x,n)
```

The add part at the end is used to give "default definitions" for exported operations. Once you have a multiplication operation *, you can define exponentiation for positive integer exponents using repeated multiplication:

$$x^n = \underbrace{x\,x\,x\,\cdots\,x}_{n \text{ times}}$$

This definition for ^ is called a *default* definition. In general, a category can give default definitions for any operation it exports. Since **SemiGroup** and all its category descendants in the hierarchy export ^, any descendant category may redefine ^ as well.

A domain of category **SemiGroup** (such as **Integer**) may or may not choose to define its own ^ operation. If it does not, a default definition that is closest (in a "tree-distance" sense of the hierarchy) to the domain is chosen.

The part of the category definition following an "add" operation is a *capsule*, as discussed in the previous chapter. The line

```
import RepeatedSquaring(%)
```

references the package **RepeatedSquaring(%)**, that is, the package **RepeatedSquaring** that takes "this domain" as its parameter. For example, if the semigroup **Polynomial (Integer)** does not define its own exponentiation operation, the definition used may come from the package **RepeatedSquaring (Polynomial (Integer))**. The next line gives the definition in terms of expt from that package.

The default definitions are collected to form a "default package" for the category. The name of the package is the same as the category but with an ampersand ("&") added at the end. A default package always takes an additional argument relative to the category. Here is the definition of the default package **SemiGroup&** as automatically generated by FriCAS from the above definition of **SemiGroup**.

```
1  SemiGroup_&(%): Exports == Implementation where
2    %: SemiGroup
3    Exports == with
4      "^": (%, PositiveInteger) -> %
5    Implementation == add
6      import RepeatedSquaring(%)
7      x:% ^ n:PositiveInteger == expt(x,n)
```

12.7 Axioms

In the previous section you saw the complete FriCAS program defining **SemiGroup**. According to this definition, semigroups are sets with the operations * and ^.

You might ask: "Aside from the notion of default packages, isn't a category just a *macro*, that is, a shorthand equivalent to the two operations * and ^ with their types?" If a category were a macro, every time you saw the word **SemiGroup**, you would rewrite it by its list of exported operations. Furthermore, every time you saw the exported operations of **SemiGroup** among the exports of a constructor, you could conclude that the constructor exported **SemiGroup**.

A category is *not* a macro and here is why. The definition for **SemiGroup** has documentation that states:

> Category **SemiGroup** denotes the class of all multiplicative semigroups, that is, a set with an associative operation *.
>
> Axioms:
> `associative("*" : (%,%)→%)  (x*y)*z = x*(y*z)`

According to the author's remarks, the mere exporting of an operation named * and ^ is not enough to qualify the domain as a **SemiGroup**. In fact, a domain can be a semigroup only if it explicitly exports a ^ and a * satisfying the associativity axiom.

In general, a category name implies a set of axioms, even mathematical theorems. There are numerous axioms from **Ring**, for example, that are well-understood from the literature. No attempt is made to list them all. Nonetheless, all such mathematical facts are implicit by the use of the name **Ring**.

12.8 Correctness

While such statements are only comments, FriCAS can enforce their intention simply by shifting the burden of responsibility onto the author of a domain. A domain belongs to category **Ring** only if the author asserts that the domain belongs to **Ring** or to a category that extends **Ring**.

This principle of assertion is important for large user-extendable systems. FriCAS has a large library of operations offering facilities in many areas. Names such as norm and product, for example, have diverse meanings in diverse contexts. An inescapable hindrance to users would be to force those who wish to extend FriCAS to always invent new names for operations. FriCAS allows you to reuse names, and then use context to disambiguate one from another.

Here is another example of why this is important. Some languages, such as **APL**, denote the **Boolean** constants `true` and `false` by the integers `1` and `0`. You may want to let infix operators + and * serve as the logical operators or and and, respectively. But note this: **Boolean** is not a ring. The *inverse axiom* for **Ring** states:

> Every element `x` has an additive inverse `y` such that `x + y = 0`.

Boolean is not a ring since `true` has no inverse—there is no inverse element `a` such that `1 + a = 0` (in terms of booleans, `(true or a)= false`). Nonetheless, FriCAS *could* easily and correctly implement **Boolean** this way. **Boolean** simply would not assert that it is of category **Ring**. Thus the + for

Boolean values is not confused with the one for **Ring**. Since the **Polynomial** constructor requires its argument to be a ring, FriCAS would then refuse to build the domain **Polynomial(Boolean)**. Also, FriCAS would refuse to wrongfully apply algorithms to **Boolean** elements that presume that the ring axioms for + hold.

12.9 Categories as attributes

Most axioms are not computationally useful. Those that are can be explicitly expressed by using special categories that export no operations. Note: in the past, instead of categories, FriCAS used a special construct called `attribute`.

The category **CommutativeStar**, for example, is used to assert that a domain has commutative multiplication. Its definition is given by its documentation:

> A domain `R` has **CommutativeStar** if it has an operation "*": `(R,R)→R` such that `x * y = y * x`.

So, to test that a domain is known to satisfy an axiom, we just test if it has corresponding category, like **CommutativeStar** above.

Do polynomials over the integers have commutative multiplication?

```
Polynomial Integer has CommutativeStar
```

```
                                    true                                    (1)
                                                                        Boolean
```

Do matrices over the integers have commutative multiplication?

```
Matrix Integer has CommutativeStar
```

```
                                    false                                   (2)
                                                                        Boolean
```

Using categories to assert axioms and category conditions to test if axioms are satisfied, we can conditionally export and define operations for a domain depending on axioms (see Section 13.3 on page 671). Of course categories can also be asserted in a category definition.

After mentioning category **Ring** many times in this book, it is high time that we show you its definition:

```
1 Ring() : Category == Join(Rng, SemiRing, NonAssociativeRing,
2                           unitsKnown)
```

As you can see **Ring** just combines properties of other categories. So let us see **NonAssociativeRing**:

```
1 NonAssociativeRing() : Category == Join(NonAssociativeRng,
2                                         NonAssociativeSemiRing) with
3     --operations
4       characteristic : -> NonNegativeInteger
5         ++ characteristic() returns the characteristic of the ring.
6         --we can not make this a constant, since some domains are mutable
7       coerce : Integer -> %
8         ++ coerce(n) coerces the integer n to an element of the ring.
```

```
9     add
10       n : Integer
11       coerce(n) == n * 1$%
```

There is one new thing here. Look at the "$%" on the last line. This is not a typographic error! The "$" says that the 1 is to come from some domain. The "%" says that the domain is "this domain." If "%" is **Fraction(Integer)**, this line reads `coerce(n)== n * 1$Fraction(Integer)`.

Let us comment on category **unitsKnown** appearing in definition of **Ring** above. The category **unitsKnown** asserts a rather subtle mathematical fact that is normally taken for granted when working with rings.[4] Because programs can test for this category, FriCAS can correctly handle rather more complicated mathematical structures (ones that are similar to rings but do not have this category).

12.10 Parameters

Like domain constructors, category constructors can also have parameters. For example, category **MatrixCategory** is a parameterized category for defining matrices over a ring R so that the matrix domains can have different representations and indexing schemes. Its definition has the form:

```
1  MatrixCategory(R,Row,Col): Category ==
2      TwoDimensionalArrayCategory(R,Row,Col) with ...
```

The category extends **TwoDimensionalArrayCategory** with the same arguments. You cannot find **TwoDimensionalArrayCategory** in the algebraic hierarchy listing. Rather, it is a member of the data structure hierarchy, given inside the back cover of this book. In particular, **TwoDimensionalArrayCategory** is an extension of **HomogeneousAggregate** since its elements are all of one type.

The domain **Matrix(R)**, the class of matrices with coefficients from domain R, asserts that it is a member of category **MatrixCategory(R, Vector(R), Vector(R))**. The parameters of a category must also have types. The first parameter to **MatrixCategory** R is required to be a ring. The second and third are required to be domains of category **FiniteLinearAggregate(R)**.[5] In practice, examples of categories having parameters other than domains are rare.

Adding the declarations for parameters to the definition for **MatrixCategory**, we have:

```
1  R: Ring
2  (Row, Col): FiniteLinearAggregate(R)
3
4  MatrixCategory(R, Row, Col): Category ==
5      TwoDimensionalArrayCategory(R, Row, Col) with ...
```

12.11 Conditionals

As categories have parameters, the actual operations exported by a category can depend on these parameters. As an example, the operation determinant from category **MatrixCategory** is only exported

[4] With this axiom, the units of a domain are the set of elements x that each have a multiplicative inverse y in the domain. Thus 1 and -1 are units in domain **Integer**. Also, for **Fraction Integer**, the domain of rational numbers, all non-zero elements are units.

[5] This is another extension of **HomogeneousAggregate** that you can see in the data structure hierarchy.

when the underlying domain R has commutative multiplication:

```
if R has CommutativeRing then
   determinant: % -> R
```

Conditionals can also define conditional extensions of a category. Here is a portion of the definition of **QuotientFieldCategory**:

```
1 QuotientFieldCategory(R) : Category == ... with ...
2      if R has OrderedSet then OrderedSet
3      if R has IntegerNumberSystem then
4        ceiling: % -> R
5          ...
```

Think of category **QuotientFieldCategory(R)** as denoting the domain **Fraction(R)**, the class of all fractions of the form a/b for elements of R. The first conditional means in English: "If the elements of R are totally ordered (R is an **OrderedSet**), then so are the fractions a/b".

The second conditional is used to conditionally export an operation ceiling which returns the smallest integer greater than or equal to its argument. Clearly, "ceiling" makes sense for integers but not for polynomials and other algebraic structures. Because of this conditional, the domain **Fraction(Integer)** exports an operation ceiling: **Fraction Integer→Integer**, but **Fraction Polynomial Integer** does not.

Conditionals can also appear in the default definitions for the operations of a category. For example, a default definition for ceiling within the part following the "add" reads:

```
if R has IntegerNumberSystem then
    ceiling x == ...
```

Here the predicate used is identical to the predicate in the Exports part. This need not be the case. See Section 11.8 on page 654 for a more complicated example.

12.12 Anonymous Categories

The part of a category to the right of a with is also regarded as a category—an "anonymous category." Thus you have already seen a category definition in Chapter 11. The Exports part of the package **DrawComplex** (Section 11.3 on page 649) is an anonymous category. This is not necessary. We could, instead, give this category a name:

```
1 DrawComplexCategory(): Category == with
2    drawComplex: (C -> C,S,S,Boolean) -> VIEW3D
3    drawComplexVectorField: (C -> C,S,S) -> VIEW3D
4    setRealSteps: INT -> INT
5    setImagSteps: INT -> INT
6    setClipValue: DFLOAT -> DFLOAT
```

and then define **DrawComplex** by:

```
1 DrawComplex(): DrawComplexCategory == Implementation
2    where
3       ...
```

There is no reason, however, to give this list of exports a name since no other domain or package exports it. In fact, it is rare for a package to export a named category. As you will see in the next chapter, however, it is very common for the definition of domains to mention one or more categories before the `with`.

Chapter 13

Domains

We finally come to the *domain constructor*. A few subtle differences between packages and domains turn up some interesting issues. We first discuss these differences and then describe the resulting issues by illustrating a program for the **QuadraticForm** constructor. After a short example of an algebraic constructor, **CliffordAlgebra**, we show how you use domain constructors to build a database query facility.

13.1 Domains vs. Packages

Packages are special cases of domains. What is the difference between a package and a domain that is not a package? Internally, FriCAS makes no distinction. However, humans think differently about them, so we make the following definition: a domain that is not a package has the symbol "`%`" appearing somewhere among the types of its exported operations. The "`%`" denotes "this domain." If the "`%`" appears before the "`->`" in the type of a signature, it means the operation takes an element from the domain as an argument. If it appears after the "`->`", then the operation returns an element of the domain.

If no exported operations mention "`%`", then evidently there is nothing of interest to do with the objects of the domain. You might then say that a package is a "boring" domain! But, as you saw in Chapter 11, packages are a very useful notion indeed. The exported operations of a package depend solely on the parameters to the package constructor and other explicit domains.

To summarize, domain constructors are versatile structures that serve two distinct practical purposes: Those like **Polynomial** and **List** describe classes of computational objects; others, like **SortPackage**, describe packages of useful operations. As in the last chapter, we focus here on the first kind.

13.2 Definitions

The syntax for defining a domain constructor is the same as for any function in FriCAS:

DomainForm `:` *Exports* `==` *Implementation*

As this definition usually extends over many lines, a `where` expression is generally used instead.

A recommended format for the definition of a domain is:
DomainForm : Exports == Implementation where
optional type declarations
Exports == [*Category Assertions*] with
list of exported operations
Implementation == [*Add Domain*] add
[Rep := *Representation*]
list of function definitions for exported operations
Note: The brackets [ ] here denote optionality.

A complete domain constructor definition for **QuadraticForm** is shown in Figure 13.1. Interestingly, this little domain illustrates all the new concepts you need to learn.

Listing 13.1: The QuadraticForm domain.

```
)abbrev domain QFORM QuadraticForm

++ Description:
++   This domain provides modest support for
++   quadratic forms.
QuadraticForm(n, K): Exports == Implementation where
    n: PositiveInteger
    K: Field

    Exports == AbelianGroup with                          -- The exports.
      quadraticForm: SquareMatrix(n,K) -> %               -- The export quadraticForm.
        ++ \spad{quadraticForm(m)} creates a quadratic
        ++ form from a symmetric,
        ++ square matrix \spad{m}.
      matrix: % -> SquareMatrix(n,K)                      -- The export matrix.
        ++ \spad{matrix(qf)} creates a square matrix
        ++ from the quadratic form \spad{qf}.
      elt: (%, DirectProduct(n,K)) -> K                   -- The export elt.
        ++ \spad{qf(v)} evaluates the quadratic form
        ++ \spad{qf} on the vector \spad{v},
        ++ producing a scalar.

    Implementation == SquareMatrix(n,K) add               -- The definitions of the exports
      Rep := SquareMatrix(n,K)                            -- The "representation."
      quadraticForm m ==                                  -- The definition of
        not symmetric? m => error                         -- quadraticForm.
          "quadraticForm requires a symmetric matrix"
        m :: %
      matrix q == q :: Rep                                -- The definition of matrix.
      elt(q,v) == dot(v, (matrix q * v))                  -- The definition of elt.
```

A domain constructor can take any number and type of parameters. **QuadraticForm** takes a positive integer `n` and a field `K` as arguments. Like a package, a domain has a set of explicit exports and an implementation described by a capsule. Domain constructors are documented in the same way as package constructors.

Domain **QuadraticForm(n, K)**, for a given positive integer `n` and domain `K`, explicitly exports three operations:

- `quadraticForm(A)` creates a quadratic form from a matrix `A`.

- `matrix(q)` returns the matrix `A` used to create the quadratic form `q`.
- `q.v` computes the scalar $v^T Av$ for a given vector `v`.

Compared with the corresponding syntax given for the definition of a package, you see that a domain constructor has three optional parts to its definition: *Category Assertions*, *Add Domain*, and *Representation*.

13.3 Category Assertions

The *Category Assertions* part of your domain constructor definition lists those categories of which all domains created by the constructor are unconditionally members. The word "unconditionally" means that membership in a category does not depend on the values of the parameters to the domain constructor. This part thus defines the link between the domains and the category hierarchies given on the inside covers of this book. As described in Section 12.8 on page 664, it is this link that makes it possible for you to pass objects of the domains as arguments to other operations in FriCAS.

Every **QuadraticForm** domain is declared to be unconditionally a member of category **AbelianGroup**. An abelian group is a collection of elements closed under addition. Every object x of an abelian group has an additive inverse y such that $x + y = 0$. The exports of an abelian group include `0`, `+`, `-`, and scalar multiplication by an integer. After asserting that **QuadraticForm** domains are abelian groups, it is possible to pass quadratic forms to algorithms that only assume arguments to have these abelian group properties.

In Section 12.11 on page 666, you saw that **Fraction(R)**, a member of **QuotientFieldCategory(R)**, is a member of **OrderedSet** if `R` is a member of **OrderedSet**. Likewise, from the `Exports` part of the definition of **ModMonic(R, S)**,

```
UnivariatePolynomialCategory(R) with
  if R has Finite then Finite
     ...
```

you see that **ModMonic(R, S)** is a member of **Finite** if `R` is.

The `Exports` part of a domain definition is the same kind of expression that can appear to the right of an "`==`" in a category definition. If a domain constructor is unconditionally a member of two or more categories, a **Join** form is used. The `Exports` part of the definition of **FlexibleArray(S)** reads, for example:

```
Join(ExtensibleLinearAggregate(S),
     OneDimensionalArrayAggregate(S)) with...
```

13.4 A Demo

Before looking at the *Implementation* part of **QuadraticForm**, let's try some examples.

Build a domain `QF`.

```
QF := QuadraticForm(2,Fraction Integer)
```

$$\text{QuadraticForm}(2, \text{Fraction}(Integer)) \tag{1}$$

Type

Define a matrix to be used to construct a quadratic form.

```
A := matrix [[-1,1/2],[1/2,1]]
```

$$\begin{bmatrix} -1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \tag{2}$$

Matrix(Fraction (Integer))

Construct the quadratic form. A package call `$QF` is necessary since there are other **QuadraticForm** domains.

```
q : QF := quadraticForm(A)
```

$$\begin{bmatrix} -1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \tag{3}$$

QuadraticForm(2, Fraction (Integer))

Looks like a matrix. Try computing the number of rows. FriCAS won't let you.

```
nrows q
```

```
   There are 2 exposed and 2 unexposed library operations named nrows having 1
      argument(s) but none was determined to be applicable. Use HyperDoc Browse,
      or issue
                                  )display op nrows
      to learn more about the available operations. Perhaps package-calling the
      operation or using coercions on the arguments will allow you to apply the
      operation.
   Cannot find a definition or applicable library operation named nrows with
      argument type(s)
                          QuadraticForm(2,Fraction(Integer))

      Perhaps you should use "@" to indicate the required return type, or "$" to
      specify which version of the function you need.
```

Create a direct product element `v`. A package call is again necessary, but FriCAS understands your list as denoting a vector.

```
v := directProduct([2,-1])$DirectProduct(2,Fraction Integer)
```

$$[2, -1] \tag{4}$$

DirectProduct(2, Fraction (Integer))

Compute the product $v^T Av$.

```
q.v
```

$$-5 \tag{5}$$

Fraction (Integer)

What is 3 times q minus q plus q?

```
3*q-q+q
```

$$\begin{bmatrix} -3 & \frac{3}{2} \\ \frac{3}{2} & 3 \end{bmatrix} \tag{6}$$

QuadraticForm(2, Fraction (Integer))

13.5 Browse

The Browse facility of HyperDoc is useful for investigating the properties of domains, packages, and categories. From the main HyperDoc menu, move your mouse to **Browse** and click on the left mouse button. This brings up the Browse first page. Now, with your mouse pointer somewhere in this window, enter the string "quadraticform" into the input area (all lower case letters will do). Move your mouse to **Constructors** and click. Up comes a page describing **QuadraticForm** that includes a part labeled by "*Description:*". You also see the types for arguments `n` and `K` displayed as well as the fact that **QuadraticForm** returns an **AbelianGroup**.

Select **Operations** to get a list of operations for **QuadraticForm**. You can select an operation by clicking on it to get an individual page with information about that operation. Or you can select the buttons along the bottom to see alternative views or get additional information on the operations. Eventually, use  to return to the first page on **QuadraticForm**.

You can go and experiment a bit by selecting **Field** and **n** with your mouse. Going back to **Operations** you will see that **Implementations** view now works (it is disabled if some domain parameter is unspecified). Then return to the page on **QuadraticForm**.

At the bottom the **QuadraticForm** page has buttons for **Parents**, **Ancestors**, and others. Clicking on **Parents**, you see that **QuadraticForm** has **AbelianGroup** and **ConvertibleTo** as parents (note that **QuadraticForm** distributed with FriCAS is richer than the demo version presented before).

13.6 Representation

The `Implementation` part of a FriCAS capsule for a domain constructor uses the special variable **Rep** to identify the lower level data type used to represent the objects of the domain. The **Rep** for quadratic forms is **SquareMatrix(n, K)**. This means that all objects of the domain are required to be `n` by `n` matrices with elements from **K**.

The code for quadraticForm in Figure 13.1 on page 670 checks that the matrix is symmetric and then converts it to "`%`", which means, as usual, "this domain." Such explicit conversions are generally required by the compiler. Aside from checking that the matrix is symmetric, the code for this function essentially does nothing. The `m :: %` on line 28 coerces `m` to a quadratic form. In fact, the quadratic form you created in step (3) of Section 13.4 on page 671 is just the matrix you passed it in disguise! Without seeing this definition, you would not know that. Nor can you take advantage of this fact now that you do know! When we try in the next step of Section 13.4 on page 671 to regard `q` as a matrix by

asking for nrows, the number of its rows, FriCAS gives you an error message saying, in effect, "Good try, but this won't work!"

The definition for the matrix function could hardly be simpler: it just returns its argument after explicitly *coercing* its argument to a matrix. Since the argument is already a matrix, this coercion does no computation.

Within the context of a capsule, an object of "%" is regarded both as a quadratic form *and* as a matrix.[1] This makes the definition of `q.v` easy—it just calls the dot product from **DirectProduct** to perform the indicated operation.

13.7 Multiple Representations

To write functions that implement the operations of a domain, you want to choose the most computationally efficient data structure to represent the elements of your domain.

A classic problem in computer algebra is the optimal choice for an internal representation of polynomials. If you create a polynomial, say $3x^2 + 5$, how does FriCAS hold this value internally? There are many ways. FriCAS has nearly a dozen different representations of polynomials, one to suit almost any purpose. Algorithms for solving polynomial equations work most efficiently with polynomials represented one way, whereas those for factoring polynomials are most efficient using another. One often-used representation is a list of terms, each term consisting of exponent-coefficient records written in the order of decreasing exponents. For example, the polynomial $3x^2 + 5$ is represented by the list `[[e:2, c:3], [e:0, c:5]]`.

What is the optimal data structure for a matrix? It depends on the application. For large sparse matrices, a linked-list structure of records holding only the non-zero elements may be optimal. If the elements can be defined by a simple formula $f(i, j)$, then a compiled function for `f` may be optimal. Some programmers prefer to represent ordinary matrices as vectors of vectors. Others prefer to represent matrices by one big linear array where elements are accessed with linearly computable indexes.

While all these simultaneous structures tend to be confusing, FriCAS provides a helpful organizational tool for such a purpose: categories. **PolynomialCategory**, for example, provides a uniform user interface across all polynomial types. Each kind of polynomial implements functions for all these operations, each in its own way. If you use only the top-level operations in **PolynomialCategory** you usually do not care what kind of polynomial implementation is used.

Within a given domain, however, you define (at most) one representation.[2] If you want to have multiple representations (that is, several domains, each with its own representation), use a category to describe the `Exports`, then define separate domains for each representation.

13.8 Add Domain

The capsule part of `Implementation` defines functions that implement the operations exported by the domain—usually only some of the operations. In our demo in Section 13.4 on page 671, we asked for the value of `3*q-q+q`. Where do the operations `*`, `+`, and `-` come from? There is no definition for them in the capsule!

[1] In case each of "%" and "Rep" have the same named operation available, the one from "%" takes precedence. Thus, if you want the one from "Rep", you must package call it using a "$Rep" suffix.

[2] You can make that representation a **Union** type, however. See Section 2.5 on page 79 for examples of unions.

The `Implementation` part of a definition can optionally specify an "add-domain" to the left of an `add` (for **QuadraticForm**, **SquareMatrix(n,K)** is the add-domain). The meaning of an add-domain is simply this: if the capsule part of the `Implementation` does not supply a function for an operation, FriCAS goes to the add-domain to find the function. So do `*`, `+` and `-` come from **SquareMatrix(n,K)**?

13.9 Defaults

In Chapter 11, we saw that categories can provide default implementations for their operations. How and when are they used? When FriCAS finds that **QuadraticForm(2, Fraction Integer)** does not implement the operations `*`, `+`, and `-`, it goes to **SquareMatrix(2,Fraction Integer)** to find it. As it turns out, **SquareMatrix(2, Fraction Integer)** does not implement *any* of these operations!

What does FriCAS do then? Here is its overall strategy. First, FriCAS looks for a function in the capsule for the domain. If it is not there, FriCAS looks in the add-domain for the operation. If that fails, FriCAS searches the add-domain of the add-domain, and so on. If all those fail, it then searches the default packages for the categories of which the domain is a member. In the case of **QuadraticForm**, it searches **AbelianGroup**, then its parents, grandparents, and so on. If this fails, it then searches the default packages of the add-domain. Whenever a function is found, the search stops immediately and the function is returned. When all fails, the system calls `error` to report this unfortunate news to you. To find out the actual order of constructors searched for **QuadraticForm**, consult Browse: from the **QuadraticForm**, and click on **Search Path**.

Let's apply this search strategy for our example `3*q-q+q`. The scalar multiplication comes first. FriCAS finds a default implementation in **AbelianGroup&**. Remember from Section 12.6 on page 663 that **SemiGroup** provides a default definition for x^n by repeated squaring? **AbelianGroup** similarly provides a definition for $n * x$ by repeated doubling.

But the search of the defaults for **QuadraticForm** fails to find any `+` or `*` in the default packages for the ancestors of **QuadraticForm**. So it now searches among those for **SquareMatrix**. Category **MatrixCategory**, which provides a uniform interface for all matrix domains, is a grandparent of **SquareMatrix** and has a capsule defining many functions for matrices, including matrix addition, subtraction, and scalar multiplication. The default package **MatrixCategory&** is where the functions for `+` and `-` come from.

You can use Browse to discover where the operations for **QuadraticForm** are implemented. First, get the page describing **QuadraticForm**. With your mouse somewhere in this window, type a "2", press the **Tab** key, and then enter "Fraction Integer" to indicate that you want the domain **QuadraticForm(2, Fraction Integer)**. Now click on **Operations** to get a table of operations and on `*` to get a page describing the `*` operation. Finally, click on **implementation** at the bottom.

13.10 Origins

Aside from the notion of where an operation is implemented, a useful notion is the *origin* or "home" of an operation. When an operation (such as `quadraticForm`) is explicitly exported by a domain (such as **QuadraticForm**), you can say that the origin of that operation is that domain. If an operation is not explicitly exported from a domain, it is inherited from, and has as origin, the (closest) category that explicitly exports it. The operations `+` and `-` of **QuadraticForm**, for example, are inherited from **AbelianMonoid**. As it turns out, **AbelianMonoid** is the origin of virtually every `+` operation in

FriCAS!

Again, you can use Browse to discover the origins of operations. From the Browse page on **QuadraticForm**, click on **Operations**, then on **origins** at the bottom of the page.

The origin of the operation is the *only* place where on-line documentation is given. However, you can re-export an operation to give it special documentation. Suppose you have just invented the world's fastest algorithm for inverting matrices using a particular internal representation for matrices. If your matrix domain just declares that it exports **MatrixCategory**, it exports the inverse operation, but the documentation the user gets from Browse is the standard one from **MatrixCategory**. To give your version of inverse the attention it deserves, simply export the operation explicitly with new documentation. This redundancy gives inverse a new origin and tells Browse to present your new documentation.

13.11 Short Forms

In FriCAS, a domain could be defined using only an add-domain and no capsule. Although we talk about rational numbers as quotients of integers, there is no type **RationalNumber** in FriCAS. To create such a type, you could compile the following "short-form" definition:

```
1 RationalNumber() == Fraction(Integer)
```

The `Exports` part of this definition is missing and is taken to be equivalent to that of **Fraction(Integer)**. Because of the add-domain philosophy, you get precisely what you want. The effect is to create a little stub of a domain. When a user asks to add two rational numbers, FriCAS would ask **RationalNumber** for a function implementing this +. Since the domain has no capsule, the domain then immediately sends its request to **Fraction (Integer)**.

The short form definition for domains is used to define such domains as **MultivariatePolynomial**:

```
1 MultivariatePolynomial(vl: List Symbol, R: Ring) ==
2    SparseMultivariatePolynomial(R,
3       OrderedVariableList vl)
```

13.12 Example 1: Clifford Algebra

Now that we have **QuadraticForm** available, let's put it to use. Given some quadratic form Q described by an n by n matrix over a field K, the domain **CliffordAlgebra(n, K, Q)** defines a vector space of dimension 2^n over K. This is an interesting domain since complex numbers, quaternions, exterior algebras and spin algebras are all examples of Clifford algebras.

The basic idea is this: the quadratic form `Q` defines a basis $e_1, e_2 \ldots, e_n$ for the vector space K^n—the direct product of `K` with itself `n` times. From this, the Clifford algebra generates a basis of 2^n elements given by all the possible products of the e_i in order without duplicates, that is, 1, e_1, e_2, e_1e_2, e_3, e_1e_3, e_2e_3, e_1e_2, e_3, and so on.

The algebra is defined by the relations

$$\begin{aligned} e_i\ e_i &= Q(e_i) \\ e_i\ e_j &= -e_j\ e_i \quad \text{for } i \neq j \end{aligned}$$

Now look at the snapshot of its definition given in Figure 13.2. Lines 9-10 show part of the definitions of the `Exports`. A Clifford algebra over a field `K` is asserted to be a ring, an algebra over `K`, and a vector space over `K`. Its explicit exports include `e(n)`, which returns the n^{th} unit element.

Listing 13.2: Part of the CliffordAlgebra domain.

```
1  NNI ==> NonNegativeInteger
2  PI  ==> PositiveInteger
3
4  CliffordAlgebra(n,K,q): Exports == Implementation where
5      n: PI
6      K: Field
7      q: QuadraticForm(n, K)
8
9      Exports == Join(Ring, Algebra(K)) with
10       e: PI -> %
11           ...
12
13     Implementation == add
14       Qeelist :=
15         [q.unitVector(i::PI) for i in 1..n]
16       dim     :=  2^n
17       Rep     := PrimitiveArray K
18       New ==> new(dim, 0$K)$Rep
19       x + y ==
20         z := New
21         for i in 0..dim-1 repeat z.i := x.i + y.i
22         z
23       addMonomProd: (K, NNI, K, NNI, %) -> %
24       addMonomProd(c1, b1, c2, b2, z) ==  ...
25       x * y ==
26         z := New
27         for ix in 0..dim-1 repeat
28           if x.ix ~= 0 then for iy in 0..dim-1 repeat
29             if y.iy ~= 0
30             then addMonomProd(x.ix,ix,y.iy,iy,z)
31           z
32            ...
```

The `Implementation` part begins by defining a local variable `Qeelist` to hold the list of all `q.v` where `v` runs over the unit vectors from 1 to the dimension `n`. Another local variable `dim` is set to 2^n, computed once and for all. The representation for the domain is **PrimitiveArray(K)**, which is a basic array of elements from domain `K`. Line 18 defines `New` as shorthand for the more lengthy expression `new(dim, 0$K)$Rep`, which computes a primitive array of length 2^n filled with `0`'s from domain `K`.

Lines 19-22 define the sum of two elements `x` and `y` straightforwardly. First, a new array of all `0`'s is created, then filled with the sum of the corresponding elements. Indexing for primitive arrays starts at 0. The definition of the product of `x` and `y` first requires the definition of a local function **addMonomProd**. FriCAS knows it is local since it is not an exported function. The types of all local functions must be declared.

For a demonstration of **CliffordAlgebra**, see '`CliffordAlgebra`' on page 367.

13.13 Example 2: Building A Query Facility

We now turn to an entirely different kind of application, building a query language for a database.

Here is the practical problem to solve. The Browse facility of FriCAS has a database for all operations and constructors which is stored on disk and accessed by HyperDoc. For our purposes here, we regard each line of this file as having eight fields: `class`, `name`, `type`, `nargs`, `exposed`, `kind`, `origin`, and `condition`. Here is an example entry:

```
o‘determinant‘$->R‘1‘x‘d‘Matrix(R)‘has(R,commutative("*"))
```

In English, the entry means:

> The operation determinant: % → R with *1* argument, is *exposed* and is exported by *domain* **Matrix(R)** if `R has commutative("*")`.

Our task is to create a little query language that allows us to get useful information from this database.

13.13.1 A Little Query Language

First we design a simple language for accessing information from the database. We have the following simple model in mind for its design. Think of the database as a box of index cards. There is only one search operation—it takes the name of a field and a predicate (a boolean-valued function) defined on the fields of the index cards. When applied, the search operation goes through the entire box selecting only those index cards for which the predicate is true. The result of a search is a new box of index cards. This process can be repeated again and again.

The predicates all have a particularly simple form: *symbol* = *pattern*, where *symbol* designates one of the fields, and *pattern* is a "search string"—a string that may contain a "`*`" as a wildcard. Wildcards match any substring, including the empty string. Thus the pattern `"*ma*t"` matches `"mat"`, `"doormat"` and `"smart"`.

To illustrate how queries are given, we give you a sneak preview of the facility we are about to create.

Extract the database of all FriCAS operations.

```
ops := getDatabase("o")
```

8474 (1)

Database(IndexCard)

How many exposed three-argument map operations involving streams?

```
ops.(name="map").(nargs="3").(type="*Stream*")
```

3 (2)

Database(IndexCard)

As usual, the arguments of elt ("`.`") associate to the left. The first elt produces the set of all operations with name `map`. The second elt produces the set of all map operations with three arguments. The third elt produces the set of all three-argument map operations having a type mentioning **Stream**.

Another thing we'd like to do is to extract one field from each of the index cards in the box and look at the result. Here is an example of that kind of request.

What constructors explicitly export a determinant operation?

```
elt(elt(elt(elt(ops,name="determinant"),origin),sort),unique)
```

$$\begin{array}{l}\texttt{["InnerMatrixLinearAlgebraFunctions", "MatrixCategory",}\\ \texttt{"MatrixLinearAlgebraFunctions", "SquareMatrixCategory"]}\end{array} \tag{3}$$

DataList(String)

The first elt produces the set of all index cards with name `determinant`. The second elt extracts the `origin` component from each index card. Each origin component is the name of a constructor which directly exports the operation represented by the index card. Extracting a component from each index card produces what we call a *datalist*. The third elt, `sort`, causes the datalist of origins to be sorted in alphabetic order. The fourth, `unique`, causes duplicates to be removed.

Before giving you a more extensive demo of this facility, we now build the necessary domains and packages to implement it.

13.13.2 The Database Constructor

We work from the top down. First, we define a database, our box of index cards, as an abstract datatype. For sake of illustration and generality, we assume that an index card is some type `S`, and that a database is a box of objects of type `S`. Here is the FriCAS program defining the **Database** domain.

```
1  PI ==> PositiveInteger
2  Database(S): Exports == Implementation where
3    S: OrderedSet with
4      elt: (%, Symbol) -> String
5      display: % -> Void
6      fullDisplay: % -> Void
7
8    Exports == with
9      elt: (%,QueryEquation) -> %                    -- Select by an equation.
10     elt: (%, Symbol) -> DataList String            -- Select by a field name.
11     "+": (%,%) -> %                                -- Combine two databases.
12     "-": (%,%) -> %                                -- Subtract one from another.
13     display: % -> Void                             -- A brief database display.
14     fullDisplay: % -> Void                         -- A full database display.
15     fullDisplay: (%,PI,PI) -> Void                 -- A selective display.
16     coerce: % -> OutputForm                        -- Display a database.
17   Implementation == add
18       ...
```

The domain constructor takes a parameter `S`, which stands for the class of index cards. We describe an index card later. Here think of an index card as a string which has the eight fields mentioned above.

First, we tell FriCAS what operations we are going to require from index cards. We need an elt to extract the contents of a field (such as `name` and `type`) as a string. For example, `c.name` returns a string that is the content of the `name` field on the index card `c`. We need to display an index card in two ways: display shows only the name and type of an operation; fullDisplay displays all fields. The display operations return no useful information and thus have return type **Void**.

Next, we tell FriCAS what operations the user can apply to the database. This part defines our little query language. The most important operation is `db . field = pattern` which returns a new database, consisting of all index cards of `db` such that the `field` part of the index card is matched by the string pattern called `pattern`. The expression `field = pattern` is an object of type **QueryEquation**

(defined in the next section).

Another elt is needed to produce a **DataList** object. Operation + is to merge two databases together; - is used to subtract away common entries in a second database from an initial database. There are three display functions. The fullDisplay function has two versions: one that prints all the records, the other that prints only a fixed number of records. A coerce to **OutputForm** creates a display object.

The `Implementation` part of **Database** is straightforward.

```
1   Implementation == add
2     s: Symbol
3     Rep := List S
4     elt(db,equation) == ...
5     elt(db,key) == [x.key for x in db]::DataList(String)
6     display(db) ==  for x in db repeat display x
7     fullDisplay(db) == for x in db repeat fullDisplay x
8     fullDisplay(db, n, m) == for x in db for i in 1..m
9       repeat
10        if i >= n then fullDisplay x
11    x+y == removeDuplicates! merge(x,y)
12    x-y == mergeDifference(copy(x::Rep),
13                           y::Rep)$MergeThing(S)
14    coerce(db): OutputForm == (#db):: OutputForm
```

The database is represented by a list of elements of S (index cards). We leave the definition of the first elt operation (on line 4) until the next section. The second elt collects all the strings with field name *key* into a list. The display function and first fullDisplay function simply call the corresponding functions from S. The second fullDisplay function provides an efficient way of printing out a portion of a large list. The + is defined by using the existing merge operation defined on lists, then removing duplicates from the result. The - operation requires writing a corresponding subtraction operation. A package **MergeThing** (not shown) provides this.

The coerce function converts the database to an **OutputForm** by computing the number of index cards. This is a good example of the independence of the representation of a FriCAS object from how it presents itself to the user. We usually do not want to look at a database—but do care how many "hits" we get for a given query. So we define the output representation of a database to be simply the number of index cards our query finds.

13.13.3 Query Equations

The predicate for our search is given by an object of type **QueryEquation**. FriCAS does not have such an object yet so we have to invent it.

```
1   QueryEquation(): Exports == Implementation where
2     Exports == with
3       equation: (Symbol, String) -> %
4       variable: % -> Symbol
5       value:    % -> String
6
7     Implementation == add
8       Rep := Record(var:Symbol, val:String)
9       equation(x, s) == [x, s]
10      variable q == q.var
11      value    q == q.val
```

FriCAS converts an input expression of the form `a = b` to `equation(a, b)`. Our equations always

have a symbol on the left and a string on the right. The `Exports` part thus specifies an operation equation to create a query equation, and variable and value to select the left- and right-hand sides. The `Implementation` part uses **Record** for a space-efficient representation of an equation.

Here is the missing definition for the elt function of **Database** in the last section:

```
1     elt(db,eq) ==
2       field  := variable eq
3       value := value eq
4       [x for x in db | matches?(value,x.field)]
```

Recall that a database is represented by a list. Line 4 simply runs over that list collecting all elements such that the pattern (that is, `value`) matches the selected field of the element.

13.13.4 DataLists

Type **DataList** is a new type invented to hold the result of selecting one field from each of the index cards in the box. It is useful to make datalists extensions of lists—lists that have special elt operations defined on them for sorting and removing duplicates.

```
1  DataList(S:OrderedSet) : Exports == Implementation where
2    Exports == ListAggregate(S) with
3      elt: (%,"unique") -> %
4      elt: (%,"sort") -> %
5      elt: (%,"count") -> NonNegativeInteger
6      coerce: List S -> %
7
8    Implementation ==  List(S) add
9      Rep := List S
10     elt(x,"unique") == removeDuplicates(x)
11     elt(x,"sort") == sort(x)
12     elt(x,"count") == #x
13     coerce(x:List S) == x :: %
```

The `Exports` part asserts that datalists belong to the category **ListAggregate**. Therefore, you can use all the usual list operations on datalists, such as first, rest, and concat. In addition, datalists have four explicit operations. Besides the three elt operations, there is a coerce operation that creates datalists from lists.

The `Implementation` part needs only to define four functions. All the rest are obtained from **List(S)**.

13.13.5 Index Cards

An index card comes from a file as one long string. We define functions that extract substrings from the long string. Each field has a name that is passed as a second argument to elt.

```
1  IndexCard() == Implementation where
2    Exports == with
3      elt: (%, Symbol) -> String
4      display: % -> Void
5      fullDisplay: % -> Void
6      coerce: String -> %
7    Implementation == String add ...
```

We leave the `Implementation` part to the reader. All operations involve straightforward string manipulations.

13.13.6 Creating a Database

We must not forget one important operation: one that builds the database in the first place! We'll name it getDatabase and put it in a package. This function is implemented by calling the Common LISP function `getBrowseDatabase(s)` to get appropriate information from Browse. This operation takes a string indicating which lines you want from the database: `"o"` gives you all operation lines, and `"k"`, all constructor lines. Similarly, `"c"`, `"d"`, and `"p"` give you all category, domain and package lines respectively.

```
1  OperationsQuery(): Exports == Implementation where
2    Exports == with
3      getDatabase: String -> Database(IndexCard)
4
5    Implementation == add
6      getDatabase(s) == getBrowseDatabase(s)$Lisp
```

We do not bother creating a special name for databases of index cards. **Database(IndexCard)** will do. Notice that we used the package **OperationsQuery** to create, in effect, a new kind of domain: **Database(IndexCard)**.

13.13.7 Putting It All Together

To create the database facility, you put all these constructors into one file.[3] At the top of the file put `)abbrev` commands, giving the constructor abbreviations you created.

```
1  )abbrev domain  ICARD   IndexCard
2  )abbrev domain  QEQUAT  QueryEquation
3  )abbrev package MTHING  MergeThing
4  )abbrev domain  DLIST   DataList
5  )abbrev domain  DBASE   Database
6  )abbrev package OPQUERY OperationsQuery
```

With all this in **alql.spad**, for example, compile it using

```
)compile alql
```

and then load each of the constructors:

```
)library ICARD QEQUAT MTHING DLIST DBASE OPQUERY
```

You are ready to try some sample queries.

13.13.8 Example Queries

Our first set of queries give some statistics on constructors in the current FriCAS system.

[3] You could use separate files, but we are putting them all together because, organizationally, that is the logical thing to do.

How many constructors does FriCAS have?

```
ks := getDatabase "k"
```

$$1286 \tag{1}$$

Database(IndexCard)

Break this down into the number of categories, domains, and packages.

```
[ks.(kind=k) for k in ["c","d","p"]]
```

$$[273,\ 443,\ 570] \tag{2}$$

List (Database(IndexCard))

What are all the domain constructors that take 5 parameters?

```
elt(ks.(kind="d").(nargs="5"),name)
```

$$\begin{array}{l}\texttt{["FractionalIdealAsModule", "IndexedJetBundle", "InnerIndexedTwoDimensionalArray",}\\ \texttt{"ModularField", "ModularRing", "RadicalFunctionField", "ResidueRing", "TensorProduct"]}\end{array} \tag{3}$$

DataList(String)

How many constructors have "Matrix" in their name?

```
mk := ks.(name="*Matrix*")
```

$$41 \tag{4}$$

Database(IndexCard)

What are the names of those that are domains?

```
elt(mk.(kind="d"),name)
```

$$\begin{array}{l}\texttt{["ComplexDoubleFloatMatrix", "DenavitHartenbergMatrix",}\\ \texttt{"DirectProductMatrixModule", "DoubleFloatMatrix", "I16Matrix", "I32Matrix",}\\ \texttt{"I8Matrix", "IndexedMatrix", "LieSquareMatrix", "LinearMultivariateMatrixPencil",}\\ \texttt{"Matrix", "RectangularMatrix", "SparseEchelonMatrix", "SquareMatrix",}\\ \texttt{"ThreeDimensionalMatrix", "U16Matrix", "U32Matrix", "U8Matrix"]}\end{array} \tag{5}$$

DataList(String)

How many operations are there in the library?

```
o := getDatabase "o"
```

$$8474 \tag{6}$$

Database(IndexCard)

Break this down into categories, domains, and packages.

```
[o.(kind=k) for k in ["c","d","p"]]
```

[2078, 2968, 3428] (7)

List (Database(IndexCard))

The query language is helpful in getting information about a particular operation you might like to apply. While this information can be obtained with Browse, the use of the query database gives you data that you can manipulate in the workspace.

How many operations have "eigen" in the name?

```
eigens := o.(name="*eigen*")
```

9 (8)

Database(IndexCard)

What are their names?

```
elt(eigens,name)
```

["eigenMatrix", "eigenvalues", "eigenvalues", "eigenvalues", "eigenvector", "eigenvector", "eigenvectors", "eigenvectors", "eigenvectors"] (9)

DataList(String)

Where do they come from?

```
elt(elt(elt(eigens,origin),sort),unique)
```

["EigenPackage", "InnerEigenPackage", "RadicalEigenPackage"] (10)

DataList(String)

The operations + and - are useful for constructing small databases and combining them. However, remember that the only matching you can do is string matching. Thus a pattern such as "*Matrix*" on the type field matches any type containing **Matrix**, **MatrixCategory**, **SquareMatrix**, and so on.

How many operations mention "Matrix" in their type?

```
tm := o.(type="*Matrix*")
```

393 (11)

Database(IndexCard)

How many operations come from constructors with "Matrix" in their name?

```
fm := o.(origin="*Matrix*")
```

316 (12)

Database(IndexCard)

How many operations are in `fm` but not in `tm`?

```
fm-tm
```

244 (13)

Database(IndexCard)

Display the first 5 operations that both mention "Matrix" in their type and come from a constructor having "Matrix" in their name.

```
fullDisplay(fm-%, 1, 5)
```

```
   * : (%, Matrix(R))->% from LinearMultivariateMatrixPencil(R) (unexposed)
   * : (Matrix(D), %)->% from SparseEchelonMatrix(C, D) (unexposed)
   * : (Matrix(Fraction(D)), %)->%
       from SparseEchelonMatrix(C, D) if D has IntegralDomain (unexposed)
   * : (Matrix(R), %)->% from LinearMultivariateMatrixPencil(R) (unexposed)
   ^ : (Matrix(R), NonNegativeInteger)->Matrix(R)
       from StorageEfficientMatrixOperations(R) (unexposed)
```

How many operations involve matrices?

```
m := tm+fm
```

637 (15)

Database(IndexCard)

Display 4 of them.

```
fullDisplay(m, 202, 205)
```

```
   elRow1! : (M, Integer, Integer)->M from MatrixLinearAlgebraFunctions(R, Row, Col, M)
   elRow2! : (M, R, Integer, Integer)->M from MatrixLinearAlgebraFunctions(R, Row, Col, M)
   element : (M, Integer, Integer)->M from MatrixManipulation(R, Row, Col, M) (unexposed)
   elimZeroCols! : (%)->Void from SparseEchelonMatrix(C, D) (unexposed)
```

How many distinct names of operations involving matrices are there?

```
elt(elt(elt(m,name),unique),count)
```

363 (17)

PositiveInteger

Chapter 14

Browse

This chapter discusses the Browse component of HyperDoc. We suggest you invoke FriCAS and work through this chapter, section by section, following our examples to gain some familiarity with Browse.

14.1 The Front Page: Searching the Library

To enter Browse, click on **Browse** on the top level page of HyperDoc to get the *front page* of Browse.

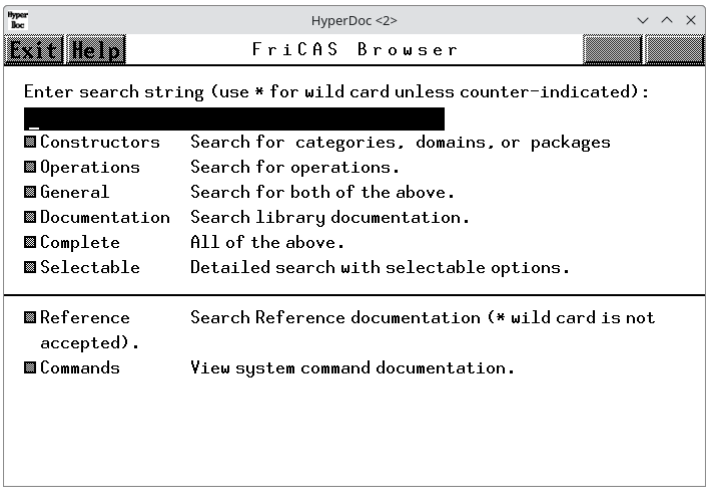


Figure 14.1: The Browse front page.

To use this page, you first enter a *search string* into the input area at the top, then click on one of the buttons below. We show the use of each of the buttons by example.

Constructors

First enter the search string `Matrix` into the input area and click on **Constructors**. What you get is the *constructor page* for **Matrix**. We show and describe this page in detail in Section 14.2 on page 691. By convention, FriCAS does a case-insensitive search for a match. Thus `matrix` is just as good as `Matrix`, has the same effect as `MaTrix`, and so on. We recommend that you generally use lowercase letters for names however. A search string with only capital letters has a special meaning (see Section 14.3.3 on page 704).

Click on  to return to the Browse front page.

Use the symbol "`*`" in search strings as a *wildcard*.

A wildcard matches any substring, including the empty string. For example, enter the search string `*matrix*` into the input area and click on **Constructors**.[1] What you get is a table of all constructors whose names contain the string "`matrix`."

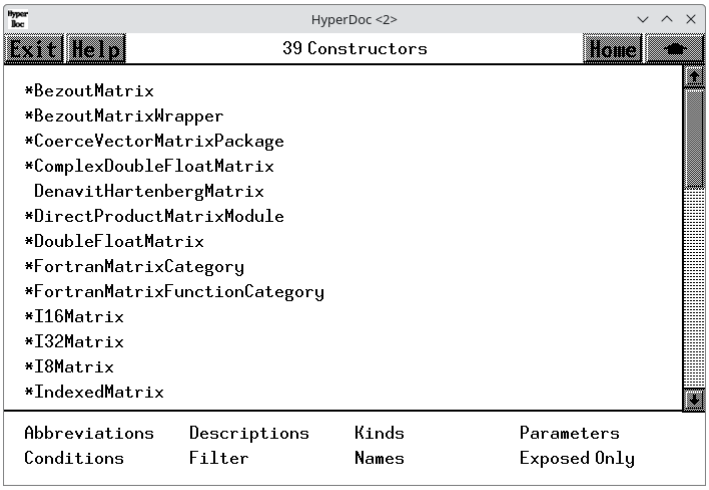


Figure 14.2: Table of exposed constructors matching `*matrix*`.

All constructors containing the string are listed, whether *exposed* or *unexposed*. You can hide the names of the unexposed constructors by clicking on the *=**unexposed** button in the *Views* panel at the bottom of the window. (The button will change to **exposed** *only*.)

One of the names in this table is **Matrix**. Click on **Matrix**. What you get is again the constructor page for **Matrix**. As you see, Browse gives you a large network of information in which there are many ways to reach the same pages.

Again click on the  to return to the table of constructors whose names contain `matrix`. Below the table is a *Views* panel. This panel contains buttons that let you view constructors in different ways. To learn about views of constructors, skip to Section 14.2.2 on page 698.

Click on  to return to the Browse front page.

[1] To get only categories, domains, or packages, rather than all constructors, you can click on the corresponding button to the right of **Constructors**.

Operations

Enter `*matrix` into the input area and click on **Operations**. This time you get a table of *operations* whose names end with `matrix` or `Matrix`.

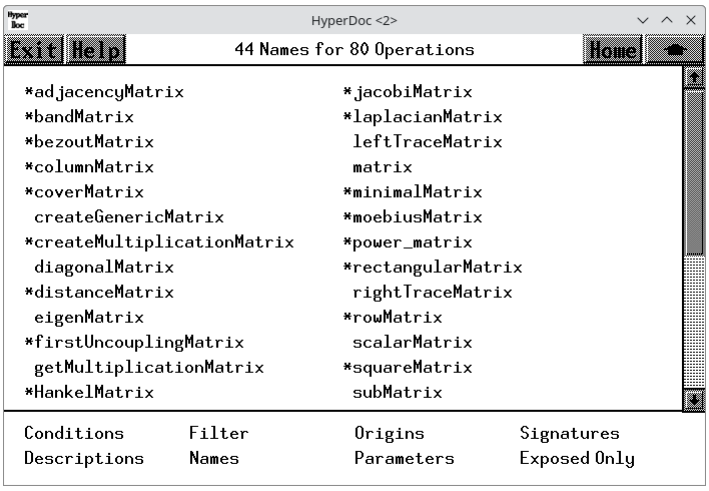


Figure 14.3: Table of operations matching `*matrix`.

If you select an operation name, you go to a page describing all the operations in FriCAS of that name. At the bottom of an operation page is another kind of *Views* panel, one for operation pages. To learn more about these views, skip to Section 14.3.2 on page 701.

Click on  to return to the Browse front page.

General

This button does a general search for all constructor and operation names matching the search string. Enter the search string `*matrix*` into the input area. Click on **General** to find all constructs that have `matrix` as a part of their name.

The summary gives you all the names under a heading when the number of entries is less than 10.

Click on  to return to the Browse front page.

Documentation

Again enter the search key `*matrix*` and this time click on **Documentation**. This search matches any constructor and operation name whose documentation contains a substring matching `matrix`.

Click on  to return to the Browse front page.

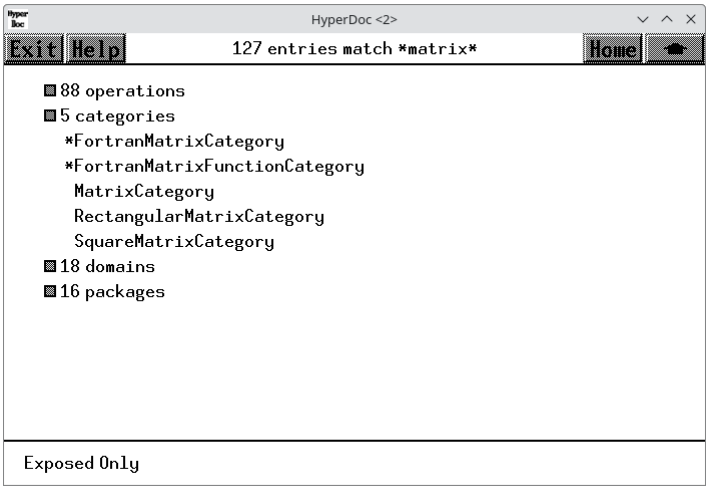


Figure 14.4: Table of all constructs matching `*matrix*`.

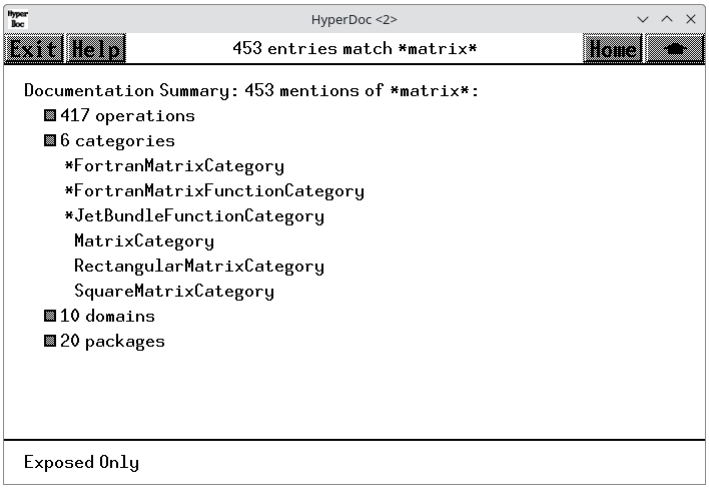


Figure 14.5: Table of constructs with documentation matching `*matrix*`.

Complete

This search combines both **General** and **Documentation**.

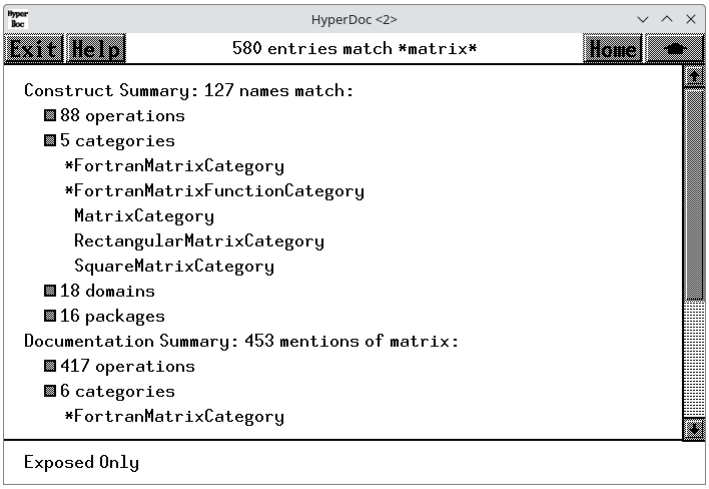


Figure 14.6: Table summarizing complete search for pattern `*matrix*`.

14.2 The Constructor Page

In this section we look in detail at a constructor page for domain **Matrix**. Enter `matrix` into the input area on the main Browse page and click on **Constructors**.

The header part tells you that **Matrix** takes one argument called `R` that must be a domain of category **AbelianMonoid**. Just what domains can be arguments of **Matrix**? To find this out, click on the `R` on the second line of the heading. What you get is a table of all acceptable domain parameter values of `R`, or a table of *abelian monoids* in FriCAS.

Click on  to return to the constructor page for **Matrix**.

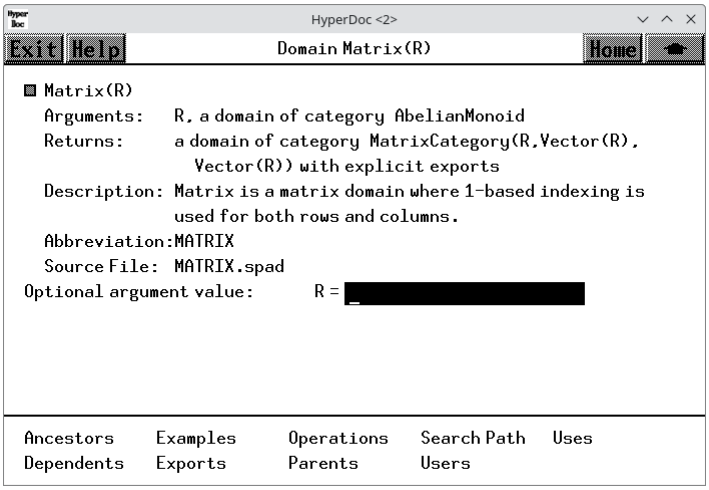


Figure 14.7: Constructor page for **Matrix**.

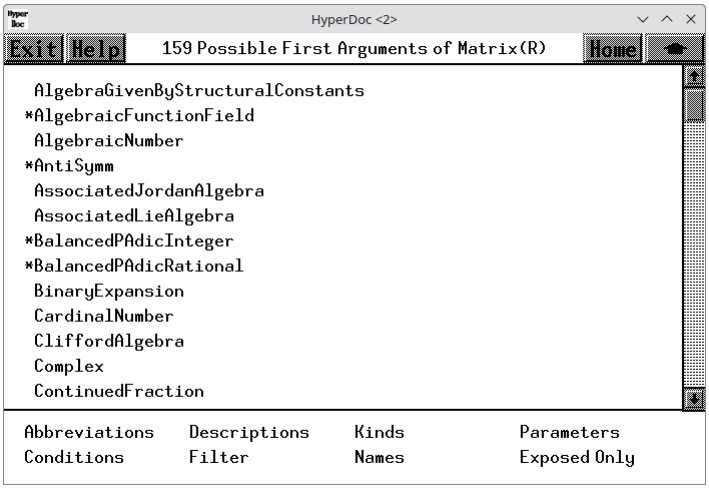


Figure 14.8: Table of acceptable domain parameters to **Matrix**.

There is also a brief description of constructor **Matrix** and its abbreviation **MATRIX**.

If you have access to the source code of FriCAS, this header part also gives you the name of the source file containing the definition of **Matrix**. Click on it to pop up an editor window containing the source code of **Matrix**.

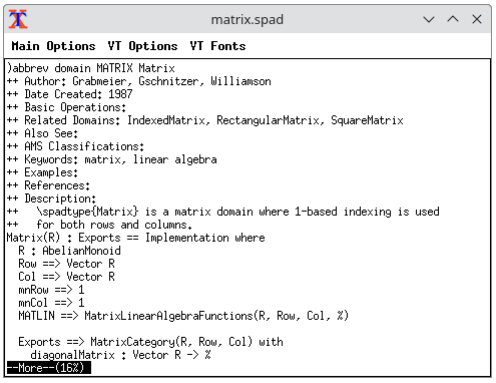


Figure 14.9: Source code for **Matrix**.

We recommend that you leave the editor window up while working through this chapter as you occasionally may want to refer to it.

14.2.1 Constructor Page Buttons

We examine each button on this page in order.

Operations

Click here to get a table of operations exported by **Matrix**. You may wish to widen the window to have multiple columns as below.

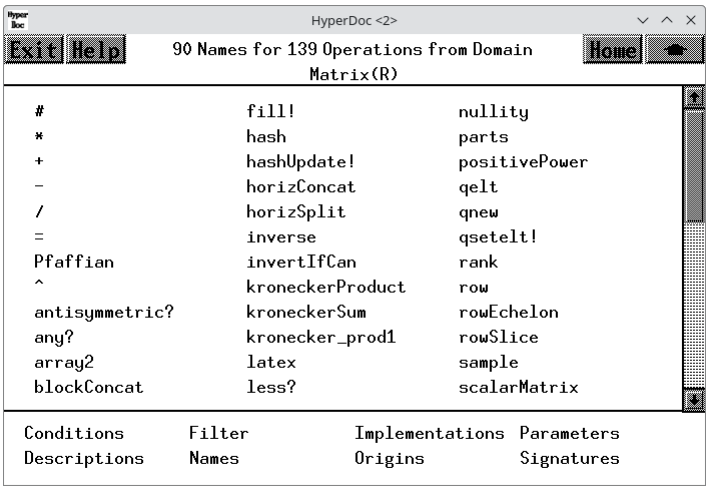


Figure 14.10: Table of operations from **Matrix**.

If you click on an operation name, you bring up a description page for the operations. For a detailed description of these pages, skip to Section 14.3.2 on page 701.

Examples

Click here to get an *examples page* with examples of operations to create and manipulate matrices.

Read through this section. Try selecting the various buttons. Notice that if you click on an operation name, such as new, you bring up a description page for that operation from **Matrix**.

Example pages have several examples of FriCAS commands. Each example has an active button to its left. Click on it! A pre-computed answer is pasted into the page immediately following the command. If you click on the button a second time, the answer disappears. This button thus acts as a toggle: "now you see it; now you don't."

Note also that the FriCAS commands themselves are active. If you want to see FriCAS execute the command, then click on it! A new FriCAS window appears on your screen and the command is executed.

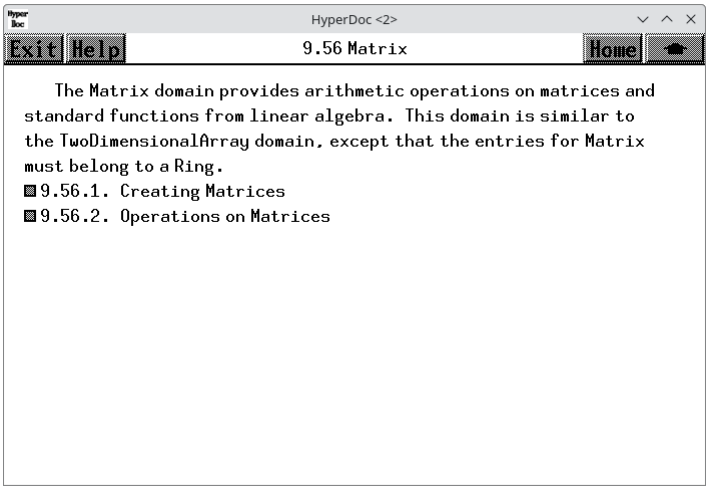


Figure 14.11: Example page for **Matrix**.

Exports

Click here to see a page describing the exports of **Matrix** exactly as described by the source code.

As you see, **Matrix** declares that it exports all the operations and categories exported by category **MatrixCategory(R, Row, Col)**. In addition, two operations, diagonalMatrix and invertIfCan, are explicitly exported.

To learn a little about the structure of FriCAS, we suggest you do the following exercise. Otherwise, go on to the next section. **Matrix** explicitly exports only two operations. The other operations are thus exports of **MatrixCategory**. In general, operations are usually not explicitly exported by a domain. Typically they are *inherited* from several different categories. Let's find out from where the operations of **Matrix** come.

1. Click on **MatrixCategory**, then on **Exports**. Here you see that **MatrixCategory** explicitly exports many matrix operations. Also, it inherits its operations from **TwoDimensionalArrayCategory**.

2. Click on **TwoDimensionalArrayCategory**, then on **Exports**. Here you see explicit operations dealing with rows and columns. In addition, it inherits operations from **HomogeneousAggregate**.

3. Click on  repeatedly to return to the constructor page for **Matrix**.

Parents

The parents of a domain are the same as the categories mentioned under the **Exports** button on the first page. Domain **Matrix** has two parents but in general a domain can have any number.

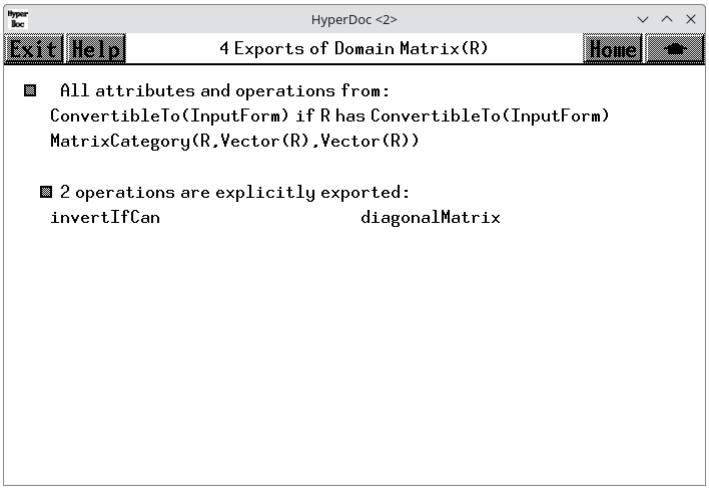


Figure 14.12: Exports of **Matrix**.

Ancestors

The *ancestors* of a constructor consist of its parents, the parents of its parents, and so on. Did you perform the exercise in the last section under **Exports**? If so, you see here all the categories you found while ascending the **Exports** chain for **Matrix**.

Dependents

The *dependents* of a constructor are those *domains* or *packages* that mention that constructor either as an argument or in its *exports*.

If you click on **Dependents** two entries may surprise you: **RectangularMatrix** and **SquareMatrix**. This happens because **Matrix**, as it turns out, appears in signatures of operations exported by these domains.

Search Path

The term *search path* refers to the *search order* for functions. If you are an expert user or curious about how the FriCAS system works, try the following exercise. Otherwise, you best skip this button and go on to **Users**.

Clicking on **Search Path** gives you a list of domain constructors: **InnerIndexedTwoDimensionalArray**, **MatrixCategory&**, **TwoDimensionalArrayCategory&**, **HomogeneousAggregate&**, **Aggregate&**, **Evalable&**, **SetCategory&**, **Hashable&**, **InnerEvalable&**, **BasicType&**. What are these constructors and how are they used?

We explain by an example. Suppose you create a matrix using the interpreter, then ask for its rank. FriCAS must then find a function implementing the rank operation for matrices. The first place FriCAS looks for rank is in the **Matrix** domain.

If not there, the search path of **Matrix** tells FriCAS where else to look. Associated with the matrix domain are ten other search path domains. Their order is important. FriCAS first searches the first one, **InnerIndexedTwoDimensionalArray**. If not there, it searches the second **MatrixCategory&**. And so on.

Where do these *search path constructors* come from? The source code for **Matrix** contains this syntax for the *function body* of **Matrix**:[2]

```
InnerIndexedTwoDimensionalArray(R,mnRow,mnCol,Row,Col)
   add ...
```

where the "`...`" denotes all the code that follows. In English, this means: "The functions for matrices are defined as those from **InnerIndexedTwoDimensionalArray** domain augmented by those defined in '`...`'," where the latter take precedence.

This explains **InnerIndexedTwoDimensionalArray**. The other names, those with names ending with an ampersand "`&`" are *default packages* for categories to which **Matrix** belongs. Default packages are ordered by the notion of "closest ancestor."

Users

A user of **Matrix** is any constructor that uses **Matrix** in its implementation. For example, **Complex** is a user of **Matrix**; it exports several operations that take matrices as arguments or return matrices as values.[3]

Uses

A *benefactor* of **Matrix** is any constructor that **Matrix** uses in its implementation. This information, like that for clients, is gathered from run-time structures.[4]

Cross reference pages for categories have some different buttons on them. Starting with the front page of Browse, search for **Ring** and enter its constructor page. Here are buttons **Parents** and **Ancestors** similar to the notion for domains, except for categories the relationship between parent and child is defined through *category extension*.

Children

Category hierarchies go both ways. There are children as well as parents.

Descendants

These are children, children of children, and so on.

[2] **InnerIndexedTwoDimensionalArray** is a special domain implemented for matrix-like domains to provide efficient implementations of two-dimensional arrays. For example, domains of category **TwoDimensionalArrayCategory** can have any integer as their `minIndex`. Matrices and other members of this special "inner" array have their `minIndex` defined as `1`.

[3] A constructor is a user of **Matrix** if it handles any matrix. For example, a constructor having internal (unexported) operations dealing with matrices is also a user.

[4] The benefactors exclude constructors such as **PrimitiveArray** whose operations macro-expand and so vanish from sight!

Category hierarchies are complicated by the fact that categories take parameters. Where a parameterized category fits into a hierarchy *may* depend on values of its parameters. In general, the set of categories in FriCAS forms a *directed acyclic graph*, that is, a graph with directed arcs and no cycles.

Domains

This produces a table of all domain constructors that can possibly be rings (members of category **Ring**). Some domains are unconditional rings. Others are rings for some parameters and not for others. To find out which, select the **conditions** button in the views panel. For example, **DirectProduct(n, R)** is a ring if `R` is a ring.

14.2.2 Views Of Constructors

Below every constructor table page is a *Views* panel. As an example, click on **Uses** from the constructor page of **Matrix** to produce a short table of constructor names.

The *Views* panel is at the bottom of the page. Two items, *names* and *conditions,* are in italics. Others are active buttons. The active buttons are those that give you useful alternative views on this table of constructors. Once you select a view, you notice that the button turns off (becomes italicized) so that you cannot reselect it.

names

This view gives you a table of names. Selecting any of these names brings up the constructor page for that constructor.

abbrs

This view gives you a table of abbreviations, in the same order as the original constructor names. Abbreviations are in capitals and they can be used interchangeably with constructor names in input areas.

kinds

This view organizes constructor names into the three kinds: categories, domains and packages.

parameters

This view presents constructors with the arguments. This view of the benefactors of **Matrix** shows that **Matrix** uses as many as ten different **List** domains in its implementation.

filter

This button is used to refine the list of names or abbreviations. Starting with the *names* view, enter `m*` into the input area and click on **filter**. You then get a shorter table with only the names beginning

with `m`.

descriptions

This gives you documentation for each of the constructors.

conditions

This page organizes the constructors according to predicates. The view is not available for your example page since all constructors are unconditional. For a table with conditions, return to the **Ancestors** page for **Matrix**, click on **conditions** in the view panel. This page shows you that **CoercibleTo(OutputForm)** and **SetCategory** are ancestors of **Matrix(R)** only if `R` belongs to category **SetCategory**.

14.2.3 Giving Parameters to Constructors

Notice the input area at the bottom of the constructor page. If you leave this blank, then the information you get is for the domain constructor **Matrix(R)**, that is, **Matrix** for an arbitrary underlying domain `R`.

In general, however, the exports and other information *do* usually depend on the actual value of `R`. For example, **Matrix** exports the inverse operation only if the domain `R` is a **Field**. To see this, try this from the main constructor page:

1. Enter `Integer` into the input area at the bottom of the page.
2. Click on **Operations**, producing a table of operations. Note the number of operation names that appear at the top of the page.
3. Click on  to return to the constructor page.
4. Use the **Delete** or **Backspace** keys to erase `Integer` from the input area.
5. Click on **Operations** to produce a new table of operations. Look at the number of operations you get. This number is greater than what you had before. Find, for example, the operation inverse.
6. Click on **inverse** to produce a page describing the operation inverse. At the bottom of the description, you notice that the **Conditions** line says "`R` has **Field**." This operation is *not* exported by **Matrix(Integer)** since **Integer** is not a *field*.

   Try putting the name of a domain such as **Fraction Integer** (which is a field) into the input area, then clicking on **Operations**. As you see, the operation inverse is exported.

14.3 Miscellaneous Features of Browse

14.3.1 The Description Page for Operations

From the constructor page of **Matrix**, click on **Operations** to bring up the table of operations for **Matrix**.

Find the operation **inverse** in the table and click on it. This takes you to a page showing the documentation for this operation.

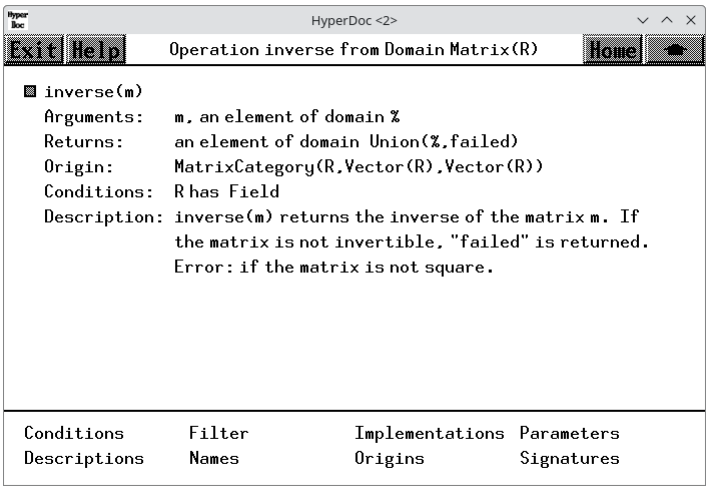


Figure 14.13: Operation inverse from **Matrix**.

Here is the significance of the headings you see.

Arguments

This lists each of the arguments of the operation in turn, paraphrasing the *signature* of the operation. As for signatures, a "%" is used to designate *this domain*, that is, **Matrix(R)**.

Returns

This describes the return value for the operation, analogous to the **Arguments** part.

Origin

This tells you which domain or category explicitly exports the operation. In this example, the *Origin* is **MatrixCategory**.

Conditions

This tells you that the operation is exported by **Matrix(R)** only if "R has **Field**," that is, "R is a member of category **Field**." When no **Conditions** part is given, the operation is exported for all values of R.

Description

Here are the "++" comments that appear in the source code of its *Origin*, here **Matrix**. You find these comments in the source code for **Matrix**.

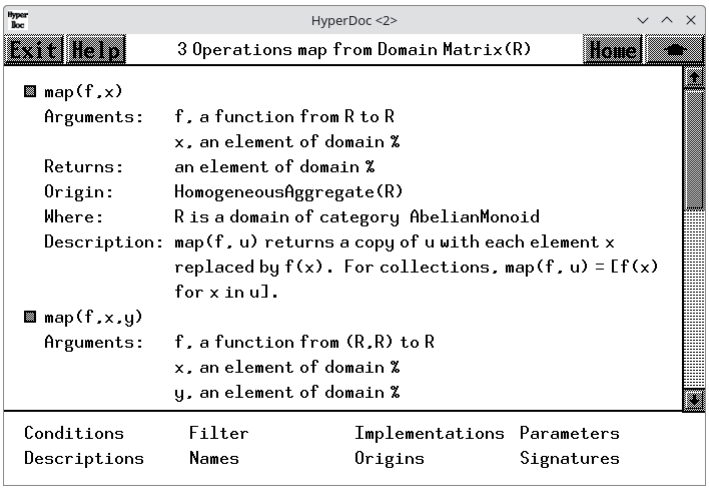


Figure 14.14: Operations map from **Matrix**.

Click on  to return to the table of operations. Click on **map**. Here you find three different operations named map. This should not surprise you. Operations are identified by name and *signature*. There are three operations named map, each with different signatures. What you see is the *descriptions* view of the operations. If you like, select the button in the heading of one of these descriptions to get *only* that operation.

Where

This part qualifies domain parameters mentioned in the arguments to the operation.

14.3.2 Views of Operations

We suggest that you go to the constructor page for **Matrix** and click on **Operations** to bring up a table of operations with a *Views* panel at the bottom.

names

This view lists the names of the operations. Unlike constructors, however, there may be several operations with the same name. The heading for the page tells you the number of unique names and the number of distinct operations when these numbers are different.

filter

As for constructors, you can use this button to cut down the list of operations you are looking at. Click on **filter** and enter, for example, `m*` into the input area, then click on **filter**. As usual, any logical expression is permitted. For example, use

```
*! or *?
```

to get a list of destructive operations and predicates.

descriptions

This gives you the most information: a detailed description of all the operations in the form you have seen before. Every other button summarizes these operations in some form.

signatures

This views the operations by showing their signatures.

parameters

This views the operations by their distinct syntactic forms with parameters.

origins

This organizes the operations according to the constructor that explicitly exports them.

conditions

This view organizes the operations into conditional and unconditional operations.

usage

This button is only available if your user-level is set to *development*. The **usage** button produces a table of constructors that reference this operation.[5]

implementation

This button is only available if your user-level is set to *development*. If you enter values for all domain parameters on the constructor page, then the **implementation** button can be clicked. This button tells you what domains or packages actually implement the various operations.[6]

[5] FriCAS requires an especially long time to produce this table, so anticipate this when requesting this information.
[6] This button often takes a long time; expect a delay while you wait for an answer.

With your user-level set to *development*, we suggest you try this exercise. Return to the main constructor page for **Matrix**, then enter `Integer` into the input area at the bottom as the value of `R`. Then click on **Operations** to produce a table of operations. Click on **implementation**. After some delay, you get a page describing what implements each of the matrix operations, organized by the various domains and packages.

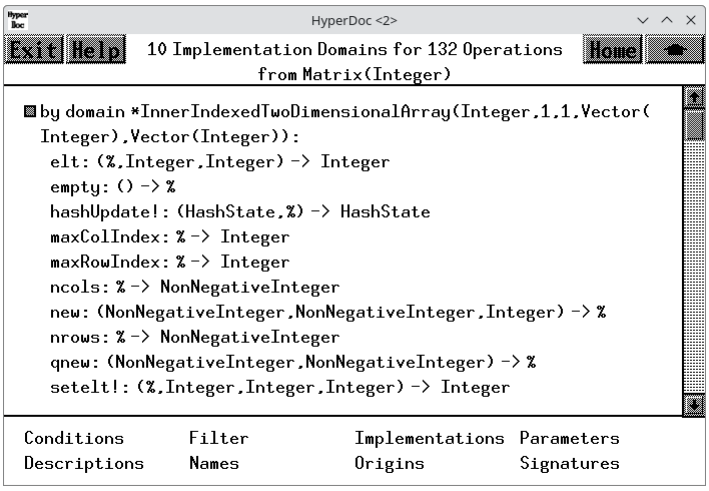


Figure 14.15: Implementation domains for **Matrix**.

14.3.3 Capitalization Convention

When entering search keys for constructors, you can use capital letters to search for abbreviations. For example, enter `UTS` into the input area and click on **Constructors**. Up comes a page describing **UnivariateTaylorSeries** whose abbreviation is **UTS**.

Constructor abbreviations always have three or more capital letters. For short constructor names (six letters or less), abbreviations are not generally helpful as their abbreviation is typically the constructor name in capitals. For example, the abbreviation for **Matrix** is **MATRIX**.

Abbreviations can also contain numbers. For example, **POLY2** is the abbreviation for constructor **PolynomialFunctions2**. For default packages, the abbreviation is the same as the abbreviation for the corresponding category with the "&" replaced by "-". For example, for the category default package **MatrixCategory&** the abbreviation is **MATCAT-** since the corresponding category **MatrixCategory** has abbreviation **MATCAT**.

Chapter 15

What's New in FriCAS

15.1 Release Notes

FriCAS information can be found online at http://fricas.github.io.

FriCAS 1.4.0

- new constructors: IndexedVariables, LeafIndexingTree.
- improved speed of expressions.
- faster algebra bootstrap.
- new HyperDoc code.
- new code for printing types.
- complex 'abs' now returns real result.
- various improvements to Spad compiler.
- command handling is now mostly case sensitive
- local libdb is no longer supported.

Bug fixes, in particular:

- fixed series expanders for 'acsc', 'acsch', 'asec' and 'asech'.
- fixed some division by zero errors during integration.
- fixed 'order' in DihedralGroup.
- fixed 'principalIdeal' in EuclideanDomain.
- order implied by Comparable is now honest linear order.
- Ring now exports Algebra(Fraction(Integer)) only in characteristic 0.

FriCAS 1.3.13

- new constructors: BrowserInformation, TexParser, ModularU32MatrixOperations, AIntervalCategory, AInterval, PadeContinuousFraction.
- changes to conditions and exports.
- new implementation of IndexCard.
- better support for pure Lisp build.

Bug fixes, in particular:

- fixed wrong initialization of number of rows and columns for two dimensional arrays.
- corrected result of integrals in terms of elliptic functions.
- corrected stirling numbers at 0.
- fixed map between matrices with differing minimal indices.
- interpreter now avoids strange types.
- more consistent handling of pathnames.

FriCAS 1.3.12

- Added Berkowitz algorithm for computing characteristic polynomial in general case.
- 'rootSplit' and 'rootFactor' now handle nested roots.
- Terminal emulator used by FriCAS is now settable via environment variables.

Bug fixes, in particular:

- Fixed several problems with handling type conditions in HyperDoc.
- Fixed coercion from expressions to algebraic numbers.
- Fixed arithmetic in FourierSeries.

FriCAS 1.3.11

- Implemented numeric 'riemannZeta'.
- Improved 'radicalSolve' for degree 4 polynomials.
- Improved FriCAS Book.
- Old factorizers for univariate polynomials over finite fields are removed.
- 'simplifyExp' supports more kinds of expressions.
- Small improvements to Spad compiler.

Bug fixes, in particular:

- Implemented inverse trigonometric and hyperbolic functions for power series at branch points.
- HyperDoc launched by ')hd' now properly exits when FriCAS quits.
- ')display op groebner' now works.
- Prevent exits from page viewer when content is less than one page.
- Fixed HyperDoc logical search.
- Various build fixes.

FriCAS 1.3.10

- New package for root denesting.
- New convenience package containing Unicode symbols.
- Improvements to runtime accounting subsystem, in particular FriCAS can now print info about storage use.
- Improved handling of roots during integration.
- Improved 'simplifyExp'.
- Domain Pi is renamed to PiDomain.
- Added Lisp version to FriCAS banner.

Bug fixes, in particular:

- Disabled splitting of roots by default simplifications.
- Fixed various build problems.
- Fixed wrong creation of sparse power series.
- Fixed problem with derivatives of unevaluated definite integrals.

FriCAS 1.3.9

- Distributed binary includes support for jfricas.
- Construct of tagged unions by specifying tag.
- New category Hashable.
- AlgebraGivenByStructuralConstants now more general and has FreeModuleCategory.
- Rename 'escape' in Character to 'underscore'.

Bug fixes, in particular:

- Fixed power series expander for 'polylog' at 0.
- Fixed storing complex data, in particular streams in files.
- Fixed various build problems, in particular build with sbcl-2.3.2 and newer.
- Main integrator no longer uses 'real'.

FriCAS 1.3.8

- Improvements to integrator, in particular integrator can now express some integrals in terms of elliptic integrals.
- More specialized array domains.
- Better handling of cyclotomic polynomials. Roots of cyclotomic polynomials are now presented in trigonometric form.
- Preliminary support for timeouts (only when using sbcl).
- 'iterate' is now implemented for Spad.

Bug fixes, in particular:

- Fixes for handling of kernels.
- Fixes for 'elt' when handling segments with increments.
- Fixed handling of predicates in rewrite rules.
- Fixed build with clisp.

FriCAS 1.3.7

- Added new formatting framework.
- The FRICASsys binary will use its parent directory when FRICAS environment variable is not set.
- Polynomial rings are now of FreeModuleCategory.
- Removed remains of obsolete commands.
- Small improvements to integrator.

Bug fixes, in particular:

- Fixed various build problems.
- Fixed tests for bad reduction during polynomial factorization and GCD.
- Worked around buffer overflow with large process identifiers.
- Removed non-Unicode default for sbcl.
- Fixed saving and restoring arrays.

FriCAS 1.3.6

- Small improvements to integrator and limits.
- Generalized a few domains and packages.
- Two new convolutions for quantum probability.
- Main FriCAS environment variable is now called FRICAS and main executable is called FRICAS-sys.

Bug fixes, in particular:

- Better error detection for numeric elementary functions.
- Fixed TeX output of formal derivatives.
- Fixed input form of formal derivatives.

FriCAS 1.3.5

- Added free noncommutative field.
- Added factorization in free algebra.
- Improved coercion to InputForm.
- Removed cycle related functions from Tree and BinaryTreeCategory.

Bug fixes, in particular:

- Improved portablity to Windows and Mac OSX.
- Fixed input form of formal derivatives.
- Fixed coercion of polynomials to patterns.
- Fixed comparison with signed floating point zero.

FriCAS 1.3.4

- Implemented 'sqrt' for prime fields.
- Improved computation of characteristic polynomial.
- Added conversion from list of elements of a free group to group presentation.
- Added 'extendedLLL'.
- Removed FreeAbelianGroup.
- Removed several old misfeatures.

Bug fixes, in particular:

- Fixed limit of Fresnel functions.
- Fixed few operations for matrices with specialised type.
- Fixed 'factor' for polynomials over finite fields.

FriCAS 1.3.3

- Added LLL reduction.
- New domain IntegerLocalizedAtPrime.
- Implemented numeric 'ellipticPi'.
- Improved Texmacs interface.
- Added 'gbasisExtend', removed VarSet from Groebner package interfaces.

Bug fixes, in particular:

- Fixed compatibility with sbcl-1.4.5.
- Fixed 'write' to Postscript file.
- Removed unsound power simplification.
- Recursion depth when resolving types is now limited (avoids crashes).
- Fixed handling of leading coefficient in gcd and square-free factorization.
- Build fixes

FriCAS 1.3.2

- Todd-Coxeter enumeration works also on cosets of subgroups.
- Handle some integrals in terms of incomplete Gamma with irrational first argument.
- Improvement to expressions: added symbolic 'conjugate', added derivatives of 'box' and 'paren'.
- Output now contains more spaces.
- Improvements to simplifying routines.

Bug fixes, in particular:

- Fixed a few glitches in printing operations.
- Fixed linear algebra with empty matrices.
- Avoided crash computing some determinants over GF(2).

FriCAS 1.3.1

- Categories with associative multiplication are now subcategories of categories with nonassociative multiplication.
- Inlining optimization in now effective also in command line (interpreter) compiler.
- Added conversions between finitely presented groups and permutation groups (Todd-Coxeter algorithm) and back.
- Removed special handling of coercion of String to OutputForm from Spad compiler.
- Former FramedModule is renamed to FractionalIdealAsModule. Added new FramedModule.
- Whole interpreter is now included in executable (no need to load parts before use).

Bug fixes, in particular:

- Fixed build with sbcl-1.3.13.
- Limits using the name of variable in limit point work now.
- A few output fixes.
- Several integrator fixes.
- Removed wrong interpreter transform of ' ='.
- Fixed compilation of type parameters containing non-type values.
- Plots sometimes used single precision. Now they should always use double precision.

FriCAS 1.3.0

- Several domains and categories are more general, in particular matrices, indexed products and direct product.
- ')show' now evaluates predicates.
- Improved integrator, handles few more 'erf' cases and more algebraic functions. Result should be simpler.
- Added support for using FriCAS as ECL shared library.
- Polynomial factorization uses Kaltofen-Shoup method when applicable.
- '$createLocalLibDb' defaults to false.
- Simpler, more predictable equality for algebraic numbers (no longer uses 'trueEqual').
- Renamed LinearlyExplicitRingOver to LinearlyExplicitOver.
- Renamed 'length' in Tuple to '#'.
- Removed argumentless 'random'.

Bug fixes, in particular:

- Fixed several build problems.
- Handle scripted symbols in DeRhamComplex.
- Handle empty matrices in more places.
- Fixed unparse of negative integers.
- No longer crashes on quoted expressions in types.

FriCAS 1.2.7

- New package implementing van Hoej factorization algorithm for LODO-s.
- Gcd over Expression(Integer) now uses modular method.
- Improvements to integrator, in particular trigonometric functions are consistently integrated via transformation to complex exponentials.
- Some categories and domains are more general. In particular OrderedFreeMonoid is removed, as ordered case is handled by FreeMonoid.
- Category Monad in renamed to Magma. Domain Magma is renamed to FreeMagma.

Bug fixes, in particular:

- Coercion of square matrices to polynomials is fixed.
- Problem with division by 0 in derivative of 'ellipticPi' is fixed.
- Division in Ore algebras used to cause infinite loop when coefficients were power series.

FriCAS 1.2.6

- Polynomial factorization is available for larger class of base rings.
- Improvements to integrator.
- 'normalize' can be applied to list of expressions.
- Eigenvalues can be computed over larger range of base fields.
- Common denominator package handles now multivariate polynomials.
- More uniform break (error) handling.

Bug fixes, in particular:

- 'distribute' handles 'box' operator.
- Fixed problem with guessing over multivariate polynomials.
- Fixed hashcode handling for Void in Aldor.

FriCAS 1.2.5

- Several improvements to integrator.
- Improvements to handling of series, in particular new function 'prodiag' to compute infinite products, 'series' and 'coefficients' for multivariate Taylor series, new 'laurent' function which builds Laurent series from order and stream of coefficients.
- GMP should now work with sbcl on all platforms and with Clozure CL on all platforms except for Power PC.
- Added a few domains for discrete groups.
- Extended GCD in Ore algebras can now return coefficients of both GCD and LCM.
- New function for computing integrals of solutions of linear differential operators.
- ')savesystem' command is now removed.
- Continuation lines which begins like commands are no longer treated as commands.

Bug fixes, in particular:

- Fixed printing of scripted symbols.
- Fixed 'totalDegreeSorted' (affected Groebner bases).
- Fixed few problems with Hensel lifting (including SF bug 47).
- Fixed 'series' in UnivariateLaurentSeriesConstructor.
- Fixed 'order' in SparseUnivariatePowerSeries.
- Printing of series now respect 'showall' setting, cyclic series are detected.
- Fixed problem with interpreter preferring Union to base type.

FriCAS 1.2.4

- New cylindrical decomposition package.
- New GnuDraw package for plotting via gnuplot.
- Texmacs interface now handles Cork symbols.
- Added double precision versions of several special functions (needed for plotting).
- Nopile mode for Spad is changed to be more convenient.
- 'stringMatch' is removed (was broken beyond repair).

Bug fixes, in particular:

- Fixed interpreter assignment to parts of nested aggregates (issue 376).

- Fixed interpreter coercion from Equation to Boolean (issue 359).
- Fix printing of '%i' in types (issue 132).
- Disabled incorrect shortcut during coercion (issue 29).
- Difference of intervals now agrees with definition as interval operation.
- Avoid overwriting loop limit and increment.
- Fix a polynomial gcd failure due to bad reduction.
- Avoid mangling unevaluated algebraic integrals.
- Fix integration of unevaluated derivatives.
- Restore parser handling of '
/' and '/
'.
- Properly escape strings and symbols in TeXFormat.
- Fix toplevel multiparameter macros.
- Fix problem with missing parentheses around plexes.
- Avoid crash when printing error message from '-eval'.
- Redirect I/O when running programs from Clozure CL.

FriCAS 1.2.3

- Improved integration in terms of 'Ei' and 'erf'.
- Classical orthogonal polynomials may be used as expressions.
- More cases of generalized indexing for two dimensional arrays.
- Value of 'lambertW' at '-1/e' is now simplified.
- FriCAS now knows that formal derivatives are commutative.
- 'setelt' is renamed to 'setelt!'.
- ')read' now creates intermediate files in current directory.
- Continuation characters in comments are now respected.
- In Spad '$Lisp' calls now must have a type.
- In Spad `error` did only minimal checking of its argument. Now argument to `error` must be a **String** or **OutputForm** or a literal list of **OutputForm**-s.

Bug fixes, in particular:

- Input lines with empty continuation are no longer lost.

- Types like "failed" now consistently use string quotes in output form.
- Fixed pattern matching using `%i` in patterns.
- Fixed ')display op coerce'.
- Fixed ')version' command.
- Fixed crash when printing '%'.
- Fix a buffer overflow in HyperDoc.
- Fixed HyperDoc errors in 'Dependants' and 'Users'.
- HyperDoc browser better handles constructors with parameters.

FriCAS 1.2.2

- Improvements to 'integrate': better handling of algebraic integrals, new routine which handles some integrals containing 'lambertW'.
- Improvements to 'limit', now Gruntz algorithm knows about a few tractable functions.
- Smith form of sparse integer matrices is now much more efficient.
- Generalized indexing for two dimensional arrays.
- Pile/nopile mode is now restored after ')read' or ')compile'. Piling rules now accept some forms of multiline lists.
- Eliminated version checking in generated code. Note: this change means that Spad code compiled by earlier FriCAS versions will not run in FriCAS 1.2.2.
- Updated Aldor interface to work with free Aldor.

Bug fixes, in particular:

- Interpreter can now handle complicated mutually recursive functions.
- Spad compiler should now correctly handle 'has' inside a function.
- Fixed derivatives of Whittaker functions.

FriCAS 1.2.1

- Improvements to 'integrate': a new routine for integration in terms of Ei, better handling of algebraic integrals.
- Implemented 'erfi'.
- Derivatives of 'asec', 'asech', 'acsc' and 'acsch' use different formula so that numeric evaluation of derivative will take correct branch on real axis.
- Linear dependence package is changed to be consistent with linear solvers.

- It is now possible to extract empty submatrices.
- Changed default style of 3D graphics.
- Support for building Mac OS application bundle.

Bug fixes, in particular:

- fixed few cases of wrong or unevaluated integrals.
- better zero test during limit computation avoids division by zero.
- fixed buffer overflow problems in view3D.
- 'reducedSystem' on empty input returns basis of correct size.

FriCAS 1.2.0

- New MatrixManipulation package.
- New ParallelIntegrationTools package.
- Gruntz algorithm is now used also for finite one-sided limits.
- FriCAS has now true 2-dimensional arrays (previously they were emulated using vectors of vectors).
- Speedups in some matrix operations and in arithmetic with algebraic expressions.
- FreeModule is now more general, it allows Comparable as second argument.
- Changed Spad parser, it now uses common scanner with interpreter. Spad language is now closer to interpreter language and Aldor. 'leave' is removed, 'free', 'generate' and 'goto' are now keywords. Pile rules changed slightly, they should be more intuitive now. Error messages from Spad parser should be slightly better.

Bug fixes, in particular:

- Fixed a few build problems.
- Eliminated division by 0 during 'normalize'.
- 'nthRootIfCan' removes leading zeros from generalized series (this avoids problems with power series expanders).
- Fixed corruption of formal derivatives.
- Fixed two problems with Fortran output.
- Fixed ')untrace' and ')undo'. Fixed ')trace' with ECL.
- Fixed problem with calling efricas if user's default shell is (t)csh.

FriCAS 1.1.8

- Improvements of pattern matching integrator, it can now integrate in terms of Fresnel integrals and better handles integrals in terms of Si and Ci.
- Better integration of symbolic derivatives.
- Better normalization of Liouvillian functions.
- New package for computing limits using Gruntz algorithm.
- Faster removal of roots from denominators.
- New domains for multivariate Ore algebras and partial differential operators.
- New package for noncommutative Groebner bases.
- New domain for univariate power series with arbitrary exponents.
- New special functions: Shi and Chi.
- Several aggregates (in particular tables) allow more general parameter types.
- New domain for hash tables using equality from underlying domain.

Bug fixes, in particular:

- Fixed problem with gcd failing due to bad reduction.
- Fixed series of 'acot' and Puiseux series of several special functions.
- Fixed wrong factorization of differential operators.
- Fixed build problem on recent Mac OS X.

FriCAS 1.1.7

- Improved integration in terms of special functions.
- Updated new graphics framework and graph theory package.
- Added routines for numerical evaluation of several special functions.
- Added modular method for computing polynomial gcd over algebraic extensions.
- Derivatives of fresnelC and fresnelS are changed to agree with established convention.
- When printing floats, groups of digits are now separated by underscores (previously they were separated by spaces).
- Added C code for removing directories, this speeds up full build and should avoid build problems on Mac OSX.

Bug fixes, in particular:

- Series expansion now handle poles of Gamma.
- Fixed derivatives of meijerG.

FriCAS 1.1.6

- Added experimental graph theory package.
- Added power series expanders for Weierstrass elliptic functions at 0.
- New functions: kroneckerProduct and kroneckerSum for matrices, numeric weierstrassInvariants and modularInvariantJ, symbolic Jacobi Zeta, double float numeric elliptic integrals.
- New domains for vectors and matrices of unsigned 8 and 16 bit integers.
- Changes to Spad compiler: underscores which are not needed as escape are now significant in Spad names and strings, macros with parameters are supported, added partial support for exceptions, braces can be used for grouping.
- A few speedups.
- Reduced disc space usage during build.

Bug fixes, in particular:

- Fixed eval of hypergeometricF at 0
- Fixed problem with scope of macros.
- Worked around problems with opening named pipes in several Lisp implementations.
- Fixed a problem with searching documentation via HyperDoc.
- Fixed build problem on Mac OSX.

FriCAS 1.1.5

- Added numeric version of lambertW.
- New function 'rootFactor' which tries to write roots of products as products of roots.
- 'try', 'catch' and 'finally' are now Spad keywords.
- Experimental support for using gmp with Clozure CL (64-bit Intel/Amd only).
- New categories CoercibleFrom and ConvertibleFrom. New domain for ordinals up to epsilon0. New domain for matrices of machine integers. New package for solving linear equations written as expressions (faster than general expression solver).
- Functions exported by Product() are now called 'construct', 'first' and 'second' (instead of 'makeprod', 'selectfirst' and 'selectsecond' respectively).
- Some functions are now much faster, in particular bivariate factorization over small finite fields.
- When using sbcl FriCAS now tries to preload statistical profiler.

Bug fixes, in particular:

- Fixed handling of Control-C in FriCAS compiled by recent sbcl.

- Fixed HyperDoc crash due to bad handling of '#'.
- Fixed power series expanders for elliptic integrals.
- Fixed 'possible wild ramification' problem with algebraic integrals.
- 'has' in interpreter now correctly handles %.
- Spad compiler can now handle single => at top level of a function.
- Fixed few problems with conditional types in Spad compiler.

FriCAS 1.1.4

- New domains for combinatorial probability theory by Franz Lehner.
- Improved integration of algebraic functions.
- Initial support for semirings.
- Updated framework for theory of computations.
- In Spad parser **, ^ and → are now right-associative.
- Spad parser no longer transforms relational operators.
- Join of categories is faster which speeds up Spad compiler.

Bug fixes, in particular:

- Retraction of 'rootOf' from **Expression(Integer)** to **AlgebraicNumber** works now.
- Attempt to print error message about invalid type no longer crash (SF 2977357).
- Fixed few problems in Spad compiler dealing with conditional exports.
- HyperDoc now should find all function descriptions (previously it missed several).

FriCAS 1.1.3

- Added "jet bundle" framework by Werner Seiler and Joachim Schue, which includes completion procedure and symmetry analysis for PDE.
- Better splitting of group representations (added Holt-Rees improvement to meatAxe).
- Added numeric versions of some elliptic integrals and few more elliptic functions.
- Speeded up FFCGP (finite fields via Zech logarithms).
- New experimental flag (off by default, set via setSimplifyDenomsFlag) which if on causes removal of irrationalities from denominators. Usually it causes slowdown, but on some examples gives huge speedup. It may go away in future (when no longer needed).
- Added experimental framework for theory of computations.

Bug fixes, in particular:

- Numerical solutions of polynomial systems have now required accuracy (SF 2418832).
- Fixed problem with crashes during tracing.
- Fixed a problem with nested iteration (SF 3016806).
- Eliminated stack overflow when concatenating long lists.

FriCAS 1.1.2

- Experimental Texmacs interface and Texmacs format output.
- Guessing package can now guess algebraic dependencies.
- Expansion into Taylor series and limits now work for most special functions.
- Spad to Aldor translator is removed.
- Spad compiler no longer allows to denote sets using braces.

Bug fixes, in particular:

- Fixed few cases where elementary integrals were returned unevaluated or produced wrong results.
- Unwanted numerical evaluation should be no longer a problem (FriCAS interpreter now very strongly prefers symbolic evaluation over numerical evaluation).
- Fixed a truncation bug in guessing package which caused loss of some correct solutions.
- TeX and MathML format should correctly put parentheses around and inside sums and products.
- Fixed few problems with handling of Unicode.

FriCAS 1.1.1

- New graphics framework.
- Support for using GMP with sbcl on 32/64 bit AMD/Intel processors (to activate it one must use '–with-gmp' option to configure).
- Improvements to integration and normalization. In particular integrals containing multiple non-nested roots should now work much faster. Also FriCAS now can compute more integrals of Liouvillian functions.
- Several new special functions.
- Improvements to efricas.
- Looking for default init file FriCAS now first tries to use '.fricas.input' and only if that fails it looks for '.axiom.input'.

Bug fixes, in particular:

- Numeric atan, asin and acos took wrong branch.
- WeierstrassPreparation package did not work.
- Saving and restoring history should be now more reliable.
- Fixed two bugs in Spad compiler related to conditional compilation.
- Fixed a problem with rational reconstruction which affected guessing package.

FriCAS 1.1.0

- New domains and packages: **VectorSpaceBasis** domain, **DirichletRing** domain, 3D graphic output in Wavefront .obj format, specialized machine precision numeric vectors and matrices (faster than general vectors and matrices), Html output.
- Support Clifford algebras corresponding to non-diagonal matrix, added new operations.
- 'normalize' now tries to simplify logarithms of algebraic constants.
- New functions: Fresnel integrals, carmichaelLambda.
- Speed improvements: several polynomial operations are faster, faster multiplication in Ore algebras, faster computation of strong generating set for permutation groups, faster coercions.
- Several improvements to the guessing package (in particular new option Somos for restricting attention to Somos-like sequences

Bug fixes, in particular:

- FriCAS can now compute multiplicative inverse of a power series with constant term not equal to 1.
- Fixed a problem with passing interpreter functions to algebra.
- Two bugs causing crashes in HyperDoc interface are fixed.
- FriCAS now ignores sign when deciding if number is prime.
- A failing coercion that used to crash FriCAS is now detected.
- 'has' test sometimes gave wrong result.
- Plotting fixes.

FriCAS 1.0.9

- Speed improvements to polynomial multiplication, power series multiplication, guessing package and coercion of polynomials to expressions.
- Domains for tensor products.
- **Complex(Integer)** is now **UniqueFactorizationDomain**.

- Types in interpreter are now of type 'Type' (instead of 'Domain') and categories in interpreter are of type 'Category' (instead of 'Subdomain(Domain)').
- Interpreter functions can now return 'Type'.
- New function for files: 'flush'.
- Spad compiler: return in nested functions and nested functions returning functions.

Bug fixes, in particular:

- Several fixes to guessing package.
- Avoid crash when unparsing equations.
- Equation solver accepts more solutions.
- Fixed handling of **Tuple** in Spad parser.
- Fixed miscompilation of record constructor by Spad compiler.

FriCAS 1.0.8

- Improved version of guessing package. It can now handle much larger problems than before. Added ability to guess functional substitution equations.
- Experimental support for build using CMU CL
- Various speed improvements including faster indexing for two dimensional arrays
- By default FriCAS build tries to use sbcl.
- Building no longer require patch.

Bug fixes, in particular:

- correct definition of random() for matrices
- conditionals in .input files work again
- Spad compiler now recognizes more types as equal
- fixed problem with pattern-matching quote

FriCAS 1.0.7

- Comparisons between elements of the Expression domain are undefined. Earlier versions gave confusing results for expressions like `%e < %pi` – now FriCAS will complain about `<` being undefined.
- A domain for general quaternions was added.
- Equality in **Any** is now more reasonable – it uses equality from underlying domain if available.

- Messages about loading of components are switched off by default.
- Release build benefits from parallel make.
- In Spad code a single quote now means that the following token is a symbol.
- Reorganization of algebra sources, in particular several types have changed (this may affect users Spad code).

Bug fixes, in particular:

- Categories with default package can be used just after definition (fixes 1.0.6 regression).
- Plots involving 0 or 1 work now.
- Numbers in radix bigger than 10 appear correctly in TeX output.
- Fixed browser crashes when displaying some domains.
- Fix horizontal display of fractions.
- Allow local domains in conditionals (in Spad code).
- Fixed problem with splitting polynomials and nested extensions.

FriCAS 1.0.6

- the axiom script is no longer installed (use fricas script instead)
- some undesirable simplification are no longer done by default, for example now asin(sin(3)) is left unevaluated
- support lambda expressions using "`+->`" syntax and nested functions in Spad
- better configure, support for Dragonfly BSD
- faster bootstrap, also parallel (this does not affect speed of release build)

Several bug fixes, in particular:

- fixed a regression introduced in 1.0.4 which caused equality for nested products to sometimes give wrong result
- corrected fixed output of floating point numbers,
- operations on differential operators like symmetric power work now
- fixed crashes related to coercing power series
- functions returning Void can be traced

FriCAS 1.0.5

- improvement to normalize function, it performs now much stronger simplifications than before
- better integration: due to improved normalize FriCAS can now integrate many functions that it previously considered unintegrable
- improvement to Martin Rubey guessing package, for example it can now guess differential equation for the generating function of integer partitions
- better support for using type valued functions
- several bug fixes

FriCAS 1.0.4

- significant speedups for some operations (for example definite integration)
- support for building algebra using user-defined optimization settings
- support for mouse wheel in HyperDoc browser
- included support for interfacing with Aldor
- new optional Emacs mode and efricas script to run FriCAS inside emacs
- better unparse
- removed support for attributes (replaced by empty categories) and use of colon for type conversions in Spad code
- a few bug fixes

FriCAS 1.0.3

- added multiple precision Gamma and logGamma functions
- better line editing
- removed some undocumented and confusing constructs from Spad language
- added new categories for semiring and ordered semigroup, direct product of monoids is now a monoid
- internal cleanups and restructurings
- a few bug fixes

FriCAS 1.0.2

- ')nopiles' command gives conventional syntax
- added pfaffian function
- ECL support
- Graphics and Hyperdoc work using openmcl or ECL
- Output may be now delimited by user defined markers
- Experimental support for using as a Lisp library
- Spad compiler is now significantly faster
- Several bug fixes

FriCAS 1.0.1

- Graphics and Hyperdoc work using sbcl or clisp
- Builds under Cygwin (using Cygwin clisp)
- MathML support contributed by Arthur C. Ralfs
- Help files created by Tim Daly
- Added SPADEDIT script
- Full release caches all generated HyperDoc pages
- Bug fixes, including implementing some missing functions and build fixes

FriCAS 1.0.0

The 1.0 release is the first release of FriCAS. Below we list main differences compared to AXIOM September 2006.

Numerous bug fixes (in particular HyperDoc is now fully functional on Unix systems).

FriCAS includes guessing package written by Martin Rubey. This package provides unique ability to guess formulas for sequences of numbers or polynomials.

Some computation, in particular involving Expression domain, should be much faster. FriCAS goes through its testsuite in only half of the time needed by AXIOM September 2006.

Spad compilation is faster (in some cases 2 times faster).

FriCAS is much more portable than AXIOM September 2006. It can be built on Linux, many Unix systems (for example Mac OS X and Solaris 10) and Windows. It can be built on top of gcl, sbcl, clisp or openmcl (gcl and sbcl based FriCAS is fully functional, clisp or openmcl based one lacks graphic support).

Many unused or non-working parts are removed from FriCAS. In particular FriCAS does not contain support for NAG numerical library.

FriCAS can be built from sources using only a few pre-generated Lisp files for bootstrap – only to bootstrap Shoe translator. This means that modifying FriCAS algebra is now much easier.

15.2 Changes to Spad language

1. `$` as name of current domain is no longer supported, use `%` instead.
2. Attributes are no longer supported, use niladic categories with no exports instead.
3. Floating point numbers without leading zero are no longer supported, so instead of `.01` use `0.01`
4. Anonymous functions using `#1`, `#2`, etc. are no longer supported, to define anonymous functions use $\mapsto$.
5. Braces no longer construct sets. So instead of `{'sin, 'cos}::Set(Symbol)` use `set(['sin, 'cos])$Set(Symbol)`.
6. Old Spad used colon (`:`) to denote conversion, like `pretend` but performing even less checking. This is no longer supported, use `::` or `pretend` instead.
7. There was an alternative spelling for brackets and braces, in FriCAS this is no longer supported, so one has to write brackets and braces as is.
8. **SubsetCategory** was handled in special way by the compiler. This is no longer supported.
9. Old Spad compiler used to transform relational operators `~=,<=,>,>=` in ways which are correct for linear order, but may conflict with other uses (as partial order or when generating **OutputForm**). FriCAS no longer performs this transformation. Similarly, Spad parser no longer treats `^` and `^=` in special way.
10. Quote in old Spad allowed to insert arbitrary literal Lisp data, FriCAS only allows symbols after quote. Code using old behavior needs to be rewritten, however it seems that this feature was almost unused, so this should be no problem.
11. Old Spad treated statement consisting just of constructor name (with arguments if needed) as request to import the constructor. FriCAS requires `import` keyword.
12. In FriCAS `**`, `^`, $\rightarrow$ are right associative. Also, right binding power of $\mapsto$ is increased, which allows more natural writing of code.
13. Few non-working experimental features are removed, in particular partial support for APL-like syntax.
14. FriCAS implemented parametric macros in the Spad compiler.
15. FriCAS allows simplified form for exporting constants (without `constant` keyword).
16. FriCAS added partial support for exception handling (currently only `finally` part).
17. The `leave` construct is removed from FriCAS. Use `break` instead.
18. `div` is no longer a keyword. `free`, `generate`, `goto` are FriCAS keywords.
19. '$Lisp' calls now must have a type
20. `error` did only minimal checking of its argument. Now argument to `error` must be a **String** or **OutputForm** or a literal list of **OutputForm**-s.

There are also library changes that affect user code:

1. `**` lost its definition as exponentiation, use `^` instead.
2. `^` is no longer used as negation (it means exponentiation now) and `^=` no longer means inequality, use `not` and `~=` instead.
3. `setelt` is renamed to `setelt!`.
4. Operator properties are now symbols and not strings, so instead of `has?(op, "even")` use `has?(op, 'even)`
5. There is new category **Comparable**, several constructors that asserted **OrderedSet** now only assert **Comparable**.

15.3 Online Information

FriCAS information can be found online at

- http://fricas.github.io – The official documentation of FriCAS including the API of the FriCAS library.
- https://github.com/fricas/fricas – The official git repository.
- http://wiki.fricas.org – A wiki site related to FriCAS.
- http://fricas.sourceforge.net – The old homepage of FriCAS.
- http://sourceforge.net/p/fricas/code/HEAD/tree/ – The old source code repository.

15.4 Old News about AXIOM Version 2.x

Many things have changed in this version of AXIOM and we describe many of the more important topics here.

15.4.1 The NAG Library Link

Content removed, NAGLink is no longer included in FriCAS.

15.4.2 Interactive Front-end and Language

The `leave` keyword has been replaced by the `break` keyword for compatibility with the new AXIOM extension language. See section Section 5.4.3 on page 132 for more information.

Curly braces are no longer used to create sets. Instead, use `set` followed by a bracketed expression. For example,

```
set [1,2,3,4]
```

$$\{1,\ 2,\ 3,\ 4\} \tag{1}$$

```
                                                                Set( PositiveInteger )
```

Curly braces are now used to enclose a block (see section Section 5.2 on page 126 for more information). For compatibility, a block can still be enclosed by parentheses as well.

New coercions to and from type **Expression** have been added. For example, it is now possible to map a polynomial represented as an expression to an appropriate polynomial type.

Various messages have been added or rewritten for clarity.

15.4.3 Library

The **FullPartialFractionExpansion** domain has been added. This domain computes factor-free full partial fraction expansions. See section '`FullPartialFractionExpansion`' on page 429 for examples.

We have implemented the Bertrand/Cantor algorithm for integrals of hyperelliptic functions. This brings a major speedup for some classes of algebraic integrals.

We have implemented a new (direct) algorithm for integrating trigonometric functions. This brings a speedup and an improvement in the answer quality.

The SmallFloat domain has been renamed **DoubleFloat** and SmallInteger has been renamed **SingleInteger**. The new abbreviations as **DFLOAT** and **SINT**, respectively. We have defined the macro SF, the old abbreviation for SmallFloat, to expand to **DoubleFloat** and modified the documentation and input file examples to use the new names and abbreviations. You should do the same in any private FriCAS files you have.

We have made improvements to the differential equation solvers and there is a new facility for solving systems of first-order linear differential equations. In particular, an important fix was made to the solver for inhomogeneous linear ordinary differential equations that corrected the calculation of particular solutions. We also made improvements to the polynomial and transcendental equation solvers including the ability to solve some classes of systems of transcendental equations.

The efficiency of power series have been improved and left and right expansions of `tan(f(x))` at `x =` a pole of `f(x)` can now be computed. A number of power series bugs were fixed and the **GeneralUnivariatePowerSeries** domain was added. The power series variable can appear in the coefficients and when this happens, you cannot differentiate or integrate the series. Differentiation and integration with respect to other variables is supported.

A domain was added for representing asymptotic expansions of a function at an exponential singularity.

For limits, the main new feature is the exponential expansion domain used to treat certain exponential singularities. Previously, such singularities were treated in an *ad hoc* way and only a few cases were covered. Now AXIOM can do things like

```
limit( (x+1)^(x+1)/x^x - x^x/(x-1)^(x-1), x = %plusInfinity)
```

in a systematic way. It only does one level of nesting, though. In other words, if `f` is a function with a pole, we can handle `exp(f)`, but not `exp(exp(f))`.

The computation of integral bases has been improved through careful use of Hermite row reduction. A P-adic algorithm for function fields of algebraic curves in finite characteristic has also been developed.

Miscellaneous: There is improved conversion of definite and indefinite integrals to **InputForm**; binomial coefficients are displayed in a new way; some new simplifications of radicals have been implemented; the

operation complexForm for converting to rectangular coordinates has been added; symmetric product operations have been added to **LinearOrdinaryDifferentialOperator**.

15.4.4 HyperDoc

The buttons on the titlebar and scrollbar have been replaced with ones which have a 3D effect. You can change the foreground and background colors of these "controls" by including and modifying the following lines in your **.Xdefaults** file.

```
Axiom.hyperdoc.ControlBackground: White
Axiom.hyperdoc.ControlForeground: Black
```

For various reasons, HyperDoc sometimes displays a secondary window. You can control the size and placement of this window by including and modifying the following line in your **.Xdefaults** file.

```
Axiom.hyperdoc.FormGeometry: =950x450+100+0
```

This setting is a standard X Window System geometry specification: you are requesting a window 950 pixels wide by 450 deep and placed in the upper left corner.

Some key definitions have been changed to conform more closely with the CUA guidelines. Press F9 to see the current definitions.

Input boxes (for example, in the Browser) now accept paste-ins from the X Window System. Use the second button to paste in something you have previously copied or cut. An example of how you can use this is that you can paste the type from a FriCAS computation into the main Browser input box.

15.4.5 Documentation

We describe here a few additions to the on-line version of the AXIOM book which you can read with HyperDoc.

A section has been added to the graphics chapter, describing how to build two-dimensional graphs from lists of points. An example is given showing how to read the points from a file. See section Section 7.1.9 on page 211 for details.

A further section has been added to that same chapter, describing how to add a two-dimensional graph to a viewport which already contains other graphs. See section Section 7.1.10 on page 219 for details.

Chapter 3 and the on-line HyperDoc help have been unified.

An explanation of operation names ending in "?" and "!" has been added to the first chapter. See the end of the section Section 1.3.6 on page 28 for details.

An expanded explanation of using predicates has been added to the sixth chapter. See the example involving **evenRule** in the middle of the section Section 6.21 on page 187 for details.

Documentation for the `)compile`, `)library` and `)load` commands has been greatly changed. This reflects the ability of the `)compile` to now invoke the AXIOM-XL compiler, the impending deletion of the `)load` command and the new `)library` command. The `)library` command replaces `)load` and is compatible with the compiled output from both the old and new compilers.

15.4.6 AXIOM-XL compiler - Enhancements and Additions

Content removed - AXIOM-XL (now using name *Aldor*) is a separate project.

15.4.7 New polynomial domains and algorithms

Univariate polynomial factorization over the integers has been enhanced by updates to the **GaloisGroupFactorizer** type and friends from Frederic Lehobey (Frederic.Lehobey@lifl.fr, University of Lille I, France).

The package constructor **PseudoRemainderSequence** provides efficient algorithms by Lionel Ducos (Lionel.Ducos@mathlabo.univ-poitiers.fr, University of Poitiers, France) for computing sub-resultants. This leads to a speed up in many places in FriCAS where sub-resultants are computed (polynomial system solving, algebraic factorization, integration).

Based on this package, the domain constructor **NewSparseUnivariatePolynomial** extends the constructor **SparseUnivariatePolynomial**. In a similar way, the **NewSparseMultivariatePolynomial** extends the constructor **SparseUnivariatePolynomial**; it also provides some additional operations related to polynomial system solving by means of triangular sets.

Several domain constructors implement regular triangular sets (or regular chains). Among them **RegularTriangularSet** and **SquareFreeRegularTriangularSet**. They also implement an algorithm by Marc Moreno Maza (marc@nag.co.uk, NAG) for computing triangular decompositions of polynomial systems. This method is refined in the package **LazardSetSolvingPackage** in order to produce decompositions by means of Lazard triangular sets. For the case of polynomial systems with finitely many solutions, these decompositions can also be computed by the package **LexTriangularPackage**.

The domain constructor **RealClosure** by Renaud Rioboo (Renaud.Rioboo@lip6.fr, University of Paris 6, France) provides the real closure of an ordered field. The implementation is based on interval arithmetic. Moreover, the design of this constructor and its related packages allows an easy use of other codings for real algebraic numbers.

Based on triangular decompositions and the **RealClosure** constructor, the package **ZeroDimensionalSolvePackage** provides operations for computing symbolically the real or complex roots of polynomial systems with finitely many solutions.

Polynomial arithmetic with non-commutative variables has been improved too by a contribution of Michel Petitot (Michel.Petitot@lifl.fr, University of Lille I, France). The domain constructors **XRecursivePolynomial** and **XDistributedPolynomial** provide recursive and distributed representations for these polynomials. They are the non-commutative equivalents for the **SparseMultivariatePolynomial** and **DistributedMultivariatePolynomial** constructors. The constructor **LiePolynomial** implement Lie polynomials in the Lyndon basis. The constructor **XPBWPolynomial** manage polynomials with non-commutative variables in the Poincaré-Birkhoff-Witt basis from the Lyndon basis. This allows to compute in the Lie Group associated with a free nilpotent Lie algebra by using the **LieExponentials** domain constructor.

15.4.8 Enhancements to HyperDoc and Graphics

From this version of AXIOM onwards, the pixmap format used to save graphics images in color and to display them in HyperDoc has been changed to the industry-standard XPM format. See `ftp://koala.inria.fr/pub/xpm`.

15.4.9 Enhancements to NAGLink

Content removed - NAGLink is no longer included in FriCAS.

15.4.10 Enhancements to the Lisp system

Content removed - no longer relevant since FriCAS runs on different Lisp systems.

Appendix A

FriCAS System Commands

This chapter describes system commands, the command-line facilities used to control the FriCAS environment. The first section is an introduction and discusses the common syntax of the commands available.

A.1 Introduction

System commands are used to perform FriCAS environment management. Among the commands are those that display what has been defined or computed, set up multiple logical FriCAS environments (frames), clear definitions, read files of expressions and commands, show what functions are available, and terminate FriCAS.

Some commands are restricted: the commands

```
)set userlevel interpreter
)set userlevel compiler
)set userlevel development
```

set the user-access level to the three possible choices. All commands are available at `development` level and the fewest are available at `interpreter` level. The default user-level is `interpreter`. In addition to the `)set` command (discussed in Section A.20 on page 753) you can use the HyperDoc settings facility to change the *user-level*.

Each command listing begins with one or more syntax pattern descriptions plus examples of related commands. The syntax descriptions are intended to be easy to read and do not necessarily represent the most compact way of specifying all possible arguments and options; the descriptions may occasionally be redundant.

All system commands begin with a right parenthesis which should be in the first available column of the input line (that is, immediately after the input prompt, if any). System commands may be issued directly to FriCAS or be included in **.input** files.

A system command *argument* is a word that directly follows the command name and is not followed or preceded by a right parenthesis. A system command *option* follows the system command and is directly

preceded by a right parenthesis. Options may have arguments: they directly follow the option. This example may make it easier to remember what is an option and what is an argument:

)syscmd *arg1 arg2*)opt1 *opt1arg1 opt1arg2*)opt2 *opt2arg1* ...

In the system command descriptions, optional arguments and options are enclosed in brackets ("[" and "]"). If an argument or option name is in italics, it is meant to be a variable and must have some actual value substituted for it when the system command call is made. For example, the syntax pattern description

)read *fileName* []quietly]

would imply that you must provide an actual file name for *fileName* but need not use the)quietly option. Thus

```
)read matrix.input
```

is a valid instance of the above pattern.

System command names and options may be abbreviated and may be in upper or lower case. The case of actual arguments may be significant, depending on the particular situation (such as in file names). System command names and options may be abbreviated to the minimum number of starting letters so that the name or option is unique. Thus

```
)s Integer
```

is not a valid abbreviation for the)set command, because both)set and)show begin with the letter "s". Typically, two or three letters are sufficient for disambiguating names. In our descriptions of the commands, we have used no abbreviations for either command names or options.

In some syntax descriptions we use a vertical line "|" to indicate that you must specify one of the listed choices. For example, in

```
)set output fortran on | off
```

only on and off are acceptable words for following boot. We also sometimes use "..." to indicate that additional arguments or options of the listed form are allowed. Finally, in the syntax descriptions we may also list the syntax of related commands.

A.2)abbreviation

User Level Required: compiler

Command Syntax:

)abbreviation query [*nameOrAbbrev*]
)abbreviation category *abbrev fullname* []quiet]
)abbreviation domain *abbrev fullname* []quiet]
)abbreviation package *abbrev fullname* []quiet]

)abbreviation remove *nameOrAbbrev*

Command Description:

This command is used to query, set and remove abbreviations for category, domain and package constructors. Every constructor must have a unique abbreviation. This abbreviation is part of the name of the subdirectory under which the components of the compiled constructor are stored. Furthermore, by issuing this command you let the system know what file to load automatically if you use a new constructor. Abbreviations must start with a letter and then be followed by up to seven letters or digits. Any letters appearing in the abbreviation must be in uppercase.

When used with the `query` argument, this command may be used to list the name associated with a particular abbreviation or the abbreviation for a constructor. If no abbreviation or name is given, the names and corresponding abbreviations for *all* constructors are listed.

The following shows the abbreviation for the constructor **List**:

```
)abbreviation query List
```

The following shows the constructor name corresponding to the abbreviation **NNI**:

```
)abbreviation query NNI
```

The following lists all constructor names and their abbreviations.

```
)abbreviation query
```

To add an abbreviation for a constructor, use this command with `category`, `domain` or `package`. The following add abbreviations to the system for a category, domain and package, respectively:

```
)abbreviation domain   SET Set
)abbreviation category COMPCAT  ComplexCategory
)abbreviation package  LIST2MAP ListToMap
```

If the `)quiet` option is used, no output is displayed from this command. You would normally only define an abbreviation in a library source file. If this command is issued for a constructor that has already been loaded, the constructor will be reloaded next time it is referenced. In particular, you can use this command to force the automatic reloading of constructors.

To remove an abbreviation, the `remove` argument is used. This is usually only used to correct a previous command that set an abbreviation for a constructor name. If, in fact, the abbreviation does exist, you are prompted for confirmation of the removal request. Either of the following commands will remove the abbreviation **VECTOR2** and the constructor name **VectorFunctions2** from the system:

```
)abbreviation remove VECTOR2
)abbreviation remove VectorFunctions2
```

Also See: '`)compile`' in Section A.7 on page 738 and '`)library`' in Section A.14 on page 749.

A.3)boot

User Level Required: development

Command Syntax:

`)boot` *bootExpression*

Command Description:

This command is used by FriCAS system developers to execute expressions written in the BOOT language. For example,

```
)boot times3(x) == 3*x
```

creates and compiles the Common LISP function "times3" obtained by translating the BOOT code.

Also See: '`)fin`' in Section A.10 on page 745, '`)lisp`' in Section A.15 on page 750, '`)set`' in Section A.20 on page 753, and '`)system`' in Section A.24 on page 756.

A.4)cd

User Level Required: interpreter

Command Syntax:

`)cd` *directory*

Command Description:

This command sets the FriCAS working current directory. The current directory is used for looking for input files (for `)read`), FriCAS library source files (for `)compile`), saved history environment files (for `)history )restore`), compiled FriCAS library files (for `)library`), and files to edit (for `)edit`). It is also used for writing spool files (via `)spool`), writing history input files (via `)history )write`) and history environment files (via `)history )save`),and compiled FriCAS library files (via `)compile`).

If issued with no argument, this command sets the FriCAS current directory to your home directory. If an argument is used, it must be a valid directory name. Except for the "`)`" at the beginning of the command, this has the same syntax as the operating system `cd` command.

Also See: '`)compile`' in Section A.7 on page 738, '`)edit`' in Section A.9 on page 744, '`)history`' in Section A.13 on page 747, '`)library`' in Section A.14 on page 749, '`)read`' in Section A.19 on page 752, and '`)spool`' in Section A.22 on page 754.

A.5)close

User Level Required: interpreter

Command Syntax:

```
)close
)close )quietly
```

Command Description:

This command is used to close down interpreter client processes. Such processes are started by HyperDoc to run FriCAS examples when you click on their text. When you have finished examining or modifying the example and you do not want the extra window around anymore, issue

```
)close
```

to the FriCAS prompt in the window.

If you try to close down the last remaining interpreter client process, FriCAS will offer to close down the entire FriCAS session and return you to the operating system by displaying something like

```
This is the last FriCAS session. Do you want to kill FriCAS?
```

Type "y" (followed by the Return key) if this is what you had in mind. Type "n" (followed by the Return key) to cancel the command.

You can use the `)quietly` option to force FriCAS to close down the interpreter client process without closing down the entire FriCAS session.

Also See: '`)quit`' in Section A.18 on page 751 and '`)pquit`' in Section A.17 on page 751.

A.6)clear

User Level Required: interpreter

Command Syntax:

```
)clear all
)clear completely
)clear properties all
)clear properties obj1 [obj2 ...]
)clear value all
)clear value obj1 [obj2 ...]
)clear mode all
)clear mode obj1 [obj2 ...]
```

Command Description:

This command is used to remove function and variable declarations, definitions and values from the workspace. To empty the entire workspace and reset the step counter to 1, issue

```
)clear all
```

To remove everything in the workspace but not reset the step counter, issue

```
)clear properties all
```

To remove everything about the object `x`, issue

```
)clear properties x
```

To remove everything about the objects `x, y` and `f`, issue

```
)clear properties x y f
```

The word `properties` may be abbreviated to the single letter "`p`".

```
)clear p all
)clear p x
)clear p x y f
```

All definitions of functions and values of variables may be removed by either

```
)clear value all
)clear v all
```

This retains whatever declarations the objects had. To remove definitions and values for the specific objects `x, y` and `f`, issue

```
)clear value x y f
)clear v x y f
```

To remove the declarations of everything while leaving the definitions and values, issue

```
)clear mode all
)clear m all
```

To remove declarations for the specific objects `x, y` and `f`, issue

```
)clear mode x y f
)clear m x y f
```

The `)display names` and `)display properties` commands may be used to see what is currently in the workspace.

The command

```
)clear completely
```

does everything that `)clear all` does, and also clears the internal system function and constructor caches.

Also See: '`)display`' in Section A.8 on page 743, '`)history`' in Section A.13 on page 747, and '`)undo`' in Section A.26 on page 760.

A.7)compile

User Level Required: compiler

Command Syntax:

`)compile`
`)compile` *fileName*

)compile *fileName*.as
)compile *directory/fileName*.as
)compile *fileName*.ao
)compile *directory/fileName*.ao
)compile *fileName*.al
)compile *directory/fileName*.al
)compile *fileName*.lsp
)compile *directory/fileName*.lsp
)compile *fileName*.spad
)compile *directory/fileName*.spad
)compile *fileName*)new
)compile *fileName*)old
)compile *fileName*)quiet
)compile *fileName*)noquiet
)compile *fileName*)moreargs
)compile *fileName*)onlyargs
)compile *fileName*)break
)compile *fileName*)nobreak
)compile *fileName*)library
)compile *fileName*)nolibrary
)compile *fileName*)vartrace
)compile *fileName*)constructor *nameOrAbbrev*

Command Description:

You use this command to invoke the Aldor library compiler or the FriCAS system compiler. The)compile system command is actually a combination of FriCAS processing and a call to the Aldor compiler. It is performing double-duty, acting as a front-end to both the Aldor compiler and the FriCAS system compiler. (The FriCAS system compiler is written in Boot and is an integral part of the FriCAS environment. The Aldor compiler is written in C and executed by the operating system when called from within FriCAS.)

The command compiles files with file extensions *.as, .ao* and *.al* with the Aldor compiler and files with file extension *.spad* with the FriCAS system compiler. It also can compile files with file extension *.lsp*. These are assumed to be Lisp files generated by the Aldor compiler. If you omit the file extension, the command looks to see if you have specified the)new or)old option. If you have given one of these options, the corresponding compiler is used. Otherwise, the command first looks in the standard system directories for files with extension *.as, .ao* and *.al* and then files with extension *.spad*. The first file found has the appropriate compiler invoked on it. If the command cannot find a matching file, an error message is displayed and the command terminates.

We now describe the options for the Aldor compiler.

The first thing)compile does is look for a source code filename among its arguments. Thus

```
)compile mycode.as
)compile /u/jones/as/mycode.as
)compile mycode
```

all invoke `)compiler` on the file `/u/jones/as/mycode.as` if the current FriCAS working directory is `/u/jones/as.` (Recall that you can set the working directory via the `)cd` command. If you don't set it explicitly, it is the directory from which you started FriCAS.)

This is frequently all you need to compile your file. This simple command:

1. invokes the Aldor compiler and produces Lisp output,
2. calls the Lisp compiler if the Aldor compilation was successful,
3. uses the `)library` command to tell FriCAS about the contents of your compiled file and arrange to have those contents loaded on demand.

Should you not want the `)library` command automatically invoked, call `)compile` with the `)nolibrary` option. For example,

```
)compile mycode.as )nolibrary
```

The general description of Aldor command line arguments is in the Aldor documentation. The default options used by the `)compile` command can be viewed and set using the `)set compiler args` FriCAS system command. The current defaults are

```
-O -Fasy -Fao -Flsp -lfricas -Mno-ALDOR_W_WillObsolete -DFriCAS
-Y $FRICAS/algebra -I $FRICAS/algebra
```

These options mean:

- `-O`: perform all optimizations,
- `-Fasy`: generate a `.asy` file,
- `-Fao`: generate a `.ao` file,
- `-Flsp`: generate a `.lsp` (Lisp) file,
- `-lfricas`: use the `fricas` library `libfricas.al`,
- `-Mno-ALDOR_W_WillObsolete`: do not display messages about older generated files becoming obsolete, and
- `-DFriCAS`: define the global assertion `FriCAS` so that the Aldor libraries for generating stand-alone code are not accidentally used with FriCAS.

To supplement these default arguments, use the `)moreargs` option on `)compile.` For example,

```
)compile mycode.as )moreargs "-v"
```

uses the default arguments and appends the `-v` (verbose) argument flag. The additional argument specification **must be enclosed in double quotes.**

To completely replace these default arguments for a particular use of `)compile`, use the `)onlyargs` option. For example,

```
)compile mycode.as )onlyargs "-v -O"
```

only uses the `-v` (verbose) and `-O` (optimize) arguments. The argument specification **must be enclosed in double quotes.** In this example, Lisp code is not produced and so the compilation output will not be available to FriCAS.

To completely replace the default arguments for all calls to `)compile` within your FriCAS session, use `)set compiler args`. For example, to use the above arguments for all compilations, issue

```
)set compiler args "-v -O"
```

Make sure you include the necessary `-l` and `-Y` arguments along with those needed for Lisp file creation. As above, **the argument specification must be enclosed in double quotes.**

By default, the `)library` system command *exposes* all domains and categories it processes. This means that the FriCAS interpreter will consider those domains and categories when it is trying to resolve a reference to a function. Sometimes domains and categories should not be exposed. For example, a domain may just be used privately by another domain and may not be meant for top-level use. The `)library` command should still be used, though, so that the code will be loaded on demand. In this case, you should use the `)nolibrary` option on `)compile` and the `)noexpose` option in the `)library` command. For example,

```
)compile mycode.as )nolibrary
)library mycode )noexpose
```

Once you have established your own collection of compiled code, you may find it handy to use the `)dir` option on the `)library` command. This causes `)library` to process all compiled code in the specified directory. For example,

```
)library )dir /u/jones/as/quantum
```

You must give an explicit directory after `)dir`, even if you want all compiled code in the current working directory processed, e.g.

```
)library )dir .
```

The `)compile` command works with several file extensions. We saw above what happens when it is invoked on a file with extension `.as`. A `.ao` file is a portable binary compiled version of a `.as` file, and `)compile` simply passes the `.ao` file onto Aldor. The generated Lisp file is compiled and `)library` is automatically called, just as if you had specified a `.as` file.

A `.al` file is an archive file containing `.ao` files. The archive is created (on Unix systems) with the `ar` program. When `)compile` is given a `.al` file, it creates a directory whose name is based on that of the archive. For example, if you issue

```
)compile mylib.al
```

the directory `mylib.axldir` is created. All members of the archive are unarchived into the directory and `)compile` is called on each `.ao` file found. It is your responsibility to remove the directory and its contents, if you choose to do so.

A `.lsp` file is a Lisp source file, presumably, in our context, generated by Aldor when called with the `-Flsp` option. When `)compile` is used with a `.lsp` file, the Lisp file is compiled and `)library` is called. You must also have present a `.asy` generated from the same source file.

The following are descriptions of options for the FriCAS system compiler.

You can compile category, domain, and package constructors contained in files with file extension *.spad*. You can compile individual constructors or every constructor in a file.

The full filename is remembered between invocations of this command and `)edit` commands. The sequence of commands

```
)compile matrix.spad
)edit
)compile
```

will call the compiler, edit, and then call the compiler again on the file **matrix.spad.** If you do not specify a *directory,* the working current directory (see Section A.4 on page 736) is searched for the file. If the file is not found, the standard system directories are searched.

If you do not give any options, all constructors within a file are compiled. Each constructor should have an `)abbreviation` command in the file in which it is defined. We suggest that you place the `)abbreviation` commands at the top of the file in the order in which the constructors are defined. The list of commands serves as a table of contents for the file.

The `)library` option causes directories containing the compiled code for each constructor to be created in the working current directory. The name of such a directory consists of the constructor abbreviation and the **.NRLIB** file extension. For example, the directory containing the compiled code for the **MATRIX** constructor is called **MATRIX.NRLIB.** The `)nolibrary` option says that such files should not be created. The default is `)library.` Note that the semantics of `)library` and `)nolibrary` for the Aldor compiler and for the FriCAS system compiler are completely different.

The `)vartrace` option causes the compiler to generate extra code for the constructor to support conditional tracing of variable assignments. (see Section A.25 on page 756). Without this option, this code is suppressed and one cannot use the `)vars` option for the trace command.

The `)constructor` option is used to specify a particular constructor to compile. All other constructors in the file are ignored. The constructor name or abbreviation follows `)constructor.` Thus either

```
)compile matrix.spad )constructor RectangularMatrix
```

or

```
)compile matrix.spad )constructor RMATRIX
```

compiles the **RectangularMatrix** constructor defined in **matrix.spad.**

The `)break` and `)nobreak` options determine what the FriCASsystem compiler does when it encounters an error. `)break` is the default and it indicates that processing should stop at the first error. The value of the `)set break` variable then controls what happens.

Also See: '`)abbreviation`' in Section A.2 on page 734, '`)edit`' in Section A.9 on page 744, and '`)library`' in Section A.14 on page 749.

A.8)display

User Level Required: interpreter

Command Syntax:

```
)display all
)display properties
)display properties all
)display properties [obj1 [obj2 ...]]
)display value all
)display value [obj1 [obj2 ...]]
)display mode all
)display mode [obj1 [obj2 ...]]
)display names
)display operations opName
```

Command Description:

This command is used to display the contents of the workspace and signatures of functions with a given name.[1]

The command

```
)display names
```

lists the names of all user-defined objects in the workspace. This is useful if you do not wish to see everything about the objects and need only be reminded of their names.

The commands

```
)display all
)display properties
)display properties all
```

all do the same thing: show the values and types and declared modes of all variables in the workspace. If you have defined functions, their signatures and definitions will also be displayed.

To show all information about a particular variable or user functions, for example, something named `d`, issue

```
)display properties d
```

To just show the value (and the type) of `d`, issue

```
)display value d
```

To just show the declared mode of `d`, issue

[1] A *signature* gives the argument and return types of a function.

```
)display mode d
```

All modemaps for a given operation may be displayed by using `)display operations`. A *modemap* is a collection of information about a particular reference to an operation. This includes the types of the arguments and the return value, the location of the implementation and any conditions on the types. The modemap may contain patterns. The following displays the modemaps for the operation complex:

```
)d op complex
```

Also See: '`)clear`' in Section A.6 on page 737, '`)history`' in Section A.13 on page 747, '`)set`' in Section A.20 on page 753, '`)show`' in Section A.21 on page 753, and '`)what`' in Section A.27 on page 761.

A.9)edit

User Level Required: interpreter

Command Syntax:

`)edit` [*filename*]

Command Description:

This command is used to edit files. It works in conjunction with the `)read` and `)compile` commands to remember the name of the file on which you are working. By specifying the name fully, you can edit any file you wish. Thus

```
)edit /u/julius/matrix.input
```

will place you in an editor looking at the file `/u/julius/matrix.input`. By default, the editor is `less` just for viewing purpose, but if you have a shell environment variable FRICASEDITOR defined, that editor will be used. When FriCAS is running under the X Window System, it will try to open a separate `xterm` running your editor if it thinks one is necessary. For example, under the Bash shell, if you issue

```
export FRICASEDITOR='emacs +$line $name'
```

then the emacs editor will be used by `)edit`.

If you do not specify a file name, the last file you edited, read or compiled will be used. If there is no "last file" you will be placed in the editor editing an empty unnamed file.

It is possible to use the `)system` command to edit a file directly. For example,

```
)system emacs /etc/rc.tcpip
```

calls `emacs` to edit the file.

Also See: '`)system`' in Section A.24 on page 756, '`)compile`' in Section A.7 on page 738, and '`)read`' in Section A.19 on page 752.

A.10)fin

User Level Required: development

Command Syntax:

```
)fin
```

Command Description:

This command is used by FriCAS developers to leave the FriCAS system and return to the underlying Common LISP system. To return to FriCAS, issue the "`(|spad|)`" function call to Common LISP.

Also See: '`)pquit`' in Section A.17 on page 751 and '`)quit`' in Section A.18 on page 751.

A.11)frame

User Level Required: interpreter

Command Syntax:

```
)frame new frameName
)frame drop [frameName]
)frame next
)frame last
)frame names
)frame import frameName [objectName1 [objectName2 ...]]
)set message frame on | off
)set message prompt frame
```

Command Description:

A *frame* can be thought of as a logical session within the physical session that you get when you start the system. You can have as many frames as you want, within the limits of your computer's storage, paging space, and so on. Each frame has its own *step number*, *environment* and *history*. You can have a variable named `a` in one frame and it will have nothing to do with anything that might be called `a` in any other frame.

Some frames are created by the HyperDoc program and these can have pretty strange names, since they are generated automatically. To find out the names of all frames, issue

```
)frame names
```

It will indicate the name of the current frame.

You create a new frame "**quark**" by issuing

```
)frame new quark
```

The history facility can be turned on by issuing either `)set history on` or `)history )on`. If the history facility is on and you are saving history information in a file rather than in the FriCAS environment

then a history file with filename **quark.axh** will be created as you enter commands. If you wish to go back to what you were doing in the "**initial**" frame, use

```
)frame next
```

or

```
)frame last
```

to cycle through the ring of available frames to get back to "**initial**".

If you want to throw away a frame (say "**quark**"), issue

```
)frame drop quark
```

If you omit the name, the current frame is dropped.

If you do use frames with the history facility on and writing to a file, you may want to delete some of the older history files. These are directories, so you may want to issue a command like `rm -r quark.axh` to the operating system.

You can bring things from another frame by using `)frame import`. For example, to bring the `f` and `g` from the frame "**quark**" to the current frame, issue

```
)frame import quark f g
```

If you want everything from the frame "**quark**", issue

```
)frame import quark
```

You will be asked to verify that you really want everything.

There are two `)set` flags to make it easier to tell where you are.

```
)set message frame on | off
```

will print more messages about frames when it is set on. By default, it is off.

```
)set message prompt frame
```

will give a prompt that looks like

```
initial (1) ->
```

when you start up. In this case, the frame name and step make up the prompt.

Also See: '`)history`' in Section A.13 on page 747 and '`)set`' in Section A.20 on page 753.

A.12)help

User Level Required: interpreter

Command Syntax:

`)help`
`)help` *commandName*
`)help syntax`

Command Description:

This command displays help information about system commands. If you issue

```
)help help
```

then this very text will be shown. You can also give the name of a system command to display information about it. For example,

```
)help clear
```

will display the description of the `)clear` system command.

The command

```
)help syntax
```

will give further information about the FriCAS language syntax.

All this material is available in the FriCAS User Guide and in HyperDoc. In HyperDoc, choose the **Commands** item from the **Reference** menu.

A.13)history

User Level Required: interpreter

Command Syntax:

`)history )on`
`)history )off`
`)history )write` *historyInputFileName*
`)history )show [`*n*`] [both]`
`)history )save` *savedHistoryName*
`)history )restore [`*savedHistoryName*`]`
`)history )reset`
`)history )change` *n*
`)history )memory`
`)history )file`
`%`

`%%(`n`)`
`)set history on | off`

Command Description:

The *history* facility within FriCAS allows you to restore your environment to that of another session and recall previous computational results. Additional commands allow you to review previous input lines and to create an **.input** file of the lines typed to FriCAS.

FriCAS saves your input and output if the history facility is turned on (which is the default). This information is saved if either of

```
)set history on
)history )on
```

has been issued. Issuing either

```
)set history off
)history )off
```

will discontinue the recording of information.

Whether the facility is disabled or not, the value of "`%`" in FriCAS always refers to the result of the last computation. If you have not yet entered anything, "`%`" evaluates to an object of type **Variable('%)**. The function "`%%`" may be used to refer to other previous results if the history facility is enabled. In that case, `%%(n)` is the output from step `n` if `n > 0`. If `n < 0`, the step is computed relative to the current step. Thus `%%(-1)` is also the previous step, `%%(-2)`, is the step before that, and so on. If an invalid step number is given, FriCAS will signal an error.

The *environment* information can either be saved in a file or entirely in memory (the default). Each frame (Section A.11 on page 745) has its own history database. When it is kept in a file, some of it may also be kept in memory for efficiency. When the information is saved in a file, the name of the file is of the form **FRAME.axh** where "**FRAME**" is the name of the current frame. The history file is placed in the current working directory (see Section A.4 on page 736). Note that these history database files are not text files (in fact, they are directories themselves), and so are not in human-readable format.

The options to the `)history` command are as follows:

`)change` n will set the number of steps that are saved in memory to n. This option only has effect when the history data is maintained in a file. If you have issued `)history )memory` (or not changed the default) there is no need to use `)history )change`.

`)on` will start the recording of information. If the workspace is not empty, you will be asked to confirm this request. If you do so, the workspace will be cleared and history data will begin being saved. You can also turn the facility on by issuing `)set history on`.

`)off` will stop the recording of information. The `)history )show` command will not work after issuing this command. Note that this command may be issued to save time, as there is some performance penalty paid for saving the environment data. You can also turn the facility off by issuing `)set history off`.

`)file` indicates that history data should be saved in an external file on disk.

`)memory` indicates that all history data should be kept in memory rather than saved in a file. Note that if you are computing with very large objects it may not be practical to keep this data in memory.

`)reset` will flush the internal list of the most recent workspace calculations so that the data structures may be garbage collected by the underlying Common LISP system. Like `)history )change`, this option only has real effect when history data is being saved in a file.

`)restore [`*savedHistoryName*`]` completely clears the environment and restores it to a saved session, if possible. The `)save` option below allows you to save a session to a file with a given name. If you had issued `)history )save jacobi` the command `)history )restore jacobi` would clear the current workspace and load the contents of the named saved session. If no saved session name is specified, the system looks for a file called **last.axh**.

`)save` *savedHistoryName* is used to save a snapshot of the environment in a file. This file is placed in the current working directory (see Section A.4 on page 736). Use `)history )restore` to restore the environment to the state preserved in the file. This option also creates an input file containing all the lines of input since you created the workspace frame (for example, by starting your FriCAS session) or last did a `)clear all` or `)clear completely`.

`)show [`*n*`] [both]` can show previous input lines and output results. `)show` will display up to twenty of the last input lines (fewer if you haven't typed in twenty lines). `)show` *n* will display up to *n* of the last input lines. `)show both` will display up to five of the last input lines and output results. `)show` *n* `both` will display up to *n* of the last input lines and output results.

`)write` *historyInputFile* creates an **.input** file with the input lines typed since the start of the session/frame or the last `)clear all` or `)clear completely`. If *historyInputFileName* does not contain a period (".") in the filename, **.input** is appended to it. For example, `)history )write chaos` and `)history )write chaos.input` both write the input lines to a file called **chaos.input** in your current working directory. If you issued one or more `)undo` commands, `)history )write` eliminates all input lines backtracked over as a result of `)undo`. You can edit this file and then use `)read` to have FriCAS process the contents.

Also See: '`)frame`' in Section A.11 on page 745, '`)read`' in Section A.19 on page 752, '`)set`' in Section A.20 on page 753, and '`)undo`' in Section A.26 on page 760.

A.14)library

User Level Required: interpreter

Command Syntax:

`)library` *libName1* `[`*libName2 ...*`]`
`)library )dir` *dirName*
`)library )only` *objName1* `[`*objlib2 ...*`]`
`)library )noexpose`

Command Description:

This command replaces the `)load` system command. The `)library` command makes available to FriCAS the compiled objects in the libraries listed.

For example, if you `)compile dopler.as` in your home directory, issue `)library dopler` to have FriCAS look at the library, determine the category and domain constructors present, update the internal database with various properties of the constructors, and arrange for the constructors to be automatically loaded when needed. If the `)noexpose` option has not been given, the constructors will be exposed (that is, available) in the current frame.

If you compiled a file with the FriCAS system compiler, you will have an *NRLIB* present, for example, *DOPLER.NRLIB,* where `DOPLER` is a constructor abbreviation. The command `)library DOPLER` will then do the analysis and database updates as above.

To tell the system about all libraries in a directory, use `)library )dir dirName` where `dirName` is an explicit directory. You may specify "." as the directory, which means the current directory from which you started the system or the one you set via the `)cd` command. The directory name is required.

You may only want to tell the system about particular constructors within a library. In this case, use the `)only` option. The command `)library dopler )only Test1` will only cause the `Test1` constructor to be analyzed, autoloaded, etc..

Finally, each constructor in a library are usually automatically exposed when the `)library` command is used. Use the `)noexpose` option if you do not want them exposed. At a later time you can use `)set expose add constructor` to expose any hidden constructors.

Also See: '`)cd`' in Section A.4 on page 736, '`)compile`' in Section A.7 on page 738, '`)frame`' in Section A.11 on page 745, and '`)set`' in Section A.20 on page 753.

A.15)lisp

User Level Required: development

Command Syntax:

`)lisp` [*lispExpression*]

Command Description:

This command is used by FriCAS system developers to have single expressions evaluated by the Common LISP system on which FriCAS is built. The *lispExpression* is read by the Common LISP reader and evaluated. If this expression is not complete (unbalanced parentheses, say), the reader will wait until a complete expression is entered.

Since this command is only useful for evaluating single expressions, the `)fin` command may be used to drop out of FriCAS into Common LISP.

Also See: '`)system`' in Section A.24 on page 756, '`)boot`' in Section A.3 on page 735, and '`)fin`' in Section A.10 on page 745.

A.16)ltrace

User Level Required: development

Command Syntax:

This command has the same arguments as options as the `)trace` command.

Command Description:

This command is used by FriCAS system developers to trace Common LISP or BOOT functions. It is not supported for general use.

Also See: '`)boot`' in Section A.3 on page 735, '`)lisp`' in Section A.15 on page 750, and '`)trace`' in Section A.25 on page 756.

A.17)pquit

User Level Required: interpreter

Command Syntax:

```
)pquit
```

Command Description:

This command is used to terminate FriCAS and return to the operating system. Other than by redoing all your computations or by using the `)history )restore` command to try to restore your working environment, you cannot return to FriCAS in the same state.

`)pquit` differs from the `)quit` in that it always asks for confirmation that you want to terminate FriCAS (the "p" is for "protected"). When you enter the `)pquit` command, FriCAS responds

Please enter **y** or **yes** if you really want to leave the interactive environment and return to the operating system:

If you respond with `y` or `yes`, you will see the message

You are now leaving the FriCAS interactive environment.
Issue the command **fricas** to the operating system to start a new session.

and FriCAS will terminate and return you to the operating system (or the environment from which you invoked the system). If you responded with something other than `y` or `yes`, then the message

You have chosen to remain in the FriCAS interactive environment.

will be displayed and, indeed, FriCAS would still be running.

Also See: '`)fin`' in Section A.10 on page 745, '`)history`' in Section A.13 on page 747, '`)close`' in Section A.5 on page 736, '`)quit`' in Section A.18 on page 751, and '`)system`' in Section A.24 on page 756.

A.18)quit

User Level Required: interpreter

Command Syntax:

```
)quit
```

```
)set quit protected | unprotected
```

Command Description:

This command is used to terminate FriCAS and return to the operating system. Other than by redoing all your computations or by using the `)history )restore` command to try to restore your working environment, you cannot return to FriCAS in the same state.

`)quit` differs from the `)pquit` in that it asks for confirmation only if the command

```
)set quit protected
```

has been issued. Otherwise, `)quit` will make FriCAS terminate and return you to the operating system (or the environment from which you invoked the system).

The default setting is `)set quit unprotected`. We suggest that you do not (somehow) assign `)quit` to be executed when you press, say, a function key.

Also See: '`)fin`' in Section A.10 on page 745, '`)history`' in Section A.13 on page 747, '`)close`' in Section A.5 on page 736, '`)pquit`' in Section A.17 on page 751, and '`)system`' in Section A.24 on page 756.

A.19)read

User Level Required: interpreter

Command Syntax:

`)read` [*fileName*]
`)read` [*fileName*] `[)quiet]` `[)ifthere]`

Command Description:

This command is used to read **.input** files into FriCAS. The command

```
)read matrix.input
```

will read the contents of the file **matrix.input** into FriCAS. The ".input" file extension is optional. See Section 4.1 on page 105 for more information about **.input** files.

This command remembers the previous file you edited, read or compiled. If you do not specify a file name, the previous file will be read.

The `)ifthere` option checks to see whether the **.input** file exists. If it does not, the `)read` command does nothing. If you do not use this option and the file does not exist, you are asked to give the name of an existing **.input** file.

The `)quiet` option suppresses output while the file is being read.

At user level "development" the `)read` command can also handle Boot and Lisp files.

Also See: '`)compile`' in Section A.7 on page 738, '`)edit`' in Section A.9 on page 744, and '`)history`' in Section A.13 on page 747.

A.20)set

User Level Required: interpreter

Command Syntax:

`)set`
`)set` *label1* `[`*... labelN*`]`
`)set` *label1* `[`*... labelN*`]` *newValue*

Command Description:

The `)set` command is used to view or set system variables that control what messages are displayed, the type of output desired, the status of the history facility, the way FriCAS user functions are cached, and so on. Since this collection is very large, we will not discuss them here. Rather, we will show how the facility is used. We urge you to explore the `)set` options to familiarize yourself with how you can modify your FriCAS working environment. There is a HyperDoc version of this same facility available from the main HyperDoc menu.

The `)set` command is command-driven with a menu display. It is tree-structured. To see all top-level nodes, issue `)set` by itself.

```
)set
```

Variables with values have them displayed near the right margin. Subtrees of selections have "`...`" displayed in the value field. For example, there are many kinds of messages, so issue `)set message` to see the choices.

```
)set message
```

The current setting for the variable that displays whether computation times are displayed is visible in the menu displayed by the last command. To see more information, issue

```
)set message time
```

This shows that time printing is on now. To turn it off, issue

```
)set message time off
```

As noted above, not all settings have so many qualifiers. For example, to change the `)quit` command to being unprotected (that is, you will not be prompted for verification), you need only issue

```
)set quit unprotected
```

Also See: '`)quit`' in Section A.18 on page 751.

A.21)show

User Level Required: interpreter

Command Syntax:

)show *nameOrAbbrev*
)show *nameOrAbbrev*)operations

Command Description: This command displays information about FriCAS domain, package and category *constructors*. If no options are given, the)operations option is assumed. For example,

```
)show POLY
)show POLY )operations
)show Polynomial
)show Polynomial )operations
```

each display basic information about the **Polynomial** domain constructor and then provide a listing of operations. Since **Polynomial** requires a **Ring** (for example, **Integer**) as argument, the above commands all refer to an unspecified ring R. In the list of operations, "$" means **Polynomial(R)**.

The basic information displayed includes the *signature* of the constructor (the name and arguments), the constructor *abbreviation*, the *exposure status* of the constructor, and the name of the *library source file* for the constructor.

If operation information about a specific domain is wanted, the full or abbreviated domain name may be used. For example,

```
)show POLY INT
)show POLY INT )operations
)show Polynomial Integer
)show Polynomial Integer )operations
```

are among the combinations that will display the operations exported by the domain **Polynomial(Integer)** (as opposed to the general *domain constructor* **Polynomial**).

Also See: ')display' in Section A.8 on page 743, ')set' in Section A.20 on page 753, and ')what' in Section A.27 on page 761.

A.22)spool

User Level Required: interpreter

Command Syntax:

)spool [*fileName*]
)spool

Command Description:

This command is used to save *(spool)* all FriCAS input and output into a file, called a *spool file.* You can only have one spool file active at a time. To start spool, issue this command with a filename. For example,

```
)spool integrate.out
```

To stop spooling, issue)spool with no filename.

If the filename is qualified with a directory, then the output will be placed in that directory. If no directory information is given, the spool file will be placed in the *current directory.* The current directory is the directory from which you started FriCAS or is the directory you specified using the `)cd` command.

Also See: '`)cd`' in Section A.4 on page 736.

A.23)synonym

User Level Required: interpreter

Command Syntax:

`)synonym`
`)synonym` *synonym fullCommand*
`)what synonyms`

Command Description:

This command is used to create short synonyms for system command expressions. For example, the following synonyms might simplify commands you often use.

```
)synonym save         history )save
)synonym restore      history )restore
)synonym mail         system mail
)synonym ls           system ls
)synonym fortran      set output fortran
```

Once defined, synonyms can be used in place of the longer command expressions. Thus

```
)fortran on
```

is the same as the longer

```
)set fortran output on
```

To list all defined synonyms, issue either of

```
)synonym
)what synonyms
```

To list, say, all synonyms that contain the substring "`ap`", issue

```
)what synonyms ap
```

Also See: '`)set`' in Section A.20 on page 753 and '`)what`' in Section A.27 on page 761.

A.24)system

User Level Required: interpreter

Command Syntax:

`)system` *cmdExpression*

Command Description:

This command may be used to issue commands to the operating system while remaining in FriCAS. The *cmdExpression* is passed to the operating system for execution.

To get an operating system shell, issue, for example, `)system sh`. When you enter the key combination, **Ctrl**–**D** (pressing and holding the **Ctrl** key and then pressing the **D** key) the shell will terminate and you will return to FriCAS. We do not recommend this way of creating a shell because Common LISP may field some interrupts instead of the shell. If possible, use a shell running in another window.

If you execute programs that misbehave you may not be able to return to FriCAS. If this happens, you may have no other choice than to restart FriCAS and restore the environment via `)history )restore`, if possible.

Also See: '`)boot`' in Section A.3 on page 735, '`)fin`' in Section A.10 on page 745, '`)lisp`' in Section A.15 on page 750, '`)pquit`' in Section A.17 on page 751, and '`)quit`' in Section A.18 on page 751.

A.25)trace

User Level Required: interpreter

Command Syntax:

`)trace`
`)trace )off`
`)trace` *function* [*options*]
`)trace` *constructor* [*options*]
`)trace` *domainOrPackage* [*options*]

where options can be one or more of

`)after` *S-expression*
`)before` *S-expression*
`)break after`
`)break before`
`)cond` *S-expression*
`)count`
`)count` *n*
`)depth` *n*
`)local` *op1* [*... opN*]
`)nonquietly`
`)nt`

)off
)only *listOfDataToDisplay*
)ops
)ops *op1* [... *opN*]
)restore
)stats
)stats reset
)timer
)varbreak
)varbreak *var1* [... *varN*]
)vars
)vars *var1* [... *varN*]
)within *executingFunction*

Command Description:

This command is used to trace the execution of functions that make up the FriCAS system, functions defined by users, and functions from the system library. Almost all options are available for each type of function but exceptions will be noted below.

To list all functions, constructors, domains and packages that are traced, simply issue

```
)trace
```

To untrace everything that is traced, issue

```
)trace )off
```

When a function is traced, the default system action is to display the arguments to the function and the return value when the function is exited. Note that if a function is left via an action such as a THROW, no return value will be displayed. Also, optimization of tail recursion may decrease the number of times a function is actually invoked and so may cause less trace information to be displayed. Other information can be displayed or collected when a function is traced and this is controlled by the various options. Most options will be of interest only to FriCAS system developers. If a domain or package is traced, the default action is to trace all functions exported.

Individual interpreter, lisp or boot functions can be traced by listing their names after)trace. Any options that are present must follow the functions to be traced.

```
)trace f
```

traces the function f. To untrace f, issue

```
)trace f )off
```

Note that if a function name contains a special character, it will be necessary to escape the character with an underscore

```
)trace _/D_,1
```

To trace all domains or packages that are or will be created from a particular constructor, give the constructor name or abbreviation after `)trace`.

```
)trace MATRIX
)trace List Integer
```

The first command traces all domains currently instantiated with **Matrix**. If additional domains are instantiated with this constructor (for example, if you have used **Matrix(Integer)** and **Matrix(Float)**), they will be automatically traced. The second command traces **List(Integer)**. It is possible to trace individual functions in a domain or package. See the `)ops` option below.

The following are the general options for the `)trace` command.

`)break after` causes a Common LISP break loop to be entered after exiting the traced function.

`)break before` causes a Common LISP break loop to be entered before entering the traced function.

`)break` is the same as `)break before`.

`)count` causes the system to keep a count of the number of times the traced function is entered. The total can be displayed with `)trace )stats` and cleared with `)trace )stats reset`.

`)count` *n* causes information about the traced function to be displayed for the first *n* executions. After the n^{th} execution, the function is untraced.

`)depth` *n* causes trace information to be shown for only *n* levels of recursion of the traced function. The command

```
)trace fib )depth 10
```

will cause the display of only 10 levels of trace information for the recursive execution of a user function **fib**.

`)math` causes the function arguments and return value to be displayed in the FriCAS monospace two-dimensional math format.

`)nonquietly` causes the display of additional messages when a function is traced.

`)nt` This suppresses all normal trace information. This option is useful if the `)count` or `)timer` options are used and you are interested in the statistics but not the function calling information.

`)off` causes untracing of all or specific functions. Without an argument, all functions, constructors, domains and packages are untraced. Otherwise, the given functions and other objects are untraced. To immediately retrace the untraced functions, issue `)trace )restore`.

`)only` *listOfDataToDisplay* causes only specific trace information to be shown. The items are listed by using the following abbreviations:

a display all arguments

v display return value

1 display first argument

2 display second argument

15 display the 15th argument, and so on

`)restore` causes the last untraced functions to be retraced. If additional options are present, they are added to those previously in effect.

`)stats` causes the display of statistics collected by the use of the `)count` and `)timer` options.

`)stats reset` resets to 0 the statistics collected by the use of the `)count` and `)timer` options.

`)timer` causes the system to keep a count of execution times for the traced function. The total can be displayed with `)trace )stats` and cleared with `)trace )stats reset`.

`)varbreak` *var1* `[`*... varN*`]` causes a Common LISP break loop to be entered after the assignment to any of the listed variables in the traced function.

`)vars` causes the display of the value of any variable after it is assigned in the traced function. Note that library code must have been compiled (see Section A.7 on page 738) using the `)vartrace` option in order to support this option.

`)vars` *var1* `[`*... varN*`]` causes the display of the value of any of the specified variables after they are assigned in the traced function. Note that library code must have been compiled (see Section A.7 on page 738) using the `)vartrace` option in order to support this option.

`)within` *executingFunction* causes the display of trace information only if the traced function is called when the given *executingFunction* is running.

The following are the options for tracing constructors, domains and packages.

`)local [`*op1* `[`*... opN*`]]` causes local functions of the constructor to be traced. Note that to untrace an individual local function, you must use the fully qualified internal name, using the escape character "`_`" before the semicolon.

```
)trace FRAC )local
)trace FRAC_;cancelGcd )off
```

`)ops` *op1* `[`*... opN*`]` By default, all operations from a domain or package are traced when the domain or package is traced. This option allows you to specify that only particular operations should be traced. The command

```
)trace Integer )ops min max _+ _-
```

traces four operations from the domain **Integer**. Since `+` and `-` are special characters, it is necessary to escape them with an underscore.

Also See: '`)boot`' in Section A.3 on page 735, '`)lisp`' in Section A.15 on page 750, and '`)ltrace`' in Section A.16 on page 750.

A.26)undo

User Level Required: interpreter

Command Syntax:

)undo
)undo *integer*
)undo *integer* [*option*]
)undo)redo

where *option* is one of

)after
)before

Command Description:

This command is used to restore the state of the user environment to an earlier point in the interactive session. The argument of an)undo is an integer which must designate some step number in the interactive session.

```
)undo n
)undo n )after
```

These commands return the state of the interactive environment to that immediately after step n. If n is a positive number, then n refers to step number n. If n is a negative number, it refers to the n^{th} previous command (that is, undoes the effects of the last $-n$ commands).

A)clear all resets the)undo facility. Otherwise, an)undo undoes the effect of)clear with options properties, value, and mode, and that of a previous undo. If any such system commands are given between steps n and $n+1$ ($n > 0$), their effect is undone for)undo m for any $0 < m \leq n$.

The command)undo is equivalent to)undo -1 (it undoes the effect of the previous user expression). The command)undo 0 undoes any of the above system commands issued since the last user expression.

```
)undo n )before
```

This command returns the state of the interactive environment to that immediately before step n. Any)undo or)clear system commands given before step n will not be undone.

```
)undo )redo
```

This command reads the file redo.input. created by the last)undo command. This file consists of all user input lines, excluding those backtracked over due to a previous)undo.

The command)history)write will eliminate the "undone" command lines of your program.

Also See: ')history' in Section A.13 on page 747.

A.27)what

User Level Required: interpreter

Command Syntax:

)what categories *pattern1* [*pattern2 ...*]
)what commands *pattern1* [*pattern2 ...*]
)what domains *pattern1* [*pattern2 ...*]
)what operations *pattern1* [*pattern2 ...*]
)what packages *pattern1* [*pattern2 ...*]
)what synonym *pattern1* [*pattern2 ...*]
)what things *pattern1* [*pattern2 ...*]
)apropos *pattern1* [*pattern2 ...*]

Command Description:

This command is used to display lists of things in the system. The patterns are all strings and, if present, restrict the contents of the lists. Only those items that contain one or more of the strings as substrings are displayed. For example,

```
)what synonym
```

displays all command synonyms,

```
)what synonym ver
```

displays all command synonyms containing the substring "ver",

```
)what synonym ver pr
```

displays all command synonyms containing the substring "ver" or the substring "pr". Output similar to the following will be displayed

```
---------------- System Command Synonyms -----------------

user-defined synonyms satisfying patterns:
      ver pr

  )apr ........................... )what things
  )apropos ....................... )what things
  )prompt ........................ )set message prompt
  )version ....................... )lisp *yearweek*
```

Several other things can be listed with the)what command:

categories displays a list of category constructors.

commands displays a list of system commands available at your user-level. Your user-level is set via the)set userlevel command. To get a description of a particular command, such as ")what", issue)help what.

`domains` displays a list of domain constructors.

`operations` displays a list of operations in the system library. It is recommended that you qualify this command with one or more patterns, as there are thousands of operations available. For example, say you are looking for functions that involve computation of eigenvalues. To find their names, try `)what operations eig`. A rather large list of operations is loaded into the workspace when this command is first issued. This list will be deleted when you clear the workspace via `)clear all` or `)clear completely`. It will be re-created if it is needed again.

`packages` displays a list of package constructors.

`synonym` lists system command synonyms.

`things` displays all of the above types for items containing the pattern strings as substrings. The command synonym `)apropos` is equivalent to `)what things`.

Also See: '`)display`' in Section A.8 on page 743, '`)set`' in Section A.20 on page 753, and '`)show`' in Section A.21 on page 753.

Appendix B

Programs for FriCAS Images

This appendix contains the FriCAS programs used to generate the images in the FriCAS Images color insert of this book. All these input files are included with the FriCAS system. To produce the images on page 6 of the FriCAS Images insert, for example, issue the command:

```
)read images6
```

These images were produced on an IBM RS/6000 model 530 with a standard color graphics adapter. The smooth shaded images were made from X Window System screen dumps. The remaining images were produced with FriCAS-generated PostScript output. The images were reproduced from slides made on an Agfa ChromaScript PostScript interpreter with a Matrix Instruments QCR camera.

B.1 images1.input

```
1  )read tknot                                    -- Read torus knot program.
2
3  torusKnot(15,17, 0.1, 6, 700)                  -- A (15,17) torus knot.
```

B.2 images2.input

These images illustrate how Newton's method converges when computing the complex cube roots of 2. Each point in the (x, y)-plane represents the complex number $x + iy$, which is given as a starting point for Newton's method. The poles in these images represent bad starting values. The flat areas are the regions of convergence to the three roots.

```
)read newton                                    -- Read the programs from
)read vectors                                   --    Chapter 10.
f := newtonStep(x^3 - 2)                        -- Create a Newton's iteration
                                                --    function for x^3 = 2.
```

The function f^n computes n steps of Newton's method.

```
clipValue := 4                                  -- Clip values with magnitude > 4.
drawComplexVectorField(f^3, -3..3, -3..3)       -- The vector field for f^3
drawComplex(f^3, -3..3, -3..3)                  -- The surface for f^3
drawComplex(f^4, -3..3, -3..3)                  -- The surface for f^4
```

B.3 images3.input

```
)r tknot
for i in 0..4 repeat torusKnot(2, 2 + i/4, 0.5, 25, 250)
```

B.4 images5.input

The parameterization of the Etruscan Venus is due to George Frances.

```
venus(a,r,steps) ==
  surf := (u:DFLOAT, v:DFLOAT): Point DFLOAT +->
    cv := cos(v)
    sv := sin(v)
    cu := cos(u)
    su := sin(u)
    x := r * cos(2*u) * cv + sv * cu
    y := r * sin(2*u) * cv - sv * su
    z := a * cv
    point [x,y,z]
  draw(surf, 0..%pi, -%pi..%pi, var1Steps==steps,
       var2Steps==steps, title == "Etruscan Venus")

venus(5/2, 13/10, 50)                           -- The Etruscan Venus
```

The Figure-8 Klein Bottle parameterization is from "Differential Geometry and Computer Graphics" by Thomas Banchoff, in *Perspectives in Mathematics,* Anniversary of Oberwolfach 1984, Birkhäuser-Verlag, Basel, pp. 43-60.

```
klein(x,y) ==
  cx := cos(x)
  cy := cos(y)
  sx := sin(x)
  sy := sin(y)
  sx2 := sin(x/2)
  cx2 := cos(x/2)
  sq2 := sqrt(2.0@DFLOAT)
```

```
  point [cx * (cx2 * (sq2 + cy) + (sx2 * sy * cy)), _
         sx * (cx2 * (sq2 + cy) + (sx2 * sy * cy)), _
         -sx2 * (sq2 + cy) + cx2 * sy * cy]

draw(klein, 0..4*%pi, 0..2*%pi, var1Steps==50,          -- Figure-8 Klein bottle
     var2Steps==50,title=="Figure Eight Klein Bottle")
```

The next two images are examples of generalized tubes.

```
)read ntube
rotateBy(p, theta) ==                                   -- Rotate a point p by
  c := cos(theta)                                       --   θ around the origin.
  s := sin(theta)
  point [p.1*c - p.2*s, p.1*s + p.2*c]

bcircle t ==                                            -- A circle in three-space.
  point [3*cos t, 3*sin t, 0]

twist(u, t) ==                                          -- An ellipse that twists
  theta := 4*t                                          --   around four times as
  p := point [sin u, cos(u)/2]                          --   t revolves once.
  rotateBy(p, theta)

ntubeDrawOpt(bcircle, twist, 0..2*%pi, 0..2*%pi,        -- Twisted Torus
             var1Steps == 70, var2Steps == 250)

twist2(u, t) ==                                         -- Create a twisting circle.
  theta := t
  p := point [sin u, cos(u)]
  rotateBy(p, theta)

cf(u,v) == sin(21*u)                                    -- Color function with 21 stripes.

ntubeDrawOpt(bcircle, twist2, 0..2*%pi, 0..2*%pi,       -- Striped Torus
  colorFunction == cf, var1Steps == 168,
  var2Steps == 126)
```

B.5 images6.input

```
gam(x,y) ==                                             -- The height and color are the
  g := Gamma complex(x,y)                               --   real and argument parts
  point [x,y,max(min(real g, 4), -4), argument g]       --   of the Gamma function,
                                                        --   respectively.

draw(gam, -%pi..%pi, -%pi..%pi,                         -- The Gamma Function
     title == "Gamma(x + %i*y)", _
     var1Steps == 100, var2Steps == 100)

b(x,y) == Beta(x,y)

draw(b, -3.1..3, -3.1 .. 3, title == "Beta(x,y)")       -- The Beta Function

atf(x,y) ==
  a := atan complex(x,y)
  point [x,y,real a, argument a]

draw(atf, -3.0..%pi, -3.0..%pi)                         -- The Arctangent function
```

B.6 images7.input

First we look at the conformal map $z \mapsto z + 1/z$.

```
)read conformal                                          -- Read program for drawing
                                                         --    conformal maps.

f z == z                                                 -- The coordinate grid for the
                                                         --    complex plane.
conformalDraw(f, -2..2, -2..2, 9, 9, "cartesian")        -- Mapping 1: Source

f z == z + 1/z                                           -- The map z ↦ z + 1/z

conformalDraw(f, -2..2, -2..2, 9, 9, "cartesian")        -- Mapping 1: Target
```

The map $z \mapsto -(z+1)/(z-1)$ maps the unit disk to the right half-plane, as shown on the Riemann sphere.

```
f z == z                                                 -- The unit disk.

riemannConformalDraw(f,0.1..0.99,0..2*%pi,7,11,"polar")  -- Mapping 2: Source

f z == -(z+1)/(z-1)                                      -- The map x ↦ −(z + 1)/(z − 1).
riemannConformalDraw(f,0.1..0.99,0..2*%pi,7,11,"polar")  -- Mapping 2: Target

riemannSphereDraw(-4..4, -4..4, 7, 7, "cartesian")       -- Riemann Sphere Mapping
```

B.7 images8.input

```
)read dhtri
)read tetra
drawPyramid 4                                            -- Sierpinsky's Tetrahedron

\index{Sierpinsky's Tetrahedron}
)read antoine
drawRings 2                                              -- Antoine's Necklace

\index{Antoine's Necklace}
)read scherk
drawScherk(3,3)                                          -- Scherk's Minimal Surface

\index{Scherk's minimal surface}
)read ribbonsNew
drawRibbons([x^i for i in 1..5], x=-1..1, y=0..2)        -- Ribbon Plot
```

B.8 conformal.input

The functions in this section draw conformal maps both on the plane and on the Riemann sphere.

```
C := Complex DoubleFloat                                 -- Complex Numbers
S := Segment DoubleFloat                                 -- Draw ranges
R3 := Point DFLOAT                                       -- Points in 3-space
```

conformalDraw*(f, rRange, tRange, rSteps, tSteps, coord)* draws the image of the coordinate grid under *f* in the complex plane. The grid may be given in either polar or Cartesian coordinates. Argument *f* is the function to draw; *rRange* is the range of the radius (in polar) or real (in Cartesian); *tRange* is

the range of θ (in polar) or imaginary (in Cartesian); *tSteps, rSteps*, are the number of intervals in the r and θ directions; and *coord* is the coordinate system to use (either `"polar"` or `"cartesian"`).

```
conformalDraw: (C -> C, S, S, PI, PI, String) -> VIEW3D
conformalDraw(f,rRange,tRange,rSteps,tSteps,coord) ==
  transformC :=                                    -- Function for changing an (x, y)
    coord = "polar" => polar2Complex               --   pair into a complex number.
    cartesian2Complex
  cm := makeConformalMap(f, transformC)
  sp := createThreeSpace()                         -- Create a fresh space.
  adaptGrid(sp, cm, rRange, tRange, rSteps, tSteps) -- Plot the coordinate lines.
  makeViewport3D(sp, "Conformal Map")              -- Draw the image.
```

riemannConformalDraw*(f, rRange, tRange, rSteps, tSteps, coord)* draws the image of the coordinate grid under f on the Riemann sphere. The grid may be given in either polar or Cartesian coordinates. Its arguments are the same as those for **conformalDraw**.

```
riemannConformalDraw:(C->C,S,S,PI,PI,String)->VIEW3D
riemannConformalDraw(f, rRange, tRange,
                     rSteps, tSteps, coord) ==
  transformC :=                                    -- Function for changing an (x, y)
    coord = "polar" => polar2Complex               --   pair into a complex number.
    cartesian2Complex
  sp := createThreeSpace()                         -- Create a fresh space.
  cm := makeRiemannConformalMap(f, transformC)
  adaptGrid(sp, cm, rRange, tRange, rSteps, tSteps) -- Plot the coordinate lines.
  curve(sp,[point [0,0,2.0@DFLOAT,0],point [0,0,2.0@DFLOAT,0]]) -- Add an invisible point at
  makeViewport3D(sp,"Map on the Riemann Sphere")   --   the north pole for scaling.

adaptGrid(sp, f, uRange, vRange,  uSteps, vSteps) == -- Plot the coordinate grid
  delU := (high(uRange) - low(uRange))/uSteps      --   using adaptive plotting for
  delV := (high(vRange) - low(vRange))/vSteps      --   coordinate lines, and draw
  uSteps := uSteps + 1; vSteps := vSteps + 1       --   tubes around the lines.
  u := low uRange
  for i in 1..uSteps repeat                        -- Draw coordinate lines in the v
    c := curryLeft(f,u)                            --   direction; curve c fixes the
    cf := (t:DFLOAT):DFLOAT +-> 0                  --   current value of u.
    makeObject(c,vRange::SEG Float,colorFunction==cf, -- Draw the v coordinate line.
      space == sp, tubeRadius == .02, tubePoints == 6)
    u := u + delU
  v := low vRange
  for i in 1..vSteps repeat                        -- Draw coordinate lines in the u
    c := curryRight(f,v)                           --   direction; curve c fixes the
    cf := (t:DFLOAT):DFLOAT +-> 1                  --   current value of v.
    makeObject(c,uRange::SEG Float,colorFunction==cf, -- Draw the u coordinate line.
      space == sp, tubeRadius == .02, tubePoints == 6)
    v := v + delV
  void()

riemannTransform(z) ==                             -- Map a point in the complex
  r := sqrt norm z                                 --   plane to the Riemann sphere.
  cosTheta := (real z)/r
  sinTheta := (imag z)/r
  cp := 4*r/(4+r^2)
  sp := sqrt(1-cp*cp)
  if r>2 then sp := -sp
  point [cosTheta*cp, sinTheta*cp, -sp + 1]

cartesian2Complex(r:DFLOAT, i:DFLOAT):C ==         -- Convert Cartesian coordinates to
  complex(r, i)                                    --   complex Cartesian form.

polar2Complex(r:DFLOAT, th:DFLOAT):C ==            -- Convert polar coordinates to
```

```
59    complex(r*cos(th), r*sin(th))                          --    complex Cartesian form.
60
61  makeConformalMap(f, transformC) ==                       -- Convert complex function f to a
62    (u:DFLOAT,v:DFLOAT):R3 +->                             --    mapping: (DFLOAT,DFLOAT)
    → R3
63      z := f transformC(u, v)                              --    in the complex plane.
64      point [real z, imag z, 0.0@DFLOAT]
65
66  makeRiemannConformalMap(f, transformC) ==                -- Convert a complex function f to a
67    (u:DFLOAT, v:DFLOAT):R3 +->                            --    mapping: (DFLOAT,DFLOAT)
    → R3
68      riemannTransform f transformC(u, v)                  --    on the Riemann sphere.
69
70  riemannSphereDraw: (S, S, PI, PI, String) -> VIEW3D      -- Draw a picture of the mapping
71  riemannSphereDraw(rRange,tRange,rSteps,tSteps,coord) ==   --    of the complex plane to
72    transformC :=                                          --    the Riemann sphere.
73      coord = "polar" => polar2Complex
74      cartesian2Complex
75    grid := (u:DFLOAT, v:DFLOAT): R3 +->                   -- Coordinate grid function.
76      z1 := transformC(u, v)
77      point [real z1, imag z1, 0]
78    sp := createThreeSpace()                               -- Create a fresh space.
79    adaptGrid(sp, grid, rRange, tRange, rSteps, tSteps)     -- Draw the flat grid.
80    connectingLines(sp,grid,rRange,tRange,rSteps,tSteps)
81    makeObject(riemannSphere,0..2*%pi,0..%pi,space==sp)    -- Draw the sphere.
82    f := (z:C):C +-> z
83    cm := makeRiemannConformalMap(f, transformC)
84    adaptGrid(sp, cm, rRange, tRange, rSteps, tSteps)      -- Draw the sphere grid.
85    makeViewport3D(sp, "Riemann Sphere")
86
87  connectingLines(sp,f,uRange,vRange,uSteps,vSteps) ==     -- Draw the lines that connect
88    delU := (high(uRange) - low(uRange))/uSteps            --    the points in the complex
89    delV := (high(vRange) - low(vRange))/vSteps            --    plane to the north pole
90    uSteps := uSteps + 1; vSteps := vSteps + 1             --    of the Riemann sphere.
91    u := low uRange
92    for i in 1..uSteps repeat                              -- For each u.
93      v := low vRange
94      for j in 1..vSteps repeat                            -- For each v.
95        p1 := f(u,v)
96        p2 := riemannTransform complex(p1.1, p1.2)         -- Project p1 onto the sphere.
97        fun := lineFromTo(p1,p2)                           -- Create a line function.
98        cf := (t:DFLOAT):DFLOAT +-> 3
99        makeObject(fun, 0..1,space==sp,tubePoints==4,      -- Draw the connecting line.
100                   tubeRadius==0.01,colorFunction==cf)
101       v := v + delV
102     u := u + delU
103   void()
104
105 riemannSphere(u,v) ==                                    -- A sphere sitting on the
106   sv := sin(v)                                           --    complex plane, with radius 1.
107   0.99@DFLOAT*(point [cos(u)*sv,sin(u)*sv,cos(v),0.0@DFLOAT])+
108     point [0.0@DFLOAT, 0.0@DFLOAT, 1.0@DFLOAT, 4.0@DFLOAT]
109
110 lineFromTo(p1, p2) ==                                    -- Create a line function
111   d := p2 - p1                                           --    that goes from p1 to p2
112   (t:DFLOAT):Point DFLOAT +->
113     p1 + t*d
```

B.9 tknot.input

Create a (p, q) torus-knot with radius r around the curve. The formula was derived by Larry Lambe.

```
)read ntube
torusKnot: (DFLOAT, DFLOAT, DFLOAT, PI, PI) -> VIEW3D
torusKnot(p, q ,r, uSteps, tSteps) ==
  knot := (t:DFLOAT):Point DFLOAT +->                    -- Function for the torus knot.
    fac := 4/(2.2@DFLOAT-sin(q*t))
    fac * point [cos(p*t), sin(p*t), cos(q*t)]
  circle := (u:DFLOAT, t:DFLOAT): Point DFLOAT +->       -- The cross section.
    r * point [cos u, sin u]
  ntubeDrawOpt(knot, circle, 0..2*%pi, 0..2*%pi,         -- Draw the circle around the knot.
               var1Steps == uSteps, var2Steps == tSteps)
```

B.10 ntube.input

The functions in this file create generalized tubes (also known as generalized cylinders). These functions draw a 2-d curve in the normal planes around a 3-d curve.

```
R3 := Point DFLOAT                                       -- Points in 3-Space
R2 := Point DFLOAT                                       -- Points in 2-Space
S := Segment Float                                       -- Draw ranges
                                                         -- Introduce types for functions for:
ThreeCurve := DFLOAT -> R3                               --   —the space curve function
TwoCurve := (DFLOAT, DFLOAT) -> R2                       --   —the plane curve function
Surface := (DFLOAT, DFLOAT) -> R3                        --   —the surface function
                                                         -- Frenet frames define a
FrenetFrame :=                                           --   coordinate system around a
   Record(value:R3,tangent:R3,normal:R3,binormal:R3)     --   point on a space curve.
frame: FrenetFrame                                       -- The current Frenet frame
                                                         --   for a point on a curve.
```

ntubeDraw*(spaceCurve, planeCurve, $u_0..u_1$, $t_0..t_1$)* draws *planeCurve* in the normal planes of *spaceCurve*. The parameter $u_0..u_1$ specifies the parameter range for *planeCurve* and $t_0..t_1$ specifies the parameter range for *spaceCurve*. Additionally, the plane curve function takes a second parameter: the current parameter of *spaceCurve*. This allows the plane curve to change shape as it goes around the space curve. See Section B.4 on page 764 for an example of this.

```
ntubeDraw: (ThreeCurve,TwoCurve,S,S) -> VIEW3D
ntubeDraw(spaceCurve,planeCurve,uRange,tRange) ==
  ntubeDrawOpt(spaceCurve, planeCurve, uRange, _
               tRange, []$List DROPT)

ntubeDrawOpt: (ThreeCurve,TwoCurve,S,S,List DROPT)
    -> VIEW3D
ntubeDrawOpt(spaceCurve,planeCurve,uRange,tRange,l) ==   -- This function is similar
                                                         --   to ntubeDraw, but takes
  delT:DFLOAT := (high(tRange) - low(tRange))/10000      --   optional parameters that it
  oldT:DFLOAT := low(tRange) - 1                         --   passes to the draw command.
  fun := ngeneralTube(spaceCurve,planeCurve,delT,oldT)
  draw(fun, uRange, tRange, l)
```

nfrenetFrame*(c, t, delT)* numerically computes the Frenet frame about the curve c at t. Parameter *delT* is a small number used to compute derivatives.

```
nfrenetFrame(c, t, delT) ==
```

```
  f0 := c(t)
  f1 := c(t+delT)
  t0 := f1 - f0                                          -- The tangent.
  n0 := f1 + f0
  b := cross(t0, n0)                                     -- The binormal.
  n := cross(b,t0)                                       -- The normal.
  ln := length n
  lb := length b
  ln = 0 or lb = 0 =>
      error "Frenet Frame not well defined"
  n := (1/ln)*n                                          -- Make into unit length vectors.
  b := (1/lb)*b
  [f0, t0, n, b]$FrenetFrame
```

ngeneralTube*(spaceCurve, planeCurve,delT, oltT)* creates a function that can be passed to the system draw command. The function is a parameterized surface for the general tube around *spaceCurve*. *delT* is a small number used to compute derivatives. *oldT* is used to hold the current value of the *t* parameter for *spaceCurve*. This is an efficiency measure to ensure that frames are only computed once for each value of *t*.

```
ngeneralTube: (ThreeCurve, TwoCurve, DFLOAT, DFLOAT) -> Surface
ngeneralTube(spaceCurve, planeCurve, delT, oldT) ==
  free frame                                             -- Indicate that frame is global.
  (v:DFLOAT, t: DFLOAT): R3 +->
    if (t ~= oldT) then                                  -- If not already computed,
      frame := nfrenetFrame(spaceCurve, t, delT)         --   compute new frame.
      oldT := t
    p := planeCurve(v, t)
    frame.value + p.1*frame.normal + p.2*frame.binormal  -- Project p into the normal plane.
```

B.11 dhtri.input

Create affine transformations (DH matrices) that transform a given triangle into another.

```
tri2tri: (List Point DFLOAT, List Point DFLOAT) -> DHMATRIX(DFLOAT)
                                                         -- Compute a DHMATRIX that
tri2tri(t1, t2) ==                                       --   transforms t1 to t2, where
  n1 := triangleNormal(t1)                               --   t1 and t2 are the vertices
  n2 := triangleNormal(t2)                               --   of two triangles in 3-space.
  tet2tet(concat(t1, n1), concat(t2, n2))

tet2tet: (List Point DFLOAT, List Point DFLOAT) -> DHMATRIX(DFLOAT)
                                                         -- Compute a DHMATRIX that
tet2tet(t1, t2) ==                                       --   transforms t1 to t2,
  m1 := makeColumnMatrix t1                              --   where t1 and t2 are the
  m2 := makeColumnMatrix t2                              --   vertices of two tetrahedrons
  m2 * inverse(m1)                                       --   in 3-space.

makeColumnMatrix(t) ==                                   -- Put the vertices of a tetra-
  m := new(4,4,0)$DHMATRIX(DFLOAT)                       --   hedron into matrix form.
  for x in t for i in 1..repeat
    for j in 1..3 repeat
      m(j,i) := x.j
    m(4,i) := 1
  m

triangleNormal(t) ==                                     -- Compute a vector normal to
  a := triangleArea t                                    --   the given triangle, whose
```

```
  p1 := t.2 - t.1                                   -- length is the square root
  p2 := t.3 - t.2                                   -- of the area of the triangle.
  c := cross(p1, p2)
  len := length(c)
  len = 0 => error "degenerate triangle!"
  c := (1/len)*c
  t.1 + sqrt(a) * c

triangleArea t ==                                   -- Compute the area of a
  a := length(t.2 - t.1)                            -- triangle using Heron's
  b := length(t.3 - t.2)                            -- formula.
  c := length(t.1 - t.3)
  s := (a+b+c)/2
  sqrt(s*(s-a)*(s-b)*(s-c))
```

B.12 tetra.input

```
)set expose add con DenavitHartenbergMatrix         -- Bring DH matrices into the
                                                    --   environment.
x1:DFLOAT := sqrt(2.0@DFLOAT/3.0@DFLOAT)            -- Set up the coordinates of the
x2:DFLOAT := sqrt(3.0@DFLOAT)/6                     --   corners of the tetrahedron.

z := 0.0@DFLOAT
h := 0.5@DFLOAT

p1 := point [-h, -x2, z]                            -- Some needed points.
p2 := point [h, -x2, z]
p3 := point [z, 2*x2, z]
p4 := point [z, z, x1]

baseTriangle  := [p2, p1, p3]                       -- The base of the tetrahedron.

mt := [h*(p2+p1), h*(p1+p3), h*(p3+p2)]             -- The "middle triangle" inscribed
                                                    --   in the base of the tetrahedron.
bt1 := [mt.1, p1, mt.2]                             -- The bases of the triangles of
bt2 := [p2, mt.1, mt.3]                             --   the subdivided tetrahedron.
bt3 := [mt.2, p3, mt.3]
bt4 := [h*(p2+p4), h*(p1+p4), h*(p3+p4)]

tt1 := tri2tri(baseTriangle, bt1)                   -- Create the transformations
tt2 := tri2tri(baseTriangle, bt2)                   --   that bring the base of the
tt3 := tri2tri(baseTriangle, bt3)                   --   tetrahedron to the bases of
tt4 := tri2tri(baseTriangle, bt4)                   --   the subdivided tetrahedron.

drawPyramid(n) ==                                   -- Draw a Sierpinsky tetrahedron
  s := createThreeSpace()                           --   with n levels of recursive
  dh := rotatex(0.0@DFLOAT)                         --   subdivision.
  drawPyramidInner(s, n, dh)
  makeViewport3D(s, "Sierpinsky Tetrahedron")

drawPyramidInner(s, n, dh) ==                       -- Recursively draw a Sierpinsky
  n = 0 => makeTetrahedron(s, dh, n)                --   tetrahedron.
  drawPyramidInner(s, n-1, dh * tt1)                -- Draw the 4 recursive pyramids.
  drawPyramidInner(s, n-1, dh * tt2)
  drawPyramidInner(s, n-1, dh * tt3)
  drawPyramidInner(s, n-1, dh * tt4)

makeTetrahedron(sp, dh, color) ==                   -- Draw a tetrahedron into the
  w1 := dh*p1                                       --   given space with the given
```

```
43    w2 := dh*p2                                    -- color, transforming it by
44    w3 := dh*p3                                    -- the given DH matrix.
45    w4 := dh*p4
46    polygon(sp, [w1, w2, w4])
47    polygon(sp, [w1, w3, w4])
48    polygon(sp, [w2, w3, w4])
49    void()
```

B.13 antoine.input

Draw Antoine's Necklace. Thank you to Matthew Grayson at IBM's T.J Watson Research Center for the idea.

```
1   )set expose add con DenavitHartenbergMatrix    -- Bring DH matrices into
2                                                  --   the environment.
3   torusRot: DHMATRIX(DFLOAT)                     -- The current transformation for
4                                                  --   drawing a sub ring.
5
6   drawRings(n) ==                                -- Draw Antoine's Necklace with n
7     s := createThreeSpace()                      --   levels of recursive subdivision.
8     dh:DHMATRIX(DFLOAT) := identity()            --   The number of subrings is 10^n.
9     drawRingsInner(s, n, dh)                     -- Do the real work.
10    makeViewport3D(s, "Antoine's Necklace")
```

In order to draw Antoine rings, we take one ring, scale it down to a smaller size, rotate it around its central axis, translate it to the edge of the larger ring and rotate it around the edge to a point corresponding to its count (there are 10 positions around the edge of the larger ring). For each of these new rings we recursively perform the operations, each ring becoming 10 smaller rings. Notice how the **DHMATRIX** operations are used to build up the proper matrix composing all these transformations.

```
12  F ==> DFLOAT
13  drawRingsInner(s, n, dh) ==                    -- Recursively draw Antoine's
14    n = 0 =>                                     --   Necklace.
15      drawRing(s, dh)
16      void()
17    t := 0.0@F                                   -- Angle around ring.
18    p := 0.0@F                                   -- Angle of subring from plane.
19    tr := 1.0@F                                  -- Amount to translate subring.
20    inc := 0.1@F                                 -- The translation increment.
21    for i in 1..10 repeat                        -- Subdivide into 10 linked rings.
22      tr := tr + inc
23      inc := -inc
24      dh' := dh*rotatez(t)*translate(tr,0.0@F,0.0@F)*  -- Transform ring in center
25             rotatey(p)*scale(0.35@F, 0.48@F, 0.4@F)   --   to a link.
26      drawRingsInner(s, n-1, dh')
27      t := t + 36.0@F
28      p := p + 90.0@F
29    void()
30
31  drawRing(s, dh) ==                             -- Draw a single ring into
32    free torusRot                                --   the given subspace,
33    torusRot := dh                               --   transformed by the given
34    makeObject(torus, 0..2*%pi, 0..2*%pi, var1Steps == 6,  --   DHMATRIX.
35               space == s, var2Steps == 15)
36
37  torus(u ,v) ==                                 -- Parameterization of a torus,
38    cu := cos(u)/6                               --   transformed by the
39    torusRot*point [(1+cu)*cos(v),(1+cu)*sin(v),(sin u)/6]  --   DHMATRIX in torusRot.
```

B.14 scherk.input

Scherk's minimal surface, defined by: $e^z \cos(x) = \cos(y)$. See: *A Comprehensive Introduction to Differential Geometry,* Vol. 3, by Michael Spivak, Publish Or Perish, Berkeley, 1979, pp. 249-252.

```
(xOffset, yOffset):DFLOAT                              -- Offsets for a single piece
                                                       --   of Scherk's minimal surface.

drawScherk(m,n) ==                                     -- Draw Scherk's minimal surface
  free xOffset, yOffset                                --   on an m by n patch.
  space := createThreeSpace()
  for i in 0..m-1 repeat
    xOffset := i*%pi
    for j in 0 .. n-1 repeat
      rem(i+j, 2) = 0 => 'iter                         -- Draw only odd patches.
      yOffset := j*%pi
      drawOneScherk(space)                             -- Draw a patch.
  makeViewport3D(space, "Scherk's Minimal Surface")

scherk1(u,v) ==                                        -- The first patch that makes
  x := cos(u)/exp(v)                                   --   up a single piece of
  point [xOffset + acos(x), yOffset + u, v, abs(v)]    --   Scherk's minimal surface.

scherk2(u,v) ==                                        -- The second patch.
  x := cos(u)/exp(v)
  point [xOffset - acos(x), yOffset + u, v, abs(v)]

scherk3(u,v) ==                                        -- The third patch.
  x := exp(v) * cos(u)
  point [xOffset + u, yOffset + acos(x), v, abs(v)]

scherk4(u,v) ==                                        -- The fourth patch.
  x := exp(v) * cos(u)
  point [xOffset + u, yOffset - acos(x), v, abs(v)]

drawOneScherk(s) ==                                    -- Draw the surface by
  makeObject(scherk1,-%pi/2..%pi/2,0..%pi/2,space==s,  --   breaking it into four
             var1Steps == 28, var2Steps == 28)         --   patches and then drawing
  makeObject(scherk2,-%pi/2..%pi/2,0..%pi/2,space==s,  --   the patches.
             var1Steps == 28, var2Steps == 28)
  makeObject(scherk3,-%pi/2..%pi/2,-%pi/2..0,space==s,
             var1Steps == 28, var2Steps == 28)
  makeObject(scherk4,-%pi/2..%pi/2,-%pi/2..0,space==s,
             var1Steps == 28, var2Steps == 28)
  void()
```

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